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DOUBLE WIJSMAN LACUNARY STATISTICAL CONVERGENCE OF ORDER α

ESRA GÜLLE*, UĞUR ULUSU

Abstract. In this paper, we introduce the concepts of Wijsman strongly p-lacunary summability of order α, Wijsman lacunary statistical convergence of order α and Hausdorff lacunary statistical convergence of order α for double set sequences. Also, we investigate some properties of these new concepts and examine the existence of some relationships between them. Furthermore, we study the relationships between these new concepts and some concepts in the literature.

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1. Introduction

The concept of statistical convergence was introduced by Fast [13] and Steinhaus [34], and later reintroduced by Schoenberg [32], independently. Then, this concept has been developmend by many researchers until recently (see, [5, 7, 8, 15, 17, 27, 35, 40]).

In [14], Freedman et al. established the connection between the strongly Cesàro summable sequences space $|\sigma_1|$ and the strongly lacunary summable sequences space N_θ defined by a lacunary sequence θ. Then, using lacunary sequence concept, Fridy and Orhan [16] defined the concept of lacunary statistical convergence. Recently, the concepts of lacunary statistical convergence of order α and strongly p-lacunary summability of order α were studied by Şengül and Et [36]. For more detail, see [12].

In [26], Pringsheim introduced the concept of convergence for double sequences. Then, Mursaleen and Edely [19] extended this concept to statistical convergence. Also, using double lacunary sequence concept, the concept of

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*Corresponding author.

lacunary statistical convergence was studied by Patterson and Savaş [25]. Recently, Çolak and Altın [9] defined the concept of statistical convergence of order α for double sequences. Also, the concepts of almost statistical and almost lacunary statistical convergence of order α for double sequences were studied by Savaş [29, 30]. More developments on double sequences can be found in [4, 6, 10, 11, 18, 20, 28, 39].

The concepts of convergence for number sequences were transferred to the concepts of convergence for set sequences by many authors. The concepts of Wijsman convergence and Hausdorff convergence are two of these transfers (see, [1, 2, 3, 43]). Nuray and Rhoades [21] extended the concepts of Wijsman convergence and Hausdorff convergence to statistical convergence for set sequences and gave some basic theorems. Then, using lacunary sequence concept, the concept of lacunary statistical convergence for set sequences was introduced by Ulusu and Nuray [41]. Recently, using ideal concept, the concept of Wijsman $\mathcal{I}$-lacunary statistical convergence of order α was studied by both Savaş [31] and Şengül and Et [37], independently.

In [22, 23, 24], Nuray et al. introduced the concepts of Wijsman strongly Cesàro summability, Wijsman statistical convergence, Wijsman strongly lacunary summability and Wijsman lacunary statistical convergence for double set sequences. Then, the concept of Hausdorff statistical convergence for double set sequences was studied by Talo et al. [38].

Lately, the concepts of Wijsman strongly p-Cesàro summability of order α, Wijsman statistical convergence of order α and Hausdorff statistical convergence of order α for double set sequences were studied by Ulusu and Gülle [42].

2. Definitions and Notations

Firstly, we recall the basic concepts that need for a good understanding of our study (see, [1, 2, 22, 23, 24, 26, 28, 33, 38, 42]).

A double sequence (x_{ij}) is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{ij} - L| < \varepsilon$, whenever $i, j > N_\varepsilon$.

Let X be any non-empty set. The function $f : \mathbb{N} \to P(X)$ is defined by $f(i) = U_i \in P(X)$ for each $i \in \mathbb{N}$, where $P(X)$ is power set of X. The sequence $\{U_i\} = \{U_1, U_2, ...\}$, which is the range's elements of f, is said to be set sequences.

Let (X, d) be a metric space. For any point $x \in X$ and any non-empty subset U of X, the distance from x to U is defined by

$$\rho(x, U) = \inf_{u \in U} d(x, u).$$

Throughout the study, (X, d) will be taken as a metric space and U, U_{ij} be any non-empty closed subsets of X.

A double sequence $\{U_{ij}\}$ is said to be Wijsman convergent to U if for each $x \in X$,

$$\lim_{i,j \to \infty} \rho(x, U_{ij}) = \rho(x, U).$$

A double sequence $\{U_{ij}\}$ is said to be Hausdorff convergent to U if

$$\lim_{i,j\to\infty}\sup_{x\in X}|\rho(x,U_{ij})-\rho(x,U)|=0.$$

A double sequence $\{U_{ij}\}$ is said to be Wijsman statistical convergent to U if for every $\varepsilon>0$ and each $x\in X$,

$$\lim_{m,n\to\infty}\frac{1}{mn}\Big|\Big\{(i,j):\ i\leq n, j\leq m,\ |\rho(x,U_{ij})-\rho(x,U)|\geq\varepsilon\Big\}\Big|=0.$$

The class of all Wijsman statistical convergent sequences denotes by simply $W(S_2)$.

A double sequence $\{U_{ij}\}$ is said to be Hausdorff statistical convergent to U if for every $\varepsilon>0$,

$$\lim_{m,n\to\infty}\frac{1}{mn}\Big|\Big\{(i,j):\ i\leq n, j\leq m,\ \sup_{x\in X}|\rho(x,U_{ij})-\rho(x,U)|\geq\varepsilon\Big\}\Big|=0.$$

A double sequence $\theta_2=\{(k_r,j_u)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0=0,\ h_r=k_r-k_{r-1}\to\infty\ \text{ and }\ j_0=0,\ \bar{h}_u=j_u-j_{u-1}\to\infty\ \text{ as }\ r,u\to\infty.$$

We use the following notations in the sequel:

$$k_{ru}=k_rj_u,\ h_{ru}=h_r\bar{h}_u,\ I_{ru}=\{(i,j):k_{r-1}<i\leq k_r\ \text{ and }\ j_{u-1}<j\leq j_u\},$$

$$q_r=\frac{k_r}{k_{r-1}}\ \text{ and }\ q_u=\frac{j_u}{j_{u-1}}.$$

Throughout the study, $\theta_2=\{(k_r,j_u)\}$ will be taken as a double lacunary sequence.

A double sequence $\{U_{ij}\}$ is said to be Wijsman lacunary convergent to U if for each $x\in X$,

$$\lim_{r,u\to\infty}\frac{1}{h_{ru}}\sum_{(i,j)\in I_{ru}}\rho(x,U_{ij})=\rho(x,U).$$

Let $0<p<\infty$. A double sequence $\{U_{ij}\}$ is said to be Wijsman strongly p-lacunary convergent to U if for each $x\in X$,

$$\lim_{r,u\to\infty}\frac{1}{h_{ru}}\sum_{(i,j)\in I_{ru}}|\rho(x,U_{ij})-\rho(x,U)|^p=0.$$

The class of all Wijsman strongly p-lacunary convergent sequences denotes by simply $[W_2N_\theta]^p$.

A double sequence $\{U_{ij}\}$ is said to be Wijsman lacunary statistically convergent to U if for every $\varepsilon>0$ and each $x\in X$,

$$\lim_{r,u\to\infty}\frac{1}{h_{ru}}\Big|\Big\{(i,j)\in I_{ru}:|\rho(x,U_{ij})-\rho(x,U)|\geq\varepsilon\Big\}\Big|=0.$$

The class of all Wijsman lacunary statistically convergent sequences denotes by simply W_2S_θ.

Let $0 < \alpha \leq 1$. A double sequence $\{U_{ij}\}$ is said to be Wijsman strongly Cesàro summable of order α to U or $W[C_2^\alpha]$-summable to U if for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{(mn)^\alpha} \sum_{i,j=1,1}^{m,n} |\rho(x,U_{ij}) - \rho(x,U)| = 0.$$

The class of all $W[C_2^\alpha]$-summable sequences denotes by simply $W[C_2^\alpha]$.

Let $0 < \alpha \leq 1$. A double sequence $\{U_{ij}\}$ is said to be Wijsman statistically convergent of order α to U or $W(S_2^\alpha)$-convergent to U if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{(mn)^\alpha} \Big| \big\{ (i,j) : \ i \leq m, j \leq n, \ |\rho(x,U_{ij}) - \rho(x,U)| \geq \varepsilon \big\} \Big| = 0.$$

The class of all $W(S_2^\alpha)$-convergent sequences denotes by simply $W(S_2^\alpha)$.

From now on, for short, we use $\rho_x(U)$ and $\rho_x(U_{ij})$ instead of $\rho(x,U)$ and $\rho(x,U_{ij})$, respectively.

3. New Concepts

In this section, we introduce the concepts of Wijsman strongly p-lacunary summability of order α, Wijsman lacunary statistical convergence of order α and Hausdorff lacunary statistical convergence of order α for double set sequences.

Definition 3.1. Let $0 < \alpha \leq 1$. A double sequence $\{U_{ij}\}$ is Wijsman lacunary summable of order α to U or $W_2N_\theta^\alpha$-summable to U if for each $x \in X$,

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}^\alpha} \sum_{(i,j)\in I_{ru}} \rho_x(U_{ij}) = \rho_x(U).$$

In this case, we write $U_{ij} \stackrel{W_2N_\theta^\alpha}{\longrightarrow} U$ or $U_{ij} \longrightarrow U\big(W_2N_\theta^\alpha\big)$.

Definition 3.2. Let $0 < \alpha \leq 1$. A double sequence $\{U_{ij}\}$ is Wijsman strongly p-lacunary summable of order α to U or $[W_2N_\theta^\alpha]^p$-summable to U if for each $x \in X$,

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}^\alpha} \sum_{(i,j)\in I_{ru}} |\rho_x(U_{ij}) - \rho_x(U)|^p = 0.$$

In this case, we write $U_{ij} \stackrel{[W_2N_\theta^\alpha]^p}{\longrightarrow} U$ or $U_{ij} \longrightarrow U\big([W_2N_\theta^\alpha]^p\big)$. If $p = 1$, then the double sequence $\{U_{ij}\}$ is simply said to be Wijsman strongly lacunary summable of order α to U and we write $U_{ij} \stackrel{[W_2N_\theta^\alpha]}{\longrightarrow} U$ or $U_{ij} \longrightarrow U\big([W_2N_\theta^\alpha]\big)$.

The class of all $[W_2N_\theta^\alpha]^p$-summable sequences will be denoted by simply $[W_2N_\theta^\alpha]^p$.

Example 3.3. Let $X = \mathbb{R}^2$ and a double sequence $\{U_{ij}\}$ be defined as following:

$$U_{ij} := \begin{cases} \Big\{(x,y) : (x-1)^2 + y^2 = \frac{1}{ij}\Big\} & ; \text{ if } (i,j) \in I_{ru},\ i \text{ and } j \text{ are square integers} \\ \{(0,1)\} & ; \text{ otherwise.} \end{cases}$$

Then, the double sequence $\{U_{ij}\}$ is Wijsman strongly lacunary summable of order α to the set $U = \{(0,1)\}$.

Remark 3.1. For $\alpha = 1$, the concepts of $W_2N_\theta^\alpha$-summability and $[W_2N_\theta^\alpha]^p$-summability coincide with the concepts of Wijsman lacunary convergence and Wijsman strongly p-lacunary convergence for double set sequences in [22], respectively.

Definition 3.4. Let $0 < \alpha \leq 1$. A double sequence $\{U_{ij}\}$ is Wijsman lacunary statistically convergent of order α to U or $W_2S_\theta^\alpha$-convergent to U if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}^\alpha} \Big| \big\{ (i,j) \in I_{ru} : |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon \big\} \Big| = 0.$$

In this case, we write $U_{ij} \xrightarrow{W_2S_\theta^\alpha} U$ or $U_{ij} \longrightarrow U\big(W_2S_\theta^\alpha\big)$.

The class of all $W_2S_\theta^\alpha$-convergent sequences will be denoted by simply $W_2S_\theta^\alpha$.

Example 3.5. Let $X = \mathbb{R}^2$ and a double sequence $\{U_{ij}\}$ be defined as following:

$$U_{ij} := \begin{cases} \big\{(x,y) : (x+i)^2 + (y-j)^2 = 1\big\} & ; \text{ if } (i,j) \in I_{ru},\ i \text{ and } j \text{ are square integers} \\ \{(1,-1)\} & ; \text{ otherwise.} \end{cases}$$

Then, the double sequence $\{U_{ij}\}$ is Wijsman lacunary statistically convergent of order α to the set $U = \{(1,-1)\}$.

Remark 3.2. For $\alpha = 1$, the concept of $W_2S_\theta^\alpha$-convergence coincides with the concept of Wijsman lacunary statistical convergence for double set sequences in [23].

Definition 3.6. Let $0 < \alpha \leq 1$. A double sequence $\{U_{ij}\}$ is Hausdorff lacunary statistically convergent of order α to U or $H_2(S_\theta^\alpha)$-convergent to U if for every $\varepsilon > 0$,

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}^\alpha} \Big| \big\{ (i,j) \in I_{ru} : \sup_{x\in X} |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon \big\} \Big| = 0.$$

In this case, we write $U_{ij} \xrightarrow{H_2(S_\theta^\alpha)} U$ or $U_{ij} \longrightarrow U\big(H_2(S_\theta^\alpha)\big)$.

The class of all $H_2(S_\theta^\alpha)$-convergent sequences will be denoted by simply $H_2(S_\theta^\alpha)$.

Remark 3.3. For $\alpha = 1$, the concept of $H_2(S_\theta^\alpha)$-convergence coincides with the concept of Hausdorff lacunary statistical convergence for double set sequences, which has not been studied till now.

4. Inclusion Theorems

In this section, firstly, we investigate some properties of the new concepts that we introduced in Section 3 and examine the existence of some relationships between them.

Theorem 4.1. *If $0 < \alpha \leq \beta \leq 1$, then $[W_2N_\theta^\alpha]^p \subseteq [W_2N_\theta^\beta]^p$ for every double lacunary sequence $\theta_2 = \{(k_r, j_u)\}$.*

Proof. Let $0 < \alpha \leq \beta \leq 1$ and suppose that $U_{ij} \stackrel{[W_2N_\theta^\alpha]^p}{\longrightarrow} U$. For each $x \in X$, we have

$$\frac{1}{h_{ru}^\beta} \sum_{(i,j)\in I_{ru}} |\rho_x(U_{ij}) - \rho_x(U)|^p \leq \frac{1}{h_{ru}^\alpha} \sum_{(i,j)\in I_{ru}} |\rho_x(U_{ij}) - \rho_x(U)|^p.$$

Hence, by our assumption, we get $U_{ij} \stackrel{[W_2N_\theta^\beta]^p}{\longrightarrow} U$. Consequently, $[W_2N_\theta^\alpha]^p \subseteq [W_2N_\theta^\beta]^p$. □

If we take $\beta = 1$ in Theorem 4.1, then we obtain the following corollary.

Corollary 4.2. *If a double sequence $\{U_{ij}\}$ is Wijsman strongly p-lacunary summable of order α to U for some $0 < \alpha \leq 1$, then the double sequence is Wijsman strongly p-lacunary summable to U, i.e., $[W_2N_\theta^\alpha]^p \subseteq [W_2N_\theta]^p$.*

Now, without proof, we shall state a theorem that gives a relation between $[W_2N_\theta^\alpha]^p$ and $[W_2N_\theta^\alpha]^q$, where $0 < \alpha \leq 1$ and $0 < p < q < \infty$.

Theorem 4.3. *Let $0 < \alpha \leq 1$ and $0 < p < q < \infty$. Then, $[W_2N_\theta^\alpha]^q \subset [W_2N_\theta^\alpha]^p$ for every double lacunary sequence $\theta_2 = \{(k_r, j_u)\}$.*

Theorem 4.4. *If $0 < \alpha \leq \beta \leq 1$, then $W_2S_\theta^\alpha \subseteq W_2S_\theta^\beta$ for every double lacunary sequence $\theta_2 = \{(k_r, j_u)\}$.*

Proof. Let $0 < \alpha \leq \beta \leq 1$ and suppose that $U_{ij} \stackrel{W_2S_\theta^\alpha}{\longrightarrow} U$. For every $\varepsilon > 0$ and each $x \in X$, we have

$$\frac{1}{h_{ru}^\beta}\Big|\Big\{(i,j) \in I_{ru} : |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon\Big\}\Big| \leq \frac{1}{h_{ru}^\alpha}\Big|\Big\{(i,j) \in I_{ru} : |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon\Big\}\Big|.$$

Hence, by our assumption, we get $U_{ij} \stackrel{W_2S_\theta^\beta}{\longrightarrow} U$. Consequently, $W_2S_\theta^\alpha \subseteq W_2S_\theta^\beta$. □

If we take $\beta = 1$ in Theorem 4.4, then we obtain the following corollary.

Corollary 4.5. *If a double sequence $\{U_{ij}\}$ is Wijsman lacunary statistically convergent of order α to U for some $0 < \alpha \leq 1$, then the double sequence is Wijsman lacunary statistically convergent to U, i.e., $W_2S_\theta^\alpha \subseteq W_2S_\theta$.*

Theorem 4.6. *Let $0 < \alpha \le \beta \le 1$ and $0 < p < \infty$. If a double sequence $\{U_{ij}\}$ is Wijsman strongly p-lacunary summable of order α to U, then the double sequence is Wijsman lacunary statistically convergent of order β to U.*

Proof. Let $0 < \alpha \le \beta \le 1$ and we suppose that the double sequence $\{U_{ij}\}$ is Wijsman strongly p-lacunary summable of order α to U. For every $\varepsilon > 0$ and each $x \in X$, we have

$$\begin{aligned}\sum_{(i,j)\in I_{ru}} |\rho_x(U_{ij}) - \rho_x(U)|^p &= \sum_{\substack{(i,j)\in I_{ru} \\ |\rho_x(U_{ij})-\rho_x(U)|\ge\varepsilon}} |\rho_x(U_{ij}) - \rho_x(U)|^p \\ &\quad + \sum_{\substack{(i,j)\in I_{ru} \\ |\rho_x(U_{ij})-\rho_x(U)|<\varepsilon}} |\rho_x(U_{ij}) - \rho_x(U)|^p \\ &\ge \sum_{\substack{(i,j)\in I_{ru} \\ |\rho_x(U_{ij})-\rho_x(U)|\ge\varepsilon}} |\rho_x(U_{ij}) - \rho_x(U)|^p \\ &\ge \varepsilon^p \left|\left\{(i,j) \in I_{ru} : |\rho_x(U_{ij}) - \rho_x(U)| \ge \varepsilon\right\}\right|\end{aligned}$$

and so

$$\begin{aligned}\frac{1}{h_{ru}^{\alpha}} \sum_{(i,j)\in I_{ru}} |\rho_x(U_{ij}) - \rho_x(U)|^p &\ge \frac{\varepsilon^p}{h_{ru}^{\alpha}} \left|\left\{(i,j) \in I_{ru} : |\rho_x(U_{ij}) - \rho_x(U)| \ge \varepsilon\right\}\right| \\ &\ge \frac{\varepsilon^p}{h_{ru}^{\beta}} \left|\left\{(i,j) \in I_{ru} : |\rho_x(U_{ij}) - \rho_x(U)| \ge \varepsilon\right\}\right|.\end{aligned}$$

Hence, by our assumption, we get that the double sequence $\{U_{ij}\}$ is Wijsman lacunary statistically convergent of order β to U. □

If we take $\beta = \alpha$ in Theorem 4.6, then we obtain the following corollary.

Corollary 4.7. *Let $0 < \alpha \le 1$ and $0 < p < \infty$. If a double sequence $\{U_{ij}\}$ is Wijsman strongly p-lacunary summable of order α to U, then the double sequence is Wijsman lacunary statistically convergent of order α to U.*

Theorem 4.8. *If $0 < \alpha \le \beta \le 1$, then $H_2(S_\theta^\alpha) \subseteq H_2(S_\theta^\beta)$ for every double lacunary sequence $\theta_2 = \{(k_r, j_u)\}$.*

Proof. Let $0 < \alpha \le \beta \le 1$ and suppose that $U_{ij} \xrightarrow{H_2(S_\theta^\alpha)} U$. For every $\varepsilon > 0$, we have

$$\begin{aligned}&\frac{1}{h_{ru}^{\beta}} \left|\left\{(i,j) \in I_{ru} : \sup_{x\in X} |\rho_x(U_{ij}) - \rho_x(U)| \ge \varepsilon\right\}\right| \\ &\qquad\le \frac{1}{h_{ru}^{\alpha}} \left|\left\{(i,j) \in I_{ru} : \sup_{x\in X} |\rho_x(U_{ij}) - \rho_x(U)| \ge \varepsilon\right\}\right|.\end{aligned}$$

Hence, by our assumption, we get $U_{ij} \overset{H_2(S_\theta^\beta)}{\longrightarrow} U$. Consequently, $H_2(S_\theta^\alpha) \subseteq H_2(S_\theta^\beta)$. □

If we take $\beta = 1$ in Theorem 4.8, then we obtain the following corollary.

Corollary 4.9. *If a double sequence $\{U_{ij}\}$ is Hausdorff lacunary statistically convergent of order α to U for some $0 < \alpha \leq 1$, then the double sequence is Hausdorff lacunary statistically convergent to U.*

Theorem 4.10. *Let $0 < \alpha \leq \beta \leq 1$. If a double sequence $\{U_{ij}\}$ is Hausdorff lacunary statistically convergent of order α to U, then the double sequence is Wijsman lacunary statistically convergent of order β to U.*

Proof. Let $0 < \alpha \leq \beta \leq 1$ and suppose that the double sequence $\{U_{ij}\}$ is Hausdorff lacunary statistically convergent of order α to U. For every $\varepsilon > 0$ and each $x \in X$, we have

$$\begin{aligned}
&\frac{1}{h_{ru}^\beta}\Big|\big\{(i,j) \in I_{ru} : |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon\big\}\Big| \\
&\qquad \leq \frac{1}{h_{ru}^\beta}\Big|\big\{(i,j) \in I_{ru} : \sup_{x\in X} |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon\big\}\Big| \\
&\qquad \leq \frac{1}{h_{ru}^\alpha}\Big|\big\{(i,j) \in I_{ru} : \sup_{x\in X} |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon\big\}\Big|.
\end{aligned}$$

Hence, by our assumption, we get that the double sequence $\{U_{ij}\}$ is Wijsman lacunary statistically convergent of order β to U. □

If we take $\beta = \alpha$ in Theorem 4.10, then we obtain the following corollary.

Corollary 4.11. *Let $0 < \alpha \leq 1$. If a double sequence $\{U_{ij}\}$ is Hausdorff lacunary statistically convergent of order α to U, then the double sequence is Wijsman lacunary statistically convergent of order α to U.*

Now, secondly, we study the relationships between the new concepts that we introduced in Section 3 and some concepts in the literature.

Theorem 4.12. *Let $0 < \alpha \leq 1$. If $\liminf_r q_r^\alpha > 1$ and $\liminf_u q_u^\alpha > 1$ for any double lacunary sequence $\theta_2 = \{(k_r, j_u)\}$, then $W[C_2^\alpha] \subseteq [W_2 N_\theta^\alpha]$.*

Proof. Let $0 < \alpha \leq 1$ and suppose that $\liminf_r q_r^\alpha > 1$ and $\liminf_u q_u^\alpha > 1$. Then, there exist $\eta > 0$ and $\mu > 0$ such that $q_r^\alpha \geq 1 + \eta$ and $q_u^\alpha \geq 1 + \mu$ for all r and u, which implies that

$$\frac{k_{ru}^\alpha}{h_{ru}^\alpha} \leq \frac{(1+\eta)(1+\mu)}{\eta\mu} \quad \text{and} \quad \frac{k_{(r-1)(u-1)}^\alpha}{h_{ru}^\alpha} \leq \frac{1}{\eta\mu}.$$

For each $x \in X$, we can write

$$\frac{1}{h_{ru}^{\alpha}} \sum_{(i,j)\in I_{ru}} |\rho_x(U_{ij}) - \rho_x(U)|$$

$$= \frac{1}{h_{ru}^{\alpha}} \sum_{m,n=1,1}^{k_r,j_u} |\rho_x(U_{mn}) - \rho_x(U)|$$

$$- \frac{1}{h_{ru}^{\alpha}} \sum_{m,n=1,1}^{k_{r-1},j_{u-1}} |\rho_x(U_{mn}) - \rho_x(U)|$$

$$= \frac{k_{ru}^{\alpha}}{h_{ru}^{\alpha}} \left(\frac{1}{k_{ru}^{\alpha}} \sum_{m,n=1,1}^{k_r,j_u} |\rho_x(U_{mn}) - \rho_x(U)| \right)$$

$$- \frac{k_{(r-1)(u-1)}^{\alpha}}{h_{ru}^{\alpha}} \left(\frac{1}{k_{(r-1)(u-1)}^{\alpha}} \sum_{m,n=1,1}^{k_{r-1},j_{u-1}} |\rho_x(U_{mn}) - \rho_x(U)| \right).$$

If $U_{ij} \overset{W[C_2^{\alpha}]}{\longrightarrow} U$, then for each $x \in X$

$$\frac{1}{k_{ru}^{\alpha}} \sum_{m,n=1,1}^{k_r,j_u} |\rho_x(U_{mn}) - \rho_x(U)| \to 0 \text{ and } \frac{1}{k_{(r-1)(u-1)}^{\alpha}} \sum_{m,n=1,1}^{k_{r-1},j_{u-1}} |\rho_x(U_{mn}) - \rho_x(U)| \to 0.$$

Thus, when the above equality is considered, for each $x \in X$ we get

$$\frac{1}{h_{ru}^{\alpha}} \sum_{(i,j)\in I_{ru}} |\rho_x(U_{ij}) - \rho_x(U)| \to 0,$$

that is, $U_{ij} \overset{[W_2 N_{\theta}^{\alpha}]}{\longrightarrow} U$. Consequently, $W[C_2^{\alpha}] \subseteq [W_2 N_{\theta}^{\alpha}]$. □

Theorem 4.13. *Let $0 < \alpha \leq 1$. If $\limsup_r q_r < \infty$ and $\limsup_u q_u < \infty$ for any double lacunary sequence $\theta_2 = \{(k_r, j_u)\}$, then $[W_2 N_{\theta}^{\alpha}] \subseteq W[C_2^{\alpha}]$.*

Proof. Let $0 < \alpha \leq 1$ and suppose that $\limsup_r q_r < \infty$ and $\limsup_u q_u < \infty$. Then, there exist $M, N > 0$ such that $q_r < M$ and $q_u < N$ for all r and u. Let $U_{ij} \overset{[W_2 N_{\theta}^{\alpha}]}{\longrightarrow} U$ and $\varepsilon > 0$ be given. Then, for each $x \in X$ we can find $R, U > 0$ and $H > 0$ such that

$$\sup_{i\geq R, j\geq U} \tau_{ij} < \varepsilon \quad \text{and} \quad \tau_{ij} < H \quad \text{for all} \quad i, j = 1, 2, \ldots$$

where

$$\tau_{ru} = \frac{1}{h_{ru}^{\alpha}} \sum_{(i,j)\in I_{ru}} |\rho_x(U_{ij}) - \rho_x(U)|.$$

If m and n are any integers satisfying $k_{r-1} < m \leq k_r$ and $j_{u-1} < n \leq j_u$ where $r > R$ and $u > U$, then for each $x \in X$ we can write

$$
\begin{aligned}
&\frac{1}{(mn)^{\alpha}}\sum_{i,j=1,1}^{m,n}|\rho_x(U_{ij})-\rho_x(U)| \\
&\qquad \le \frac{1}{k^{\alpha}_{(r-1)(u-1)}}\sum_{i,j=1,1}^{k_r,j_u}|\rho_x(U_{ij})-\rho_x(U)| \\
&\qquad = \frac{1}{k^{\alpha}_{(r-1)(u-1)}}\Bigg(\sum_{I_{11}}|\rho_x(U_{ij})-\rho_x(U)| \\
&\qquad\quad +\sum_{I_{12}}|\rho_x(U_{ij})-\rho_x(U)|+\sum_{I_{21}}|\rho_x(U_{ij})-\rho_x(U)| \\
&\qquad\quad +\sum_{I_{22}}|\rho_x(U_{ij})-\rho_x(U)| \\
&\qquad\quad +\cdots+\sum_{I_{ru}}|\rho_x(U_{ij})-\rho_x(U)|\Bigg) \\
&\qquad = \frac{h^{\alpha}_{11}}{k^{\alpha}_{(r-1)(u-1)}}\tau_{11}+\frac{h^{\alpha}_{12}}{k^{\alpha}_{(r-1)(u-1)}}\tau_{12}+\frac{h^{\alpha}_{21}}{k^{\alpha}_{(r-1)(u-1)}}\tau_{21} \\
&\qquad\quad +\frac{h^{\alpha}_{22}}{k^{\alpha}_{(r-1)(u-1)}}\tau_{22}+\cdots+\frac{h^{\alpha}_{ru}}{k^{\alpha}_{(r-1)(u-1)}}\tau_{ru} \\
&\qquad \le \sum_{i,j=1,1}^{R,U}\frac{h_{ij}}{k_{(r-1)(u-1)}}\tau_{ij}+\sum_{i,j=R+1,U+1}^{r,u}\frac{h_{ij}}{k_{(r-1)(u-1)}}\tau_{ij} \\
&\qquad \le \left(\sup_{i\ge 1,j\ge 1}\tau_{ij}\right)\frac{k_{RU}}{k_{(r-1)(u-1)}} \\
&\qquad\quad +\left(\sup_{i\ge R,j\ge U}\tau_{ij}\right)\frac{(k_r-k_R)(j_u-j_U)}{k_{(r-1)(u-1)}} \\
&\qquad \le H\,\frac{k_{RU}}{k_{(r-1)(u-1)}}+\varepsilon\, M\, N.
\end{aligned}
$$

Since $k_{r-1}, j_{u-1}\to\infty$ as $m,n\to\infty$, it follows that for each $x\in X$

$$\frac{1}{(mn)^{\alpha}}\sum_{i,j=1,1}^{m,n}|\rho_x(U_{ij})-\rho_x(U)|\to 0.$$

Thus, $U_{ij}\xrightarrow{W[C_2^{\alpha}]}U$. Consequently, $[W_2N_{\theta}^{\alpha}]\subseteq W[C_2^{\alpha}]$. $\square$

Theorem 4.14. *Let $0 < \alpha \le 1$. For any double lacunary sequence $\theta_2 = \{(k_r, j_u)\}$, if*

$$1 < \liminf_r q_r^\alpha \le \limsup_r q_r < \infty \quad \text{and} \quad 1 < \liminf_u q_u^\alpha \le \limsup_u q_u < \infty,$$

then $[W_2 N_\theta^\alpha] = W[C_2^\alpha]$.

Proof. This can be obtained from Theorem 4.12 and Theorem 4.13, immediately. □

Theorem 4.15. *Let $0 < \alpha \le 1$. If $U_{ij} \overset{[W_2 N_\theta^\alpha]}{\longrightarrow} U$ and $U_{ij} \overset{W[C_2^\alpha]}{\longrightarrow} V$, where $\{U_{ij}\} \in [W_2 N_\theta^\alpha] \cap W[C_2^\alpha]$ for any double lacunary sequence $\theta_2 = \{(k_r, j_u)\}$, then $U = V$.*

Proof. Let $0 < \alpha \le 1$, $\{U_{ij}\} \in [W_2 N_\theta^\alpha] \cap W[C_2^\alpha]$, $U_{ij} \overset{[W_2 N_\theta^\alpha]}{\longrightarrow} U$ and $U_{ij} \overset{W[C_2^\alpha]}{\longrightarrow} V$. Also, we suppose that $U \ne V$. For each $x \in X$, we have

$$\begin{aligned} \upsilon_{ru} + \tau_{ru} &= \frac{1}{h_{ru}^\alpha} \sum_{(i,j)\in I_{ru}} |\rho_x(U_{ij}) - \rho_x(U)| + \frac{1}{h_{ru}^\alpha} \sum_{(i,j)\in I_{ru}} |\rho_x(U_{ij}) - \rho_x(V)| \\ &\ge \frac{1}{h_{ru}^\alpha} \sum_{(i,j)\in I_{ru}} |\rho_x(U) - \rho_x(V)| \\ &= \frac{h_{ru}}{h_{ru}^\alpha} |\rho_x(U) - \rho_x(V)| \\ &\ge |\rho_x(U) - \rho_x(V)|, \end{aligned}$$

where

$$\upsilon_{ru} = \frac{1}{h_{ru}^\alpha} \sum_{(i,j)\in I_{ru}} |\rho_x(U_{ij}) - \rho_x(U)| \quad \text{and} \quad \tau_{ru} = \frac{1}{h_{ru}^\alpha} \sum_{(i,j)\in I_{ru}} |\rho_x(U_{ij}) - \rho_x(V)|.$$

Since $U_{ij} \overset{[W_2 N_\theta^\alpha]}{\longrightarrow} U$, then $\upsilon_{ru} \to 0$ for each $x \in X$. Thus, for each $x \in X$ and sufficiently large r and u, we must have

$$\tau_{ru} > \frac{1}{2} |\rho_x(U) - \rho_x(V)|.$$

Observe that for each $x \in X$ and sufficiently large r and u

$$\begin{aligned} \frac{1}{k_{ru}^\alpha} \sum_{m,n=1,1}^{k_r, j_u} |\rho_x(U_{mn}) - \rho_x(V)| &\ge \frac{1}{k_{ru}^\alpha} \sum_{(m,n)\in I_{ru}} |\rho_x(U_{mn}) - \rho_x(V)| \\ &= \frac{h_{ru}^\alpha}{k_{ru}^\alpha} \tau_{ru} \\ &= \left(1 - \frac{1}{q_r}\right)^\alpha \left(1 - \frac{1}{q_u}\right)^\alpha \tau_{ru} \\ &> \frac{1}{2} \left(1 - \frac{1}{q_r}\right)^\alpha \left(1 - \frac{1}{q_u}\right)^\alpha |\rho_x(U) - \rho_x(V)|. \end{aligned}$$

Since $U_{ij} \overset{W[C_2^\alpha]}{\longrightarrow} V$, for each $x \in X$ the left side of the above inequality convergent to 0. Hence, we have $q_r \to 1$ and $q_u \to 1$, and by Theorem 4.13 this implies that

$$[W_2 N_\theta^\alpha] \subseteq W[C_2^\alpha],$$

i.e.,

$$U_{ij} \overset{[W_2 N_\theta^\alpha]}{\longrightarrow} U \Rightarrow U_{ij} \overset{W[C_2^\alpha]}{\longrightarrow} U$$

and therefore for each $x \in X$, we can write

$$\frac{1}{(mn)^\alpha} \sum_{i,j=1,1}^{m,n} |\rho_x(U_{ij}) - \rho_x(U)| \to 0.$$

For each $x \in X$, we have

$$\frac{1}{(mn)^\alpha} \sum_{i,j=1,1}^{m,n} |\rho_x(U_{ij}) - \rho_x(U)| + \frac{1}{(mn)^\alpha} \sum_{i,j=1,1}^{m,n} |\rho_x(U_{ij}) - \rho_x(V)|$$
$$\geq \frac{mn}{(mn)^\alpha} \, |\rho_x(U) - \rho_x(V)| > 0.$$

Since both terms on the left side of the above inequality convergent to 0, then for each $x \in X$ we get

$$|\rho_x(U) - \rho_x(V)| = 0.$$

This situation causes a contradiction to our assumption. Consequently, we get $U = V$. □

Theorem 4.16. *Let $0 < \alpha \leq 1$. If $\liminf_r q_r^\alpha > 1$ and $\liminf_u q_u^\alpha > 1$ for any double lacunary sequence $\theta_2 = \{(k_r, j_u)\}$, then $U_{ij} \overset{W(S_2^\alpha)}{\longrightarrow} U$ implies that $U_{ij} \overset{W_2 S_\theta^\alpha}{\longrightarrow} U$.*

Proof. Let $0 < \alpha \leq 1$ and suppose that $\liminf_r q_r^\alpha > 1$ and $\liminf_u q_u^\alpha > 1$. Then, there exists $\eta, \mu > 0$ such that $q_r^\alpha \geq 1 + \eta$ and $q_u^\alpha \geq 1 + \mu$ for all r and u, which implies that

$$\frac{h_{ru}^\alpha}{k_{ru}^\alpha} \geq \frac{\eta\mu}{(1+\eta)(1+\mu)}.$$

For every $\varepsilon > 0$ and each $x \in X$, we can write

$$\frac{1}{k_{ru}^\alpha} \Big| \big\{ (i,j) : \ i \leq k_r, j \leq j_u, \ |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon \big\} \Big|$$
$$\geq \frac{1}{k_{ru}^\alpha} \Big| \big\{ (i,j) \in I_{ru} : |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon \big\} \Big|$$
$$= \frac{h_{ru}^\alpha}{k_{ru}^\alpha} \frac{1}{h_{ru}^\alpha} \Big| \big\{ (i,j) \in I_{ru} : |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon \big\} \Big|$$
$$\geq \frac{\eta\mu}{(1+\eta)(1+\mu)} \frac{1}{h_{ru}^\alpha} \Big| \big\{ (i,j) \in I_{ru} : |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon \big\} \Big|.$$

If $U_{ij} \xrightarrow{W(S_2^\alpha)} U$, then for each $x \in X$ the term on the left side of the above inequality convergent to 0 and this implies that

$$\frac{1}{h_{ru}^\alpha}\Big|\big\{(i,j) \in I_{ru} : |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon\big\}\Big| \to 0.$$

Thus, we get $U_{ij} \xrightarrow{W_2 S_\theta^\alpha} U$. □

Theorem 4.17. *Let $0 < \alpha \leq 1$. If $\limsup_r q_r < \infty$ and $\limsup_u q_u < \infty$ for any double lacunary sequence $\theta_2 = \{(k_r, j_u)\}$, then $U_{ij} \xrightarrow{W_2 S_\theta^\alpha} U$ implies that $U_{ij} \xrightarrow{W(S_2^\alpha)} U$.*

Proof. Let $0 < \alpha \leq 1$ and suppose that $\limsup_r q_r < \infty$ and $\limsup_u q_u < \infty$. Then, there exist $M, N > 0$ such that $q_r < M$ and $q_u < N$ for all r and u. Let $U_{ij} \xrightarrow{W_2 S_\theta^\alpha} U$ and $\varepsilon > 0$ be given, and let

$$T_{ru} := \Big|\big\{(i,j) \in I_{ru} : |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon\big\}\Big|.$$

Then, there exist $r_0, u_0 \in \mathbb{N}$ such that for every $\varepsilon > 0$, each $x \in X$ and all $r \geq r_0, u \geq u_0$

$$\frac{T_{ru}}{h_{ru}^\alpha} < \varepsilon.$$

Now, let

$$H := \max\{T_{ru} : 1 \leq r \leq r_0,\ 1 \leq u \leq u_0\},$$

and let m and n be any integers satisfying $k_{r-1} < m \leq k_r$ and $j_{u-1} < n \leq j_u$. Then, for each $x \in X$ we can write

$$\begin{aligned}
&\frac{1}{(mn)^\alpha}\Big|\big\{(i,j) :\ i \leq m,\ j \leq n, |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon\big\}\Big| \\
&\leq \frac{1}{k_{(r-1)(u-1)}^\alpha}\Big|\big\{(i,j) :\ i \leq k_r, j \leq j_u :\ |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon\big\}\Big| \\
&= \frac{1}{k_{(r-1)(u-1)}^\alpha}\big\{T_{11} + T_{12} + T_{21} + T_{22} + \cdots + T_{r_0 u_0} + \cdots + T_{ru}\big\} \\
&\leq \frac{r_0 u_0}{k_{(r-1)(u-1)}^\alpha}\left(\max_{\substack{1 \leq j \leq u_0 \\ 1 \leq i \leq r_0}}\{T_{ij}\}\right) \\
&\quad + \frac{1}{k_{(r-1)(u-1)}^\alpha}\left\{h_{r_0(u_0+1)}^\alpha \frac{T_{r_0(u_0+1)}}{h_{r_0(u_0+1)}^\alpha} + h_{(r_0+1)u_0}^\alpha \frac{T_{(r_0+1)u_0}}{h_{(r_0+1)u_0}^\alpha}\right. \\
&\quad \left. + h_{(r_0+1)(u_0+1)}^\alpha \frac{T_{(r_0+1)(u_0+1)}}{h_{(r_0+1)(u_0+1)}^\alpha} + \cdots + h_{ru}^\alpha \frac{T_{ru}}{h_{ru}^\alpha}\right\}
\end{aligned}$$

$$\leq \frac{r_0 u_0 H}{k^{\alpha}_{(r-1)(u-1)}} + \frac{1}{k^{\alpha}_{(r-1)(u-1)}} \left(\sup_{\substack{r>r_0 \\ u>u_0}} \frac{T_{ru}}{h^{\alpha}_{ru}} \right) \left(\sum_{i,j \geq r_0, u_0}^{r,u} h^{\alpha}_{ij} \right)$$

$$\leq \frac{r_0 u_0 H}{k^{\alpha}_{(r-1)(u-1)}} + \frac{1}{k_{(r-1)(u-1)}} \left(\sup_{\substack{r>r_0 \\ u>u_0}} \frac{T_{ru}}{h^{\alpha}_{ru}} \right) \left(\sum_{i,j \geq r_0, u_0}^{r,u} h_{ij} \right)$$

$$\leq \frac{r_0 u_0 H}{k^{\alpha}_{(r-1)(u-1)}} + \varepsilon \frac{(k_r - k_{r_0})(j_u - j_{u_0})}{k_{(r-1)(u-1)}}$$

$$\leq \frac{r_0 u_0 H}{k^{\alpha}_{(r-1)(u-1)}} + \varepsilon\, q_r\, q_u$$

$$\leq \frac{r_0 u_0 H}{k^{\alpha}_{(r-1)(u-1)}} + \varepsilon\, M\, N.$$

Since $k_{r-1}, j_{u-1} \to \infty$ as $m, n \to \infty$, it follows that for each $x \in X$

$$\frac{1}{(mn)^{\alpha}} \Big| \big\{ (i,j) : \ i \leq m, j \leq n : \ |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon \big\} \Big| \to 0.$$

Thus, we get $U_{ij} \overset{W(S_2^{\alpha})}{\longrightarrow} U$. □

Theorem 4.18. *Let $0 < \alpha \leq 1$. For any double lacunary sequence $\theta_2 = \{(k_r, j_u)\}$, if*

$$1 < \liminf_r q_r^{\alpha} \leq \limsup_r q_r < \infty \quad \text{and} \quad 1 < \liminf_u q_u^{\alpha} \leq \limsup_u q_u < \infty,$$

then $U_{ij} \overset{W_2 S_{\theta}^{\alpha}}{\longrightarrow} U \Leftrightarrow U_{ij} \overset{W(S_2^{\alpha})}{\longrightarrow} U$.

Proof. This can be obtained from Theorem 4.16 and Theorem 4.17, immediately. □

Theorem 4.19. *Let $0 < \alpha \leq 1$. If $\liminf_{r,u \to \infty} \frac{h^{\alpha}_{ru}}{k_{ru}} > 0$ for any double lacunary sequence $\theta_2 = \{(k_r, j_u)\}$, then $W(S_2) \subseteq W_2 S_{\theta}^{\alpha}$.*

Proof. For every $\varepsilon > 0$ and each $x \in X$, it is obvious that

$$\big\{ i \leq k_r, j \leq j_u, \ |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon \big\} \supset \big\{ (i,j) \in I_{ru} : |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon \big\}.$$

Thus, we get

$$\frac{1}{k_{ru}} \Big| \big\{ (i,j) : \ i \leq k_r, j \leq j_u, \ |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon \big\} \Big|$$

$$\geq \frac{1}{k_{ru}} \Big| \big\{ (i,j) \in I_{ru} : |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon \big\} \Big|$$

$$= \frac{h^{\alpha}_{ru}}{k_{ru}} \frac{1}{h^{\alpha}_{ru}} \Big| \big\{ (i,j) \in I_{ru} : |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon \big\} \Big|.$$

If $U_{ij} \overset{W(S_2)}{\longrightarrow} U$, then for each $x \in X$ the term on the left side of the above inequality convergent to 0 and this implies that for each $x \in X$,

$$\frac{1}{h_{ru}^{\alpha}}\Big|\big\{(i,j) \in I_{ru} : |\rho_x(U_{ij}) - \rho_x(U)| \geq \varepsilon\big\}\Big| \to 0.$$

Thus, we get $U_{ij} \overset{W_2 S_\theta^\alpha}{\longrightarrow} U$. Consequently, $W(S_2) \subseteq W_2 S_\theta^\alpha$. □

5. Conclusions and Future Work

We introduced new convergence concepts for double set sequences, also we studied the relationships between them. These concepts can also be studied for the ideal convergence and invariant convergence in the future.

6. Competing Interests

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

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Esra Gülle is working as research assistant in Department of Mathematics, Afyon Kocatepe University. She received Ph.D. in Mathematics from Afyon Kocatepe University, Afyonkarahisar, Turkey. Her area of expertise includes: Summability theory and sequences spaces through functional analysis. She has many international published papers.

Department of Mathematics, Afyon Kocatepe University, Afyonkarahisar, Turkey.
e-mail: egulle@aku.edu.tr

Uğur Ulusu is working as Associate Professor in Sivas Cumhuriyet University. He received Ph.D. in Mathematics from Afyon Kocatepe University, Afyonkarahisar, Turkey. His area of expertise includes: Summability theory and sequences spaces through functional analysis. He is the author of many international published papers.

Sivas Cumhuriyet University, 58140, Sivas, Turkey.
e-mail: ugurulusu@cumhuriyet.edu.tr