Conference Proceeding of 4th International E-Conference on Mathematical Advances and Applications (ICOMAA-2021)

# CONFERENCE PROCEEDINGS OF SCIENCE AND TECHNOLOGY

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# Preface

Dear Conference Participant,

Welcome to the International E-Conference on Mathematical Development and Applications (ICOMAA-2021) we organized the fourth. The aim of our conferences is to bring together scientists and young researchers from all over the world and their work on the fields of mathematics in a discussion environment. With this interaction, functional analysis, approach theory, differential equations and partial differential equations and the results of applications in the field of Mathematics are discussed with our valuable academics, and in mathematical developments both science and young researchers are opened. We are happe to host many prominent experts from different countries who will present the state-of-the-art in real analysis, complex analysis, harmonic and non-harmonic analysis, operator theory and spectral analysis, applied analysis.

I would like express my gratitude to those who see and appreciate our efforts and innovative steps that we have made to improve our conference every year, to our dear invited speakers and to all our participants. I owe a debt of gratitude to the Scientific committee, organizing committee, local organizing committee and for their efforts throughout this conference series.

The conference brings together about 175 participants and 11 invited speakers from 27 countries (Algeria, Albania, Azerbaijan, Canada, China, Colombia, Cyprus, Czech Republic, Finland, Germany, Greece, India, Iran, Italy, Kuwait, Malaysia, Morocco, Pakistan, Qatar, Saudi Arabia, Thailand, Tunisia, Turkey, United Arab Emirates, USA, Uzbekistan, Yemen).

**More than 50% of our participants participated from abroad.** This shows that the conference meets the criteria of being international.

The scientific committee members of ICOMAA-2021 and the external reviewers invested significant time in analyzing and assessing multiple papers, consequently, they hold and maintain a high standard of quality for this conference. The scientific program of the conference features invited talks, followed by contributed oral and poster presentations in seven parallel sessions.

The conference program represents the efforts of many people. I would like to express my gratitude to all members of the scientific committee, external reviewers, sponsors and, honorary committee for their continued support to the ICOMAA. I also thank the invited speakers for presenting their talks on current researches. Also, the success of ICOMAA depends on the effort and talent of researchers in mathematics and its applications that have written and submitted papers on a variety of topics. So, I would like to sincerely thank all participants of ICOMAA-2021 for contributing to this great meeting in many different ways. I believe and hope that each of you will get the maximum benefit from the conference.

Assoc. Prof. Dr. Yusuf ZEREN
Chairman
On behalf of the Organizing Committee

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Conference Proceeding Science and Technology, 4(1), 2021, 1–5

*Conference Proceeding of 4th International E-Conference on Mathematical Advances and Applications (ICOMAA-2021).*

# The Impact of Random Noise in the Death Rate of Prey for a Nonlinear Mathematical Model with Harvesting

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**Abstract:** This paper concentrates on a nonlinear prey predator model incorporating prey harvesting with Holling type-IV functional response. The primary objective is to theoretically analyse the population dynamics and discuss how noise incorporated in prey's death affects their interactions. The noise term translates the system of ordinary differential equations into a stochastic one. Various degrees of noise strength are applied, and deterministic and stochastic models are compared. The theoretical findings are complemented with various numerical simulations.

**Keywords:** Dynamical systems, Population dynamics, Stochastic differential equations.

## 1 Introduction

Mathematical modelling of the population dynamics trace its roots back to a seminal model independently developed by Lotka [1] and Volterra [2]. After these pioneering works, the relationship between prey and predator species has been extensively studied with various parameters, environmental factors and functional responses.

Studying different properties of the system with various functional response in the ecological systems are constantly extending with a new knowledge. First simplification on the Holling type IV functional response is presented by Sokol and Howell [3], then further investigated by many scientists, see for example [4]-[6]. Holling type IV functional response gives rise to more complex dynamics compared to Holling type I, II and III responses [4, 5, 7]. To explore the effect of human related issues, e.g hunting, many researchers have studied population dynamics with a harvesting contribution and obtained much richer dynamical behaviour [8]-[11]. Furthermore, the influence of environmental noise is one of the natural facts that should be considered in population dynamics. In this context there is also a recent trend for exploring stochastic differential equations in various context, including prey-predator systems with fear, foraging, group defence etc. See [12]-[15] for further information.

In this paper, a nonlinear population model which comprises prey harvesting with Holling type-IV functional response is examined. The model is based on the recent work by Shang *et al.* [7] with a small modification, for which a natural death rate in prey's dynamics are incorporated. Dimensionless version of the model is also described to determine relative importance of the terms in the system and to reduce the number of parameters. The steady states of the model is also determined. Since natural systems are subject to environmental noise, white Gaussian noise is incorporated into natural death rate of prey species. The impact of this noise term on the dynamics of both populations is examined. For this purpose, a particular attention is paid on a predator free axial steady state.

## 2 Mathematical model

A nonlinear population model including prey harvesting with Holling type-IV functional response considered in [7] is modified by incorporating natural death rate of prey species below:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\left(1-\frac{x}{k}\right) - \frac{m_1xy}{b+x^2} - \frac{qex}{ce+lx} - ax, \tag{1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{m_2xy}{b+x^2} - dy. \tag{2}$$

where parameters $r$, $k$, $m_1$, $m_2$, $b$, $a$, $d$ respectively represent intrinsic growth rate of prey species, carrying capacity, predator max growth ratio, conversion ratio, half saturation constant, natural death ratio of prey and natural death ratio of predator species. Besides $q$ and $e$ respectively stand for catchability coefficient and external effort associated with harvesting.

### 2.1 Dimensionless version of the model

In order to reduce the number of parameters and to get rid of dimensions of the model, the new variables are considered as follows

$$\tilde{x} = \frac{x}{x_0} \qquad \tilde{y} = \frac{y}{y_0} \qquad \tau = \frac{t}{t_0} \tag{3}$$

and introducing new parameters

$$x_0 = k \qquad t_0 = \frac{1}{d} \qquad y_0 = \frac{x_0^2}{m_1 t_0} = \frac{k^2 d}{m_1},$$
$$r = \frac{r}{d} \qquad b = \frac{b}{k^2}, \qquad q = \frac{qe}{dlk}$$
$$c = \frac{ce}{lk} \qquad m_2 = \frac{m}{dk} \qquad a = \frac{a}{d},$$

dimensionless form of the model can be written as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = rx\,(1-x) - \frac{xy}{b+x^2} - \frac{qx}{c+x} - ax = \mathcal{H}(x,y,t) \tag{4}$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{mxy}{b+x^2} - y = \mathcal{G}(x,y,t). \tag{5}$$

Here $\tilde{(\cdot)}$ is omitted for simplicity. Introducing Gaussian noise in the death rate of prey population, the model can be rewritten as

$$\mathrm{d}x = \mathcal{H}(x,y,t)\mathrm{d}t - \epsilon_1 x\, \mathrm{d}\mathcal{P}_1, \tag{6}$$
$$\mathrm{d}y = \mathcal{G}(x,y,t)\mathrm{d}t, \tag{7}$$

where $\epsilon_1 > 0$ is the strengths of the random perturbation. Furthermore, $\mathcal{P}_1$ denotes standard Brownian motion and all parameters are assumed to be positive for their biological meaning.

The deterministic version of the model presented in equations (4) and (5) has five steady states. One steady state is trivial where both prey and predator species are absent $(0,0)$, two steady states are predator free $(x_1^1, 0)$ and $(x_2^1, 0)$ which can be obtained using equation (4) for $y = 0$:

$$rx^2 - x\left(r(1-c) - a\right)x - \left(c(r-a) - q\right) = 0, \tag{8}$$

that leads to

$$x_{1,2}^1 = \frac{r(1-c) - a \pm \sqrt{\Delta}}{2r}, \tag{9}$$

with $\Delta = \left(r(1-c) - a\right)^2 + 4r\left(c(r-a) - q\right)$ . Besides the system has two coexisting states $(x_1^2, y_1^2)$ and $(x_2^2, y_2^2)$. Here the prey state can be found using equation (5) as

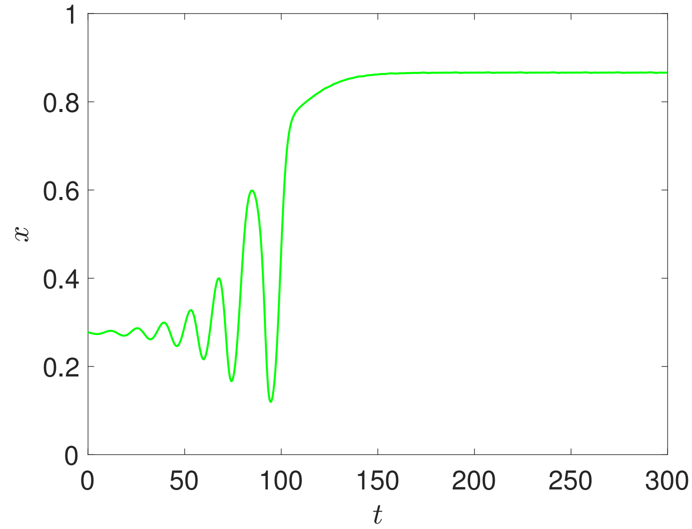
$$x_{1,2}^2 = \frac{m \pm \sqrt{m^2 - 4b}}{2}. \tag{10}$$

Substituting equation (10) into equation (4) the states for predator are found as

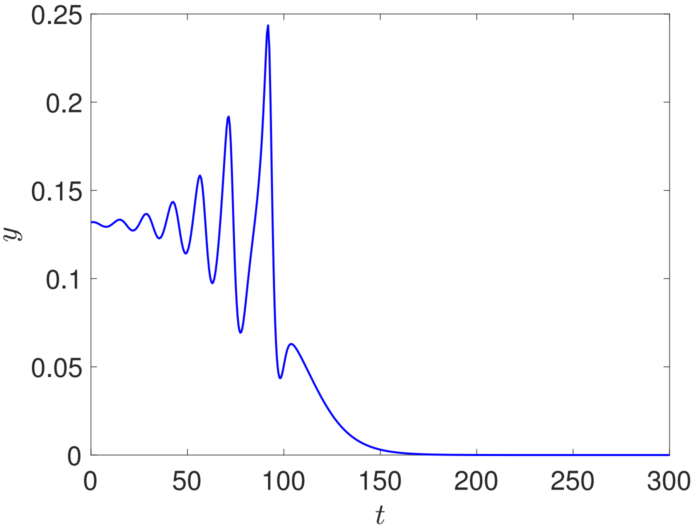
$$y_{1,2}^2 = \frac{x_{1,2}^2 + b}{x_{1,2} + c}\left[-rx_{1,2}^2 + \left(r(1-c) - a\right)x_{1,2} + c(r-a) - q\right]. \tag{11}$$

As seen in equations (8)-(11), the model has one trivial steady state and two predator free steady states and two non-trivial coexistence states. In this paper, we only concentrate on one of the predator free steady states and explore the dynamical behaviour deterministically and stochastically.

Figure 1 demonstrates deterministic time simulations of prey (A) and predator (B) density (in the absence of noise) for a stable system. The initial conditions are chosen as $(x_0, y_0) = (0.8, 0.1)$ and the system converges to a predator free steady state $(x_s, y_s) = (0.8662, 0)$. After some initial oscillations, prey species converge to its steady state and predator species wipe out from the system, e.g. after $t = 150$.



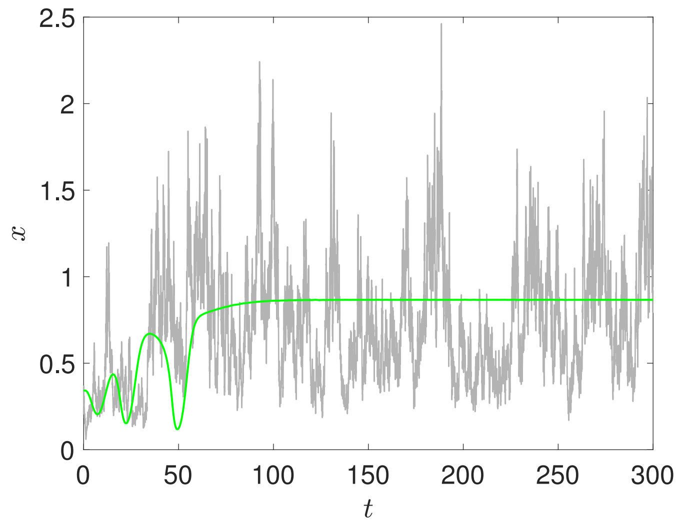

**(A)**



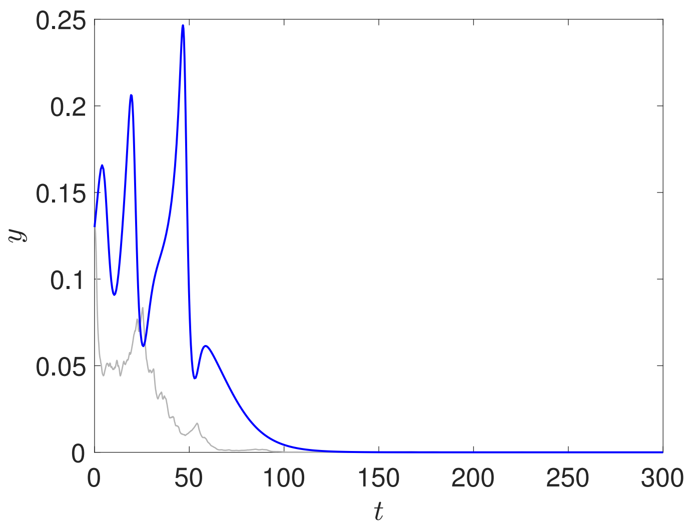

**(B)**

**Fig. 1**: Time simulations of predator free stable dynamics obtained with parameters $r = 1, b = 0.2, q = 0.1, c = 0.2, m = 1, a = 0.04$ for $t \in [0, 300]$. Prey (x) and predator (y) species are respectively presented in (A) and (B) with respect to time in the absence of noise.

Figure 2 demonstrates time evolutions of prey (A) and predator (B) density in the presence of noise term $\epsilon_1$ in the prey's death rate, where $\epsilon_1 = 0.5$. Green and blue colors correspond to deterministic case, whilst gray colors stand for the noise induced stochastic dynamics. As seen noise in the death rate of prey species has a dramatic effect on the prey density, leading to irregular oscillations in time. On the contrary, it has been observed that increasing noise gives rise to extinction of predator species in a smaller time interval, compared to deterministic case. As seen in Figure 1, predator species undergoes extinction at $t = 150$. In contrast, the extinction in the presence of noise ($\epsilon_1 = 0.5$) occurs at around $t = 60$.




**(A)**




**(B)**

**Fig. 2**: Deterministic and stochastic simulations of predator free dynamics obtained with parameters $r = 1, b = 0.2, q = 0.1, c = 0.2, m = 1, a = 0.04$ for $t \in [0, 300]$. Prey (x) and predator (y) species are respectively presented in (A) and (B) with respect to time for $\epsilon_1 = 0.5$. Gray line in both case represents noise perturbed dynamics.

Figure 3 demonstrates phase portraits of the system given by (6)-(7) in the absence and presence of noise term $\epsilon_1$ for a stable system. The black color stands for deterministic case, whilst gray color represents noise perturbed the stochastic system with various noise strengths, e.g. $\epsilon_1 = 0.01$ (A), $\epsilon_1 = 0.05$ (B), $\epsilon_1 = 0.1$ (C), $\epsilon_1 = 0.5$ (D). Here, the initial condition is chosen as $(x_0, y_0) = (0.277, 0.132)$. Besides, the red dot represents the steady state $(x_s, y_s) = (0.2764, 0.1309)$ for a coexisting system, the yellow dot represents the initial condition of the system and the blue dot stands for the case where the system converges to predator free steady state $(x_s, y_s) = (0.8662, 0)$.

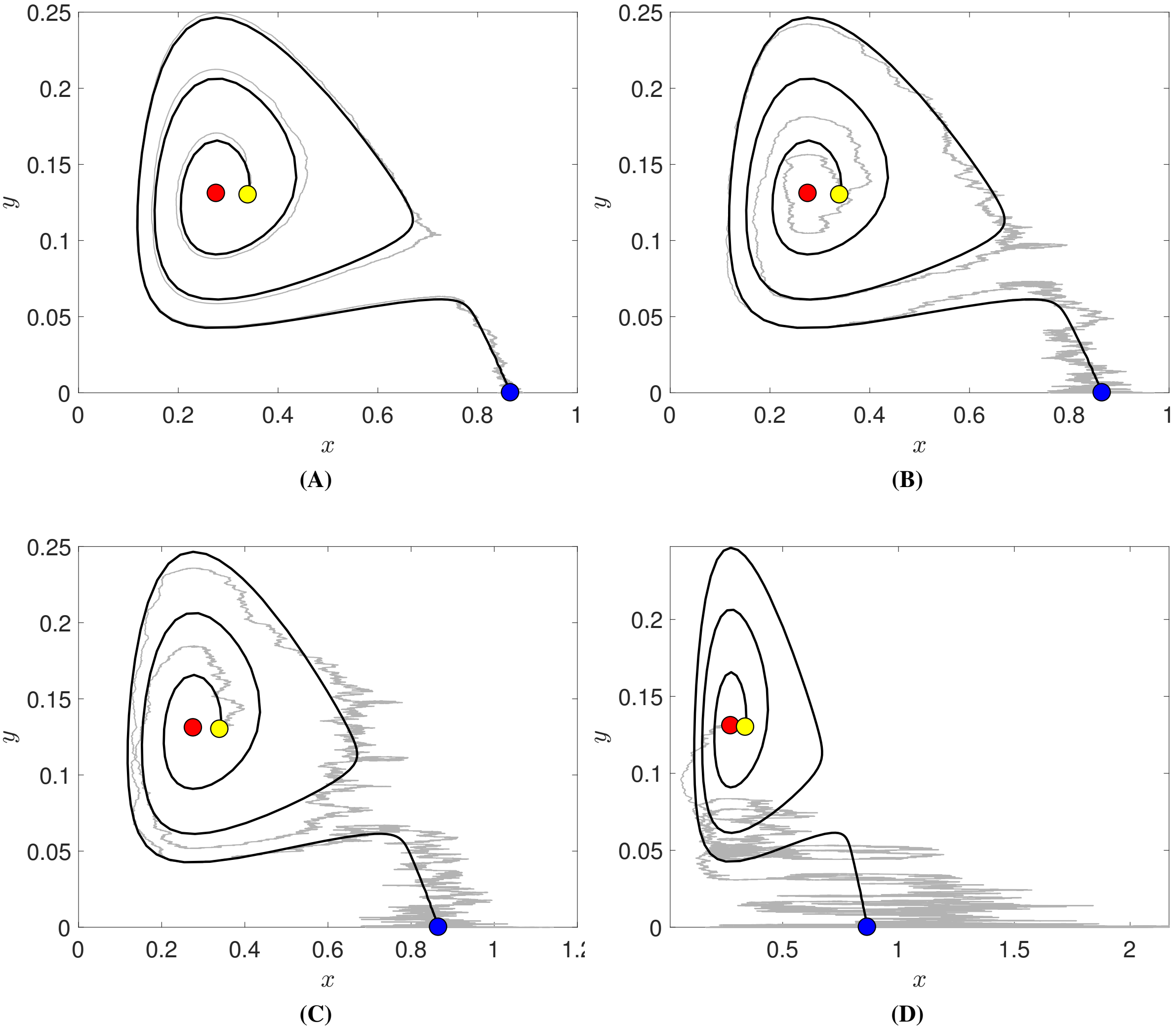



**Fig. 3**: Phase portraits of deterministic and stochastic model for various noise values $\epsilon_1 = 0.01$ (A), $\epsilon_1 = 0.05$ (B), $\epsilon_1 = 0.1$ (C), $\epsilon_1 = 0.5$ (D). Red, yellow and blue points respectively represent steady state of the system, initial point of the trajectory and final point of the trajectory.

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# On Amply Cofinitely g-Radical Supplemented Modules

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**Abstract:** In this work, some new properties of amply cofinitely g-radical supplemented modules are studied. Let $M$ be an amply cofinitely g-radical supplemented module. Then $M$ is cofinitely g-radical supplemented.

**Keywords:** Small Submodules, g-Small Submodules, Supplemented Modules, g-Supplemented Modules.

## 1 Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let $R$ be a ring and $M$ be an $R$ -module. We will denote a submodule $N$ of $M$ by $N \leq M$. Let $M$ be an $R$ -module and $N \leq M$. If $L = M$ for every submodule $L$ of $M$ such that $M = N + L$, then $N$ is called a *small submodule* of $M$ and denoted by $N \ll M$. Let $M$ be an $R$ -module and $N \leq M$. If there exists a submodule $K$ of $M$ such that $M = N + K$ and $N \cap K = 0$, then $N$ is called a *direct summand* of $M$ and it is denoted by $M = N \oplus K$. A submodule $N$ of an $R$ -module $M$ is called an *essential submodule* of $M$, denoted by $N \trianglelefteq M$, if $K = 0$ for every $K \leq M$ with $K \cap N = 0$. A submodule $K$ of an $R-$module $M$ is said to be *cofinite* if $M/K$ is finitely generated. Let $M$ be an $R$ -module and $K$ be a submodule of $M$. $K$ is called a *generalized small* (briefly, *g-small*) *submodule* of $M$ if for every $T \trianglelefteq M$ with $M = K + T$ implies that $T = M$, this is written by $K \ll_g M$. Let $M$ be an $R-$module. $M$ is called an *hollow module* if every proper submodule of $M$ is small in $M$. $M$ is called a *generalized hollow* (briefly, *g-hollow*) *module*, if every proper submodule of $M$ is g-small in $M$. $M$ is called a *local module* if $M$ has the largest submodule, i. e. a proper submodule which contains all other proper submodules. Let $U$ and $V$ be submodules of $M$. If $M = U + V$ and $V$ is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then $V$ is called a *supplement* of $U$ in $M$. $M$ is called a *supplemented module* if every submodule of $M$ has a supplement in $M$. $M$ is said to be *cofinitely supplemented* if every cofinite submodule of $M$ has a supplement in $M$. Let $M$ be an $R$-module and $U, V \leq M$. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then $V$ is called a *g-supplement* of $U$ in $M$. $M$ is said to be *g-supplemented* if every submodule of $M$ has a g-supplement in $M$. $M$ is said to be *cofinitely supplemented* if every cofinite submodule of $M$ has a g-supplement in $M$. Let $M$ be an $R-$module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, $U$ has a supplement (g-supplement) $V^{'}$ in $M$ with $V^{'} \leq V$, then we say $U$ has *ample supplements* (*g-supplements*) in $M$. If every submodule of $M$ has ample supplements (g-supplements) in $M$, then $M$ is called an *amply supplemented* (*g-supplemented*) *module*. If every cofinite submodule of $M$ has ample supplements (g-supplements) in $M$, then $M$ is called an *amply cofinitely supplemented* (*g-supplemented*) *module*. The intersection of all maximal submodules of an $R$-module $M$ is called the *radical* of $M$ and denoted by $RadM$. If $M$ have no maximal submodules, then we denote $RadM = M$.$M$ is said to be *semilocal* if $M/RadM$ is semisimple, i. e. every submodule of $M/RadM$ is a direct summand of $M/RadM$. Let $M$ be an $R$-module and $U, V \leq M$. If $M = U + V$ and $U \cap V \leq RadV$, then $V$ is called a *generalized* (*radical*) *supplement* (briefly, *Rad-supplement*) of $U$ in $M$. $M$ is said to be *generalized* (*radical*) *supplemented* (briefly, *Rad-supplemented*) if every submodule of $M$ has a generalized (radical) supplement in $M$. $M$ is said to be *cofinitely Rad-supplemented* if every cofinite submodule of $M$ has a Rad-supplement in $M$. Let $M$ be an $R-$module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, $U$ has a Rad-supplement $V^{'}$ in $M$ with $V^{'} \leq V$, then we say $U$ has *ample generalized* (*radical*) *supplements* (briefly, *ample Rad-supplements*) in $M$. If every submodule of $M$ has ample Rad-supplements in $M$, then $M$ is called an *amply generalized* (*radical*) *supplemented* (briefly, *amply Rad-supplemented*) *module*. $M$ is said to be *amply cofinitely Rad-supplemented* if every cofinite submodule of $M$ has ample Rad-supplements in $M$. The intersection of all essential maximal submodules of an $R$-module $M$ is called the *generalized radical* (briefly, *g-radical*) of $M$ and denoted by $Rad_g M$. If $M$ have no maximal essential submodules, then we denote $Rad_g M = M$. $M$ is said to be *g-semilocal* if $M/Rad_g M$ is semisimple, i. e. every submodule of $M/Rad_g M$ is a direct summand of $M/Rad_g M$. Let $M$ be an $R$-module and $U, V \leq M$. If $M = U + V$ and $U \cap V \leq Rad_g V$, then $V$ is called a *g-radical supplement* of $U$ in $M$. If every submodule of $M$ has a g-radical supplement in $M$, then $M$ is called a *g-radical supplemented module*. $M$ is said to be *cofinitely g-radical supplemented* if every cofinite submodule of $M$ has a g-radical supplement in $M$. Let $M$ be an $R-$module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, $U$ has a g-radical supplement $V^{'}$ in $M$ with $V^{'} \leq V$, then we say $U$ has *ample g-radical supplements* in $M$. If every submodule of $M$ has ample g-radical supplements in $M$, then $M$ is called an *amply g-radical supplemented module*.

More informations about (amply) supplemented modules are in [2, 12]. More details about (amply) cofinitely supplemented modules are in [1]. More informations about g-small submodules and (amply) g-supplemented modules are in [5]. More details about (amply) cofinitely g-supplemented modules are in [3]. The definition of (amply) generalized supplemented modules and some properties of them are in [11]. More details about (amply) cofinitely Rad-supplemented modules are in [10]. More informations about (amply) g-radical supplemented modules are in [4, 7] . The definition of cofinitely g-radical supplemented modules and some properties of them are in [8]. The definition of g-semilocal modules and some properties of them are in [6].

**Lemma 1.** *Let $M$ be an $R-$module.*
(1) *If $K \leq L \leq M$, then $K \trianglelefteq M$ if and only if $K \trianglelefteq L \trianglelefteq M$.*
(2) *Let $N$ be an $R-$module and $f: M \longrightarrow N$ be an $R-$module homomorphism. If $K \trianglelefteq N$, then $f^{-1}(K) \trianglelefteq M$.*
(3) *For $N \leq K \leq M$, if $K/N \trianglelefteq M/N$, then $K \trianglelefteq M$.*
(4) *If $K_1 \trianglelefteq L_1 \leq M$ and $K_2 \trianglelefteq L_2 \leq M$, then $K_1 \cap K_2 \trianglelefteq L_1 \cap L_2$.*
(5) *If $K_1 \trianglelefteq M$ and $K_2 \trianglelefteq M$, then $K_1 \cap K_2 \trianglelefteq M$.*

*Proof:* See [12, 17.3]. □

**Lemma 2.** *Let $M$ be an $R-$module. The following assertions are hold.*
(1) *If $K \leq L \leq M$, then $L \ll M$ if and only if $K \ll M$ and $L/K \ll M/K$.*
(2) *Let $N$ be an $R-$module and $f: M \longrightarrow N$ be an $R-$module homomorphism. If $K \ll M$, then $f(K) \ll N$. The converse is true if $f$ is an epimorphism and $Kef \ll M$.*
(3) *If $K \ll M$, then $\frac{K+L}{L} \ll \frac{M}{L}$ for every $L \leq M$.*
(4) *If $L \leq M$ and $K \ll L$, then $K \ll M$.*
(5) *If $K_1, K_2, ..., K_n \ll M$, then $K_1 + K_2 + ... + K_n \ll M$.*
(6) *Let $K_1, K_2, ..., K_n, L_1, L_2, ..., L_n \leq M$. If $K_i \ll L_i$ for every $i = 1, 2, ..., n$, then $K_1 + K_2 + ... + K_n \ll L_1 + L_2 + ... + L_n$.*

*Proof:* See [2, 2.2] and [12, 19.3]. □

**Lemma 3.** *Let $M$ be an $R-$module. The following assertions are hold.*
(1) $RadM = \sum_{L \ll M} L$.
(2) *Let $N$ be an $R-$module and $f: M \longrightarrow N$ be an $R-$module homomorphism. Then $f(RadM) \leq RadN$. If $Kef \leq RadM$, then $f(RadM) = Radf(M)$.*
(3) *If $N \leq M$, then $RadN \leq RadM$.*
(4) *For $K, L \leq M$, $RadK + RadL \leq Rad(K + L)$.*
(5) *$Rx \ll M$ for every $x \in RadM$.*

*Proof:* See [12, 21.5 and 21.6]. □

**Lemma 4.** *Let $M$ be an $R-$module. The following assertions are hold.*
(1) *Every small submodule in $M$ is g-small in $M$.*
(2) *If $K \leq L \leq M$ and $L \ll_g M$, then $K \ll_g M$ and $L/K \ll_g M/K$.*
(3) *Let $N$ be an $R-$module and $f: M \longrightarrow N$ be an $R-$module homomorphism. If $K \ll_g M$, then $f(K) \ll_g N$.*
(4) *If $K \ll_g M$, then $\frac{K+L}{L} \ll_g \frac{M}{L}$ for every $L \leq M$.*
(5) *If $L \leq M$ and $K \ll_g L$, then $K \ll_g M$.*
(6) *If $K_1, K_2, ..., K_n \ll_g M$, then $K_1 + K_2 + ... + K_n \ll_g M$.*
(7) *Let $K_1, K_2, ..., K_n, L_1, L_2, ..., L_n \leq M$. If $K_i \ll_g L_i$ for every $i = 1, 2, ..., n$, then $K_1 + K_2 + ... + K_n \ll_g L_1 + L_2 + ... + L_n$.*

*Proof:* See [2, 3, 9]. □

**Lemma 5.** *Let $M$ be an $R-$module. The following assertions are hold.*
(1) $RadM \leq Rad_g M$.
(2) $Rad_g M = \sum_{L \ll_g M} L$.
(3) *Let $N$ be an $R-$module and $f: M \longrightarrow N$ be an $R-$module homomorphism. Then $f(Rad_g M) \leq Rad_g N$.*
(4) *For $K, L \leq M$, $\frac{Rad_g K + L}{L} \leq Rad_g \frac{K+L}{L}$. If $L \leq Rad_g K$, then $Rad_g K/L \leq Rad(K/L)$.*
(5) *If $L \leq M$, then $Rad_g L \leq Rad_g M$.*
(6) *For $K, L \leq M$, $Rad_g K + Rad_g L \leq Rad_g(K + L)$.*
(7) *$Rx \ll_g M$ for every $x \in Rad_g M$.*

*Proof:* See [4–6]. □

## 2 Amply cofinitely g-radical supplemented modules

**Lemma 6.** *Let $V$ be a g-supplement of $U$ in $M$. Then*
(1) *If $W + V = M$ for some $W \leq U$, then $V$ is a g-supplement of $W$ in $M$.*
(2) *If every nonzero submodule of $M$ is essential in $M$, then $V$ is a supplement of $U$ in $M$.*
(3) *If $U$ is an essential maximal submodule of $M$, then $U \cap V = RadV$ is the unique essential maximal submodule of $V$.*
(4) *If $K \ll_g M$ and $U \trianglelefteq M$, then $V$ is a g-supplement of $U + K$ in $M$.*
(5) *Let $U \trianglelefteq M$ and $K \ll_g M$. Then $K \cap V \ll_g V$ and hence $Rad_g V = V \cap Rad_g M$.*
(6) *Let $U \trianglelefteq M$ and $K \leq V$. Then $K \ll_g V$ if and only if $K \ll_g M$.*
(7) *For $L \leq U$, $\frac{V+L}{L}$ is a g-supplement of $U/L$ in $M/L$.*

*Proof:* See [4–6]. □

**Lemma 7.** *Let $M$ be an $R-$module.*

(1) *If $V$ is a supplement of $U$ in $M$, then $V$ is a g-suppement of $U$ in $M$.*

(2) *If $M = U \oplus V$ then $V$ is a g-supplement of $U$ in $M$. Also $U$ is a g-supplement of $V$ in $M$.*

(3) *For $M_1, U \leq M$, if $M_1 + U$ has a g-supplement in $M$ and $M_1$ is g-supplemented, then $U$ also has a g-supplement in $M$.*

(4) *Let $M = M_1 + M_2$. If $M_1$ and $M_2$ are g-supplemented, then $M$ is also g-supplemented.*

(5) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If $M_i$ is g-supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also g-supplemented.*

(6) *If $M$ is g-supplemented, then $M/L$ is g-supplemented for every $L \leq M$.*

(7) *If $M$ is g-supplemented, then every homomorphic image of $M$ is also g-supplemented.*

(8) *If $M$ is g-supplemented, then $M/Rad_g M$ is semisimple.*

(9) *Hollow, local and g-hollow modules are g-supplemented.*

(10) *If $M$ is g-supplemented, then every finitely $M-$generated module is g-supplemented.*

(11) *${}_R R$ is g-supplemented if and only if every finitely generated $R-$module is g-supplemented.*

(12) *If $M$ is g-supplemented and every nonzero submodule of $M$ is essential in $M$, then $M$ is supplemented.*

*Proof:* See [4, 5]. □

**Lemma 8.** *Let $M$ be an $R-$module.*

(1) *If $M$ is g-supplemented, then $M$ is cofinitely g-supplemented.*

(2) *If $M$ is supplemented, then $M$ is cofinitely g-supplemented.*

(3) *If $M$ is cofinitely supplemented, then $M$ is cofinitely g-supplemented.*

(4) *If $M$ is cofinitely g-supplemented and every nonzero submodule of $M$ is essential in $M$, then $M$ is cofinitely supplemented.*

(5) *If $M$ is finitely generated and cofinitely g-supplemented, then $M$ is g-supplemented.*

(6) *For $M_1 \leq M$ and $U$ cofinite submodule of $M$, if $M_1 + U$ has a g-supplement in $M$ and $M_1$ is cofinitely g-supplemented, then $U$ also has a g-supplement in $M$.*

(7) *Let $M = \sum\limits_{i \in I} M_i$. If $M_i$ is cofinitely g-supplemented for every $i \in I$, then $M$ is also cofinitely g-supplemented.*

(8) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If $M_i$ is cofinitely g-supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also cofinitely g-supplemented.*

(9) *If $M$ is cofinitely g-supplemented, then $M/L$ is cofinitely g-supplemented for every $L \leq M$.*

(10) *If $M$ is cofinitely g-supplemented, then every homomorphic image of $M$ is also cofinitely g-supplemented.*

(11) *If $M$ is cofinitely g-supplemented, then every cofinite submodule of $M/Rad_g M$ is a direct summand of $M/Rad_g M$.*

(12) *Hollow, g-hollow and local modules are cofinitely g-supplemented.*

(13) *If $M$ is cofinitely g-supplemented, then every $M-$generated module is cofinitely g-supplemented.*

(14) *${}_R R$ is g-supplemented if and only if every generated $R-$module is cofinitely g-supplemented.*

*Proof:* See [3]. □

**Lemma 9.** *Let $M$ be an $R-$module.*

(1) *If $M$ is Rad-supplemented, then $M$ is g-radical supplemented.*

(2) *If $V$ is a g-radical supplement of $U$ in $M$ and $W + V = M$ for some $W \leq U$, then $V$ is a g-radical supplement of $W$ in $M$.*

(3) *If $U$ is an essential maximal submodule of $M$ and $V$ is a g-radical supplement of $U$ in $M$, then $U \cap V = Rad_g V$ is the unique essential maximal submodule of $V$.*

(4) *If $V$ is a g-radical supplement of $U$ in $M$ and $L \leq U$, $\frac{V+L}{L}$ is a g-radical supplement of $U/L$ in $M/L$.*

(5) *For $M_1, U \leq M$, if $M_1 + U$ has a g-radical supplement in $M$ and $M_1$ is g-radical supplemented, then $U$ also has a g-radical supplement in $M$.*

(6) *Let $M = M_1 + M_2$. If $M_1$ and $M_2$ are g-radical supplemented, then $M$ is also g-radical supplemented.*

(7) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If $M_i$ is g-radical supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also g-radical supplemented.*

(8) *If $M$ is g-radical supplemented, then $M/L$ is g-radical supplemented for every $L \leq M$.*

(9) *If $M$ is g-radical supplemented, then every homomorphic image of $M$ is also g-radical supplemented.*

(10) *If $M$ is g-radical supplemented, then $M/Rad_g M$ is semisimple.*

(11) *If $M$ is g-radical supplemented, then every finitely $M-$generated module is g-radical supplemented.*

*Proof:* See [4]. □

**Definition 1.** *Let $M$ be an $R-$module. If every cofinite submodule of $M$ has ample g-radical supplements in $M$, then $M$ is called an amply g-radical supplemented module. (See also [9])*

**Lemma 10.** *Every amply cofinitely g-radical supplemented module is cofinitely g-radical supplemented.*

*Proof:* Clear from definitions. □

**Corollary 1.** *Let $M$ be an $R-$module and $M = \sum_{i \in I} M_i$ for $M_i \leq M$. If $M_i$ is amply cofinitely g-radical supplemented for every $i \in I$, then $M$ is cofinitely g-radical supplemented.*

*Proof:* Since $M_i$ is amply cofinitely g-radical supplemented for every $i \in I$, by Lemma 10, $M_i$ is cofinitely g-radical supplemented. Then by [8, Proposition 2.5], $M$ is cofinitely g-radical supplemented. □

**Proposition 1.** *Every amply cofinitely g-supplemented module is amply cofinitely g-radical supplemented.*

 

*Proof:* Let $M$ be an amply cofinitely g-supplemented module and $U \leq M$. Let $M = U + V$ with $V \leq M$. Since $M$ is amply cofinitely g-supplemented, $U$ has a g-supplement $X$ in $M$ with $X \leq V$. Here $M = U + X$ and $U \cap X \ll_g X$. Since $U \cap X \ll_g X, U \cap X \leq Rad_g X$. Hence $X$ is a g-radical supplement of $U$ in $M$. Moreover, $X \leq V$. Hence $M$ is amply cofinitely g-radical supplemented. □

**Proposition 2.** *Every amply g-radical supplemented module is amply cofinirely g-radical supplemented.*

*Proof:* Clear from definitions. □

**Lemma 11.** *Every amply cofinitely Rad-supplemented module is amply cofinitely g-radical supplemented.*

*Proof:* Let $M$ be an amply cofinitely Rad-supplemented module and $U \leq M$. Let $M = U + V$ with $V \leq M$. Since $M$ is amply cofinitely Rad-supplemented, $U$ has a Rad-supplement $X$ in $M$ with $X \leq V$. Here $M = U + X$ and $U \cap X \leq RadX$. Since $U \cap X \leq RadX \leq Rad_g X$, $X$ is a g-radical supplement of $U$ in $M$. Moreover, $X \leq V$. Hence $M$ is amply cofinitely g-radical supplemented. □

**Corollary 2.** *Every amply cofinitely supplemented module is amply cofinitely g-radical supplemented.*

*Proof:* Clear from Lemma 11, since every amply cofinitely supplemented module is amply cofinitely Rad-supplemented. □

**Proposition 3.** *Let $M$ be an amply cofinitely g-radical supplemented $R-$module. If every nonzero submodule of $M$ is essential in $M$, then $M$ is amply cofinitely Rad-supplemented.*

*Proof:* Clear from definitions. □

**Corollary 3.** *Let $M$ be an amply cofinitely g-radical supplemented $R-$module. If every nonzero submodule of $M$ is essential in $M$, then $M$ is cofinitely Rad-supplemented.*

*Proof:* Clear from Proposition 3. □

**Lemma 12.** *Every factor module of an amply cofinitely g-radical supplemented module is amply cofinitely g-radical supplemented.*

*Proof:* Let $M$ be any amply cofinitely g-radical supplemented module and $K \leq M$. Let $U/K$ be a cofinite submodule of $M/K$ and $M/K = U/K + V/K$ with $V/K \leq M/K$. Then $M = U + V$ and since $M$ is amply cofinitely g-radical supplemented, there exists a g-radical supplement $T$ of $U$ with $T \leq V$. Then by [4, Lemma 8], $(T + K)/K$ is a g-radical supplement of $U/K$ in $M/K$. Moreover, $(T + K)/K \leq V/K$. Hence $U$ has ample g-radical supplements in $M$ and $M$ is amply cofinitely g-radical supplemented. □

**Corollary 4.** *The homomorphic image of an amply cofinitely g-radical supplemented module is amply cofinitely g-radical supplemented.*

*Proof:* Clear from Lemma 12. □

## 3 Conclusion

Amply cofinitely g-radical supplemented modules are more general than amply cofinitely supplemented modules.

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# On $\oplus - g - Rad-$ Supplemented Modules

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**Abstract:** In this work, every ring has unity and every module is unital left module. Let $M$ be an $R-$module. If every submodule of $M$ has a g-radical supplement that is a direct summand of $M$, then $M$ is called a $\oplus - g - Rad-$supplemented module (See also [9]). In this work, some properties of these modules are investigated.

**Keywords:** Essential Submodules, Small Submodules, $g-$Supplemented Modules, $\oplus-$Supplemented Modules.

## 1 Introduction

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let $R$ be a ring and $M$ be an $R$ -module. We will denote a submodule $N$ of $M$ by $N \leq M$. Let $M$ be an $R$ -module and $N \leq M$. If $L = M$ for every submodule $L$ of $M$ such that $M = N + L$, then $N$ is called a *small submodule* of $M$ and denoted by $N \ll M$. Let $M$ be an $R$ -module and $N \leq M$. If there exists a submodule $K$ of $M$ such that $M = N + K$ and $N \cap K = 0$, then $N$ is called a *direct summand* of $M$ and it is denoted by $M = N \oplus K$. For any $R-$module $M$, we have $M = M \oplus 0$. The intersection of all maximal submodules of $M$ is called the *radical* of $M$ and denoted by $RadM$. If $M$ have no maximal submodules, then it is defined $RadM = M$. $M$ is said to be *semilocal* if $M/RadM$ is semisimple. A submodule $N$ of an $R$ -module $M$ is called an *essential submodule* of $M$ and denoted by $N \trianglelefteq M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equivalently, $K = 0$ for every $K \leq M$ with $N \cap K = 0$. Let $M$ be an $R$ -module and $K$ be a submodule of $M$. $K$ is called a *generalized small* (or briefly, $g-$*small*) submodule of $M$ if for every essential submodule $T$ of $M$ with the property $M = K + T$ implies that $T = M$, then we write $K \ll_g M$ (in [12], it is called an *e-small submodule* of $M$ and denoted by $K \ll_e M$). It is clear that every small submodule is a generalized small submodule but the converse is not true in general. Let $M$ be an $R$ -module. $M$ is called a *hollow* module if every proper submodule of $M$ is small in $M$. $M$ is called a *generalized hollow* (or briefly, $g-$*hollow*) module if every proper submodule of $M$ is g-small in $M$. Here it is clear that every hollow module is generalized hollow. The converse of this statement is not always true. $M$ is called a *local module* if $M$ has the largest submodule, i.e. a proper submodule which contains all other proper submodules. $M$ is called a *generalized local* (briefly, *g-local*) if $M$ has a large proper essential submodule which contain all proper essential submodules of $M$ or $M$ have no proper essential submodules. Let $U$ and $V$ be submodules of $M$. If $M = U + V$ and $V$ is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then $V$ is called a *supplement* of $U$ in $M$. $M$ is said to be *supplemented* if every submodule of $M$ has a supplement in $M$. If every submodule of $M$ has a supplement that is a direct summand in $M$, then $M$ is called a $\oplus-$*supplemented module*. Let $M$ be an $R$ $-$module and $U, V \leq M$. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g V$, then $V$ is called a *g-supplement* of $U$ in $M$. $M$ is said to be *g-supplemented* if every submodule of $M$ has a g-supplement in $M$. $M$ is said to be $\oplus - g-$*supplemented* if every submodule of $M$ has a g-supplement that is a direct summand in $M$ (see [8]). Let $M$ be an $R$-module and $U, V \leq M$. If $M = U + V$ and $U \cap V \leq RadV$, then $V$ is called a *generalized* (*radical*) *supplement* (briefly, *Rad-supplement*) of $U$ in $M$. $M$ is said to be *generalized* (*radical*) *supplemented* (briefly, *Rad-supplemented*) if every submodule of $M$ has a Rad-supplement in $M$. $M$ is said to be *generalized* (*radical*) $\oplus-$*supplemented* (briefly, $Rad - \oplus-$*supplemented*) if every submodule of $M$ has a Rad-supplement that is a direct summand in $M$. The intersection of all essential maximal submodules of an $R-$module $M$ is called the *generalized radical* of $M$ and denoted by $Rad_g M$ (in [12], it is denoted by $Rad_e M$). If $M$ have no essential maximal submodules, then we denote $Rad_g M = M$. An $R-$module $M$ is said to be *g-semilocal* if $M/Rad_g M$ is semisimple (see [7]). Let $M$ be an $R-$module and $U, V \leq M$. If $M = U + V$ and $U \cap V \leq Rad_g V$, then $V$ is called a *generalized radical supplement* (or briefly, *g-radical supplement*) of $U$ in $M$. $M$ is said to be *generalized radical supplemented* (briefly, *g-radical supplemented*) if every submodule of $M$ has a g-radical supplement in $M$.

More informations about supplemented modules are in [1, 11]. More results about $\oplus-$supplemented modules are in [4]. More details about generalized (radical) supplemented modules are in [10]. More details about generalized (radical) $\oplus-$supplemented modules are in [2, 3]. More informations about g-supplemented modules are in [5]. More informations about g-radical supplemented modules are in [6].

**Lemma 1.** *Let $M$ be an $R-$module.*
(1) *If $K \leq L \leq M$, then $K \trianglelefteq M$ if and only if $K \trianglelefteq L \trianglelefteq M$.*
(2) *Let $N$ be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \trianglelefteq N$, then $f^{-1}(K) \trianglelefteq M$.*
(3) *For $N \leq K \leq M$, if $K/N \trianglelefteq M/N$, then $K \trianglelefteq M$.*
(4) *If $K_1 \trianglelefteq L_1 \leq M$ and $K_2 \trianglelefteq L_2 \leq M$, then $K_1 \cap K_2 \trianglelefteq L_1 \cap L_2$.*
(5) *If $K_1 \trianglelefteq M$ and $K_2 \trianglelefteq M$, then $K_1 \cap K_2 \trianglelefteq M$.*

*Proof:* See [11, 17.3]. □

**Lemma 2.** *Let $M$ be an $R-$module. The following assertions are hold.*
(1) *If $K \leq L \leq M$, then $L \ll M$ if and only if $K \ll M$ and $L/K \ll M/K$.*
(2) *Let $N$ be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \ll M$, then $f(K) \ll N$. The converse is true if $f$ is an epimorphism and $Kef \ll M$.*
(3) *If $K \ll M$, then $\frac{K+L}{L} \ll \frac{M}{L}$ for every $L \leq M$.*
(4) *If $L \leq M$ and $K \ll L$, then $K \ll M$.*
(5) *If $K_1, K_2, ..., K_n \ll M$, then $K_1 + K_2 + ... + K_n \ll M$.*
(6) *Let $K_1, K_2, ..., K_n, L_1, L_2, ..., L_n \leq M$. If $K_i \ll L_i$ for every $i = 1, 2, ..., n$, then $K_1 + K_2 + ... + K_n \ll L_1 + L_2 + ... + L_n$.*

*Proof:* See [1, 2.2] and [11, 19.3]. □

**Lemma 3.** *Let $M$ be an $R-$module. The following assertions are hold.*
(1) *Every small submodule in $M$ is g-small in $M$.*
(2) *If $K \leq L \leq M$ and $L \ll_g M$, then $K \ll_g M$ and $L/K \ll_g M/K$.*
(3) *Let $N$ be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. If $K \ll_g M$, then $f(K) \ll_g N$.*
(4) *If $K \ll_g M$, then $\frac{K+L}{L} \ll_g \frac{M}{L}$ for every $L \leq M$.*
(5) *If $L \leq M$ and $K \ll_g L$, then $K \ll_g M$.*
(6) *If $K_1, K_2, ..., K_n \ll_g M$, then $K_1 + K_2 + ... + K_n \ll_g M$.*
(7) *Let $K_1, K_2, ..., K_n, L_1, L_2, ..., L_n \leq M$. If $K_i \ll_g L_i$ for every $i = 1, 2, ..., n$, then $K_1 + K_2 + ... + K_n \ll_g L_1 + L_2 + ... + L_n$.*

*Proof:* See [5–7]. □

**Lemma 4.** *Let $M$ be an $R-$module. The following assertions are hold.*
(1) $RadM \leq Rad_g M$.
(2) $Rad_g M = \sum\limits_{L \ll_g M} L$.
(3) *Let $N$ be an $R-$module and $f : M \longrightarrow N$ be an $R-$module homomorphism. Then $f(Rad_g M) \leq Rad_g N$.*
(4) *For $K, L \leq M$, $\frac{Rad_g K + L}{L} \leq Rad_g \frac{K+L}{L}$. If $L \leq Rad_g K$, then $Rad_g K/L \leq Rad(K/L)$.*
(5) *If $L \leq M$, then $Rad_g L \leq Rad_g M$.*
(6) *For $K, L \leq M$, $Rad_g K + Rad_g L \leq Rad_g(K + L)$.*
(7) *$Rx \ll_g M$ for every $x \in Rad_g M$.*

*Proof:* See [5–7]. □

## 2 $\oplus - g-$rad supplemented modules

**Lemma 5.** *Let $V$ be a supplement of $U$ in $M$. Then*
(1) *If $W + V = M$ for some $W \leq U$, then $V$ is a supplement of $W$ in $M$.*
(2) *If $M$ is finitely generated, then $V$ is also finitely generated.*
(3) *If $U$ is a maximal submodule of $M$, then $V$ is cyclic and $U \cap V = RadV$ is the unique maximal submodule of $V$.*
(4) *If $K \ll M$, then $V$ is a supplement of $U + K$ in $M$.*
(5) *For $K \ll M$, $K \cap V \ll V$ and hence $RadV = V \cap RadM$.*
(6) *Let $K \leq V$. Then $K \ll V$ if and only if $K \ll M$.*
(7) *For $L \leq U$, $\frac{V+L}{L}$ is a supplement of $U/L$ in $M/L$.*

*Proof:* See [11, 41.1]. □

**Lemma 6.** *Let $V$ be a g-supplement of $U$ in $M$. Then*
(1) *If $W + V = M$ for some $W \leq U$, then $V$ is a g-supplement of $W$ in $M$.*
(2) *If every nonzero submodule of $M$ is essential in $M$, then $V$ is a supplement of $U$ in $M$.*
(3) *If $U$ is an essential maximal submodule of $M$, then $U \cap V = RadV$ is the unique essential maximal submodule of $V$.*
(4) *If $K \ll_g M$ and $U \trianglelefteq M$, then $V$ is a g-supplement of $U + K$ in $M$.*
(5) *Let $U \trianglelefteq M$ and $K \ll_g M$. Then $K \cap V \ll_g V$ and hence $Rad_g V = V \cap Rad_g M$.*
(6) *Let $U \trianglelefteq M$ and $K \leq V$. Then $K \ll_g V$ if and only if $K \ll_g M$.*
(7) *For $L \leq U$, $\frac{V+L}{L}$ is a g-supplement of $U/L$ in $M/L$.*

*Proof:* See [5–7]. □

**Lemma 7.** *Let $M$ be an $R-$module.*
(1) *If $M = U \oplus V$ then $V$ is a supplement of $U$ in $M$. Also $U$ is a supplement of $V$ in $M$.*
(2) *For $M_1, U \leq M$, if $M_1 + U$ has a supplement in $M$ and $M_1$ is supplemented, then $U$ also has a supplement in $M$.*
(3) *Let $M = M_1 + M_2$. If $M_1$ and $M_2$ are supplemented, then $M$ is also supplemented.*
(4) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If $M_i$ is supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also supplemented.*
(5) *If $M$ is supplemented, then $M/L$ is supplemented for every $L \leq M$.*
(6) *If $M$ is supplemented, then every homomorphic image of $M$ is also supplemented.*
(7) *If $M$ is supplemented, then $M/RadM$ is semisimple.*

(8) *Hollow and local modules are supplemented.*
(9) *If $M$ is supplemented, then every finitely $M-$generated module is supplemented.*
(10) ${}_RR$ *is supplemented if and only if every finitely generated $R-$module is supplemented.*

*Proof:* See [11, 41.2]. □

**Lemma 8.** *Let $M$ be an $R-$module.*
(1) *If $V$ is a supplement of $U$ in $M$, then $V$ is a g-suppement of $U$ in $M$.*
(2) *If $M = U \oplus V$ then $V$ is a g-supplement of $U$ in $M$. Also $U$ is a g-supplement of $V$ in $M$.*
(3) *For $M_1, U \leq M$, if $M_1 + U$ has a g-supplement in $M$ and $M_1$ is g-supplemented, then $U$ also has a g-supplement in $M$.*
(4) *Let $M = M_1 + M_2$. If $M_1$ and $M_2$ are g-supplemented, then $M$ is also g-supplemented.*
(5) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If $M_i$ is g-supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also g-supplemented.*
(6) *If $M$ is g-supplemented, then $M/L$ is g-supplemented for every $L \leq M$.*
(7) *If $M$ is g-supplemented, then every homomorphic image of $M$ is also g-supplemented.*
(8) *If $M$ is g-supplemented, then $M/Rad_gM$ is semisimple.*
(9) *Hollow, local and g-hollow modules are g-supplemented.*
(10) *If $M$ is g-supplemented, then every finitely $M-$generated module is g-supplemented.*
(11) ${}_RR$ *is g-supplemented if and only if every finitely generated $R-$module is g-supplemented.*
(12) *If $M$ is g-supplemented and every nonzero submodule of $M$ is essential in $M$, then $M$ is supplemented.*

*Proof:* See [5, 6]. □

**Lemma 9.** *Let $M$ be an $R-$module.*
(1) *If $M$ is supplemented, then $M$ is Rad-supplemented.*
(2) *If $V$ is a Rad-supplement of $U$ in $M$ and $W + V = M$ for some $W \leq U$, then $V$ is a Rad-supplement of $W$ in $M$.*
(3) *If $U$ is a maximal submodule of $M$ and $V$ is a Rad-supplement of $U$ in $M$, $U \cap V = RadV$ is the unique maximal submodule of $V$.*
(4) *If $V$ is a Rad-supplement of $U$ in $M$ and $L \leq U$, then $\frac{V+L}{L}$ is a Rad-supplement of $U/L$ in $M/L$.*
(5) *For $M_1, U \leq M$, if $M_1 + U$ has a Rad-supplement in $M$ and $M_1$ is Rad-supplemented, then $U$ also has a Rad-supplement in $M$.*
(6) *Let $M = M_1 + M_2$. If $M_1$ and $M_2$ are Rad-supplemented, then $M$ is also Rad-supplemented.*
(7) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If $M_i$ is Rad-supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also Rad-supplemented.*
(8) *If $M$ is Rad-supplemented, then $M/L$ is Rad-supplemented for every $L \leq M$.*
(9) *If $M$ is Rad-supplemented, then every homomorphic image of $M$ is also Rad-supplemented.*
(10) *If $M$ is Rad-supplemented, then $M/RadM$ is semisimple.*
(11) *If $M$ is Rad-supplemented, then every finitely $M-$generated module is Rad-supplemented.*

*Proof:* See [2, 3, 10]. □

**Lemma 10.** *Let $M$ be an $R-$module.*
(1) *If $M$ is Rad-supplemented, then $M$ is g-radical supplemented.*
(2) *If $V$ is a g-radical supplement of $U$ in $M$ and $W + V = M$ for some $W \leq U$, then $V$ is a g-radical supplement of $W$ in $M$.*
(3) *If $U$ is an essential maximal submodule of $M$ and $V$ is a g-radical supplement of $U$ in $M$, then $U \cap V = Rad_gV$ is the unique essential maximal submodule of $V$.*
(4) *If $V$ is a g-radical supplement of $U$ in $M$ and $L \leq U$, $\frac{V+L}{L}$ is a g-radical supplement of $U/L$ in $M/L$.*
(5) *For $M_1, U \leq M$, if $M_1 + U$ has a g-radical supplement in $M$ and $M_1$ is g-radical supplemented, then $U$ also has a g-radical supplement in $M$.*
(6) *Let $M = M_1 + M_2$. If $M_1$ and $M_2$ are g-radical supplemented, then $M$ is also g-radical supplemented.*
(7) *Let $M_i \leq M$ for $i = 1, 2, ..., n$. If $M_i$ is g-radical supplemented for every $i = 1, 2, ..., n$, then $M_1 + M_2 + ... + M_n$ is also g-radical supplemented.*
(8) *If $M$ is g-radical supplemented, then $M/L$ is g-radical supplemented for every $L \leq M$.*
(9) *If $M$ is g-radical supplemented, then every homomorphic image of $M$ is also g-radical supplemented.*
(10) *If $M$ is g-radical supplemented, then $M/Rad_gM$ is semisimple.*
(11) *If $M$ is g-radical supplemented, then every finitely $M-$generated module is g-radical supplemented.*

*Proof:* See [6]. □

**Lemma 11.** *Let $M$ be an $R-$module.*
(1) *If $M$ is $\oplus-$supplemented, then $M$ is supplemented.*
(2) *Let $M = M_1 \oplus M_2$. If $M_1$ and $M_2$ are $\oplus-$supplemented, then $M$ is also $\oplus-$supplemented.*
(3) *Let $M = M_1 \oplus M_2 \oplus ... \oplus M_n$. If $M_i$ is $\oplus-$supplemented for every $i = 1, 2, ..., n$, then $M$ is also $\oplus-$supplemented.*
(4) *Hollow and local modules are $\oplus-$supplemented.*
(5) *If $M$ is supplemented and $\pi-$projective, then $M$ is $\oplus-$supplemented.*

*Proof:* See [4, 11]. □

**Lemma 12.** *Let $M$ be an $R-$module.*
(1) *If $M$ is $Rad - \oplus-$supplemented, then $M$ is Rad-supplemented.*
(2) *Let $M = M_1 \oplus M_2$. If $M_1$ and $M_2$ are $Rad - \oplus-$supplemented, then $M$ is also $Rad - \oplus-$supplemented.*
(3) *Let $M = M_1 \oplus M_2 \oplus ... \oplus M_n$. If $M_i$ is $Rad - \oplus-$supplemented for every $i = 1, 2, ..., n$, then $M$ is also $Rad - \oplus-$supplemented.*

 

(4) *Hollow and local modules are* $Rad - \oplus -$*supplemented.*
(5) *If* $M$ *is* $\oplus -$*supplemented, then* $M$ *is* $\oplus -$*supplemented.*

*Proof:* See [2, 3]. □

**Definition 1.** *Let* $M$ *be an* $R-$*module. If every submodule of* $M$ *has a g-radical supplement that is a direct summand of* $M$*, then* $M$ *is called a* $\oplus -$*generalized radical supplemented (briefly* $\oplus - g - Rad - supplemented$*) module. (See also [9])*

**Lemma 13.** *Every* $\oplus - g - Rad -$*supplemented module is* $g-$*radical supplemented.*

*Proof:* Clear from definitions. □

**Corollary 1.** *Let* $M$ *be a* $\oplus - g - Rad -$*supplemented module. Then every factor module of* $M$ *is g-radical supplemented.*

*Proof:* Since $M$ is $\oplus - g - Rad -$supplemented, by Lemma 13, $M$ is g-radical supplemented. Then by [6, Lemma 9], every factor module of $M$ is g-radical supplemented. □

**Corollary 2.** *Let* $M$ *be a* $\oplus - g - Rad -$*supplemented module. Then every homomorphic image of* $M$ *is g-radical supplemented.*

*Proof:* Clear from Corollary 1. □

**Lemma 14.** *Let* $M$ *be an* $R-$*module and* $M = M_1 + M_2$*. If* $M_1$ *and* $M_2$ *are* $\oplus - g - Rad -$*supplemented, then* $M$ *is g-radical supplemented.*

*Proof:* Since $M_1$ and $M_2$ are $\oplus - g - Rad -$supplemented, by Lemma 13, these are g-radical supplemented. Then by [6, Lemma 7], $M$ is g-radical supplemented. □

**Corollary 3.** *The finite sum of* $\oplus - g - Rad -$*supplemented modules is g-radical supplemented.*

*Proof:* Clear from Lemma 14. □

**Corollary 4.** *Let* $M$ *be a* $\oplus - g - Rad -$*supplemented module. Then every finitely* $M-$*generated module is g-radical supplemented.*

*Proof:* Since $M$ is $\oplus - g - Rad -$supplemented, by Lemma 13, $M$ is g-radical supplemented. Then by [6, Lemma 10], every finitely $M-$generated module is g-radical supplemented. □

**Corollary 5.** *Let* $R$ *be a ring. If* $_RR$ *is* $\oplus - g - Rad -$*supplemented, then every finitely generated* $R-$*module is g-radical supplemented.*

*Proof:* Clear from Corollary 4. □

**Proposition 1.** *Every* $Rad - \oplus -$*supplemented module is* $\oplus - g - Rad -$*supplemented.*

*Proof:* Clear from definitions. □

**Corollary 6.** *The finite direct sum of* $Rad - \oplus -$*supplemented modules is* $\oplus - g - Rad -$*supplemented.*

*Proof:* Let $M_1, M_2, ..., M_n$ be $Rad - \oplus -$supplemented modules and $M = M_1 \oplus M_2 \oplus ... \oplus M_n$. By [3, Proposition 3.1], $M$ is $Rad - \oplus -$supplemented. Then by Proposition 1, $M$ is $\oplus - g - Rad -$supplemented. □

**Corollary 7.** *Every* $\oplus -$*supplemented module is* $\oplus - g - Rad -$*supplemented.*

*Proof:* Clear from Proposition 1, since every $\oplus -$supplemented module is $Rad - \oplus -$supplemented. □

**Corollary 8.** *The finite direct sum of* $\oplus -$*supplemented modules is* $\oplus - g - Rad -$*supplemented.*

*Proof:* Clear from Corollary 6 and Corollary 7. □

## 3 Conclusion

$\oplus - g - Rad -$supplemented modules are more general than $\oplus -$supplemented modules.

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# An Efficient Numerical Method for Solving Parabolic Partial Differential Equation

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**Abstract:** The parabolic partial differential equation has been used as a model for many situations. Therefore, the accuracy of numerical solutions is important in the literature. In this study, finite difference method is developed with Lagrange polynomials and applied to heat equation. The equation made discrete with this approach is solved by the implicit method to find the solution at each grid point. The numerical solutions were found to be more accurate when compared with the results obtained with the classical finite difference method. The results obtained were supported by tables and graphs.

**Keywords:** Finite difference method, Implicit method, Lagrange interpolation, Partial differential equation.

## 1 Introduction

Partial differential equations are used to characterize many phenomena in nature. In recent years, most physical events have been expressed through mathematical models. Solutions of such equations are important in understanding and interpreting events. A lot of work has been done in this area recently. Some of these are as follows. In [1] the Pade approach has been applied to a parabolic partial differential equation. A local mesh-free method for solving ill-posed problem was presented in [2]. Physical events are modeled with partial differential equation. An example of metrology is solved in [3]. In [4, 5], non local boundary conditional problems have been used in the quasistatic theory of thermoelasticity. Studies have been done about the accuracy of numerical solution methods [6]-[8]. Finite difference and finite element method has been used to solve similar problems [9]-[11]. Additionally, there are many numerical studies in the literature such as Taylor method [12, 13], collacation method [14], variational iteration method [15], cubic trigonometric B splines method [16], homotopy analysis method [17].

In this work, we examine the one dimensional parabolic equation [1],

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad (x,t) \in [0,L] \times [0,T] \tag{1}$$

with the initial conditions

$$u(x,0) = h(x), \tag{2}$$

and the boundary conditions

$$u(0,t) = g_0(t), \quad u(L,t) = g_1(t), \tag{3}$$

where $h$, $g_0$ and $g_1$ are functions that describe the initial boundary and boundary conditions, respectively, and T is a given positive value of the final time, $g_0$ and $g_1$ are the boundary temperatures.

We investigate the numerical solution of Eq.(1) with initial and boundary condition Eqs.(2,3) by finite difference method. Finite difference approximations are obtained by using Taylor series. In this work we used finite difference equations obtained Lagrange interpolation. By using this interpolation we approximate the first and second derivatives of any function. Here we apply these finite difference formulas to parabolic one dimensional heat equation. The finite difference method solves many partial differential equations [18]-[20].

This paper is organized as follows: Numerical scheme for the solution of Eq.(1) in Section 2. The mentioned method is applied to some examples and the usefulness of the method is supported by tables and graphics in Section 3. In Section 4, a conclusion is given.

## 2 Solution method

In this section, an approximation to the first and second derivatives of any function is constructed using quadratic interpolation polynomials. If we take the Lagrange interpolating polynomial form.

$$y(x) \approx p_2(x) = y(x_0)L_0^{(2)}(x) + y(x_1)L_1^{(2)}(x) + y(x_2)L_2^{(2)}(x) \tag{4}$$

The approximate derivatives are

 

$$y^{'}(x_0) \approx p_2^{'}(x_0) = y(x_0)(L_0^{(2)})^{'}(x_0) + y(x_1)(L_1^{(2)})^{'}(x_0) + y(x_2)(L_2^{(2)})^{'}(x_0) \tag{5}$$

$$y^{''}(x_0) \approx p_2^{''}(x_0) = y(x_0)(L_0^{(2)})^{''}(x_0) + y(x_1)(L_1^{(2)})^{''}(x_0) + y(x_2)(L_2^{(2)})^{''}(x_0) \tag{6}$$

where $(L_j^{(2)})(x)$ is Lagrange interpolation polynomials of degree 2 and $x_1 = x_0 + h$ and $x_2 = x_0 + 2h$. We evaluate the values of $(L_j^{(2)})^{'}(x_0)$ and $(L_j^{(2)})^{''}(x_0)$ in Eq.3. If we compute these values, we obtain,

$$(L_0^{(2)})^{'}(x_0) = -\frac{3}{2h}, \quad (L_1^{(2)})^{'}(x_0) = \frac{2}{h}, \quad (L_2^{(2)})^{'}(x_0) = -\frac{1}{2h}, \tag{7}$$

$$(L_0^{(2)})^{''}(x_0) = \frac{1}{h^2}, \quad (L_1^{(2)})^{''}(x_0) = -\frac{2}{h^2}, \quad (L_2^{(2)})^{''}(x_0) = \frac{1}{h^2}, \tag{8}$$

So, we have the following approximate of the first and second derivatives,

$$y^{'}(x) = \frac{1}{2h}(-y(x+2h) + 4y(x+h) - 3y(x)) \tag{9}$$

$$y^{''}(x) = \frac{1}{h^2}(y(x+2h) - 2y(x+h) + y(x)) \tag{10}$$

The relation (9) and (10) can be extended the approximations to partial derivatives to numerically solve the Eq.(1). For this proposed, we divide the spatial domain $W = [0, L]$ into the uniform mesh with $N + 1$ nodes $x_i = ih$ for $i = 0, ..., N$, where $h = \frac{L}{N}$. The time interval $[0, T]$ is divided into $F$ subintervals where the subinterval length equals $dt = \frac{T}{F}$ and the time nodes are $t_j = jdt$ for $j = 0, ..., F$. At these points, the value of the function is represented by $u(x_i, t_j) = u_i^j$. The equivalents of the above approaches in partial derivatives are as follows:

$$\frac{\partial u}{\partial x} = \frac{1}{2h}(-u_{i+2}^j + 4u_{i+1}^j - 3u_i^j) \tag{11}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h^2}(u_{i+2}^j - 2u_{i+1}^j + u_i^j) \tag{12}$$

$$\frac{\partial u}{\partial t} = \frac{1}{2dt}(-u_i^{j+2} + 4u_i^{j+1} - 3u_i^j) \tag{13}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{(dt)^2}(u_i^{j+2} - 2u_i^{j+1} + u_i^j) \tag{14}$$

If the Eq. (11)-(14) put into Eq. (1), the following discrete form is obtained for Eq.(1)

$$u_{i-1}^{j+1}(-\frac{1}{(dx)^2}) + u_i^{j+1}(\frac{2}{(dx)^2} - \frac{1}{2dt}) + u_{i+1}^{j+1}(-\frac{1}{(dx)^2}) = u_i^j(-\frac{2}{dt}) + u_i^{j-1}(\frac{3}{2dt}) + f_i^j \tag{15}$$

If $j = 0$, $u_i^{-1}$ is obtained and this means the external mesh point $(-h, jdt)$. It is called external mesh point (Figure 1). We can find the equivalent of this point with the help of derivative in boundary conditions. The boundary condition can be represented by [20].

$$\frac{u_i^1 - u_i^{-1}}{2dt} = \frac{\partial u_i^j}{\partial t} = g(x_i) \tag{16}$$

If the unknown is eliminated with Eq.(15), the following equation is achieved for $j = 0$,

$$u_{i-1}^1(-\frac{1}{(dx)^2}) + u_i^1(-\frac{2}{(dx)^2} - \frac{1}{2dt}) + u_{i+1}^1(-\frac{1}{(dx)^2}) = u_i^0(-\frac{2}{dt} + \frac{3}{2dt}) + f_i^j \tag{17}$$

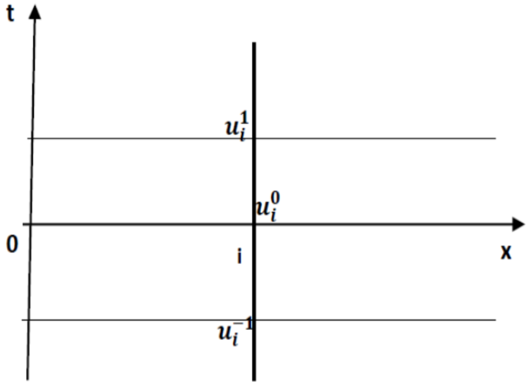


**Fig. 1**: Graph of the external mesh point

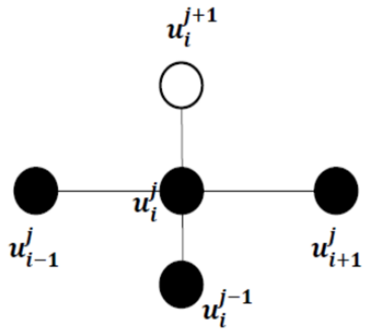


**Fig. 2**: Nodes display graph of the given scheme

If the solutions of the difference equations are all of the form

$$u_s^t = \xi^t e^{ik\Delta x} \tag{18}$$

where k is a real number, $\xi = \xi(k)$ is a complex number that depends on k. The number $\xi$ is called the amplification factor at a given number k.

To find $\xi(k)$ we substitute $u_s^t$ into Eq.(15) and get $\xi(k)$. As a von-Neumann stability analysis shows [21], the amplification factor is given by $\xi(k) < 1$ so that the finite-differencing Eq .(15) is unconditionally stable.

## 3 Examples

### *3.1 Example 1*

We consider the Eq.(1) representing the heat phenomena with the following conditions [1];

$$h(x) = e^x$$

$$g_0(t) = \frac{1}{1+t^2}, \;\; g_1(t) = \frac{e}{1+t^2}$$

$$f(x,t) = -\frac{(1+t)^2 e^x}{(1+t^2)^2}$$

The exact solution is given by $u_{exact} = \dfrac{e^x}{(1+t^2)}$. The absolute errors corresponding to different N values at different times are given in the Table 1 with$dt = 0.000001$. Table 2 shows the absolute error using the technique presented with $N = 100$ with different $dt$.

| $i$ | $N = 10$ | $N = 20$ | $N = 30$ | $N = 40$ |
|---|---|---|---|---|
| 1 | 8.8658e-5 | 3.9126e-5 | 5.9119e-7 | 1.2940e-5 |
| 2 | 8.8670e-5 | 4.6928e-5 | 4.9290e-5 | 3.8531e-5 |
| 3 | 8.8599e-5 | 3.3295e-5 | 1.0600e-6 | 3.4142e-5 |
| 4 | 8.8965e-5 | 4.8299e-5 | 3.9786e-5 | 9.2400e-7 |
| 5 | 1.2418e-4 | 1.2924e-5 | 1.2879e-5 | 1.3333e-5 |
| 6 | 1.7725e-4 | 3.9431e-5 | 1.6000e-8 | 3.4319e-5 |
| 7 | 4.9559e-5 | 5.2588e-5 | 1.3474e-5 | 1.2987e-5 |
| 8 | 1.7738e-4 | 4.8079e-5 | 1.3736e-5 | 2.0800e-7 |
| 9 | 9.8402e-5 | 5.3010e-5 | 1.4443e-5 | 2.0600e-7 |

**Table 1** Absolute Values of Errors for $u(x, t)$ with $dt = 0.000001$ with Different Values of $N$

| $i$ | $dt = 0.001$ | $dt = 0.00001$ |
|---|---|---|
| 1 | 1.8509e-3 | 1.9000e-8 |
| 2 | 2.6650e-3 | 2.3000e-8 |
| 3 | 3.3512e-3 | 6.0000e-9 |
| 4 | 3.8766e-3 | 2.0000e-9 |
| 5 | 4.2161e-3 | 1.2000e-8 |
| 6 | 4.3534e-3 | 5.0000e-9 |
| 7 | 4.2821e-3 | 1.0000e-9 |
| 8 | 4.0059e-3 | 6.0000e-9 |
| 9 | 3.5382 e-3 | 5.0000e-9 |

**Table 2** Absolute values of errors for $u(x, t)$ with $N = 100$ with different values of $dt$

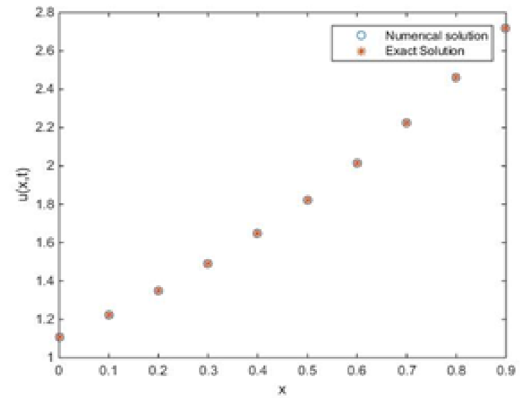


**Fig. 3**: Numerical and exact solutions at any time level with $h = 0.1$, $dt = 0.000001$

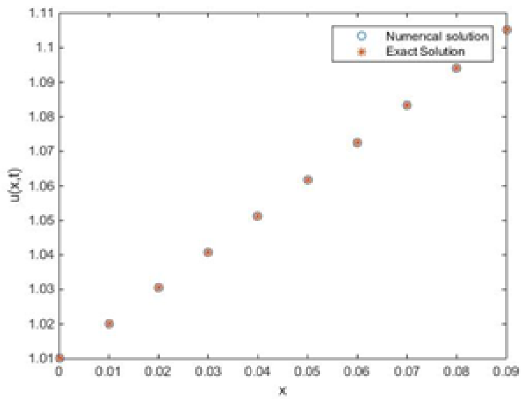


**Fig. 4**: Comparison of exact and numerical solutions for $h = 0.01$, $dt = 0.000001$

In Figure 1, numerical and exact solutions at any time level with$h = 0.1$, $dt = 0.000001$ are compared. It has been observed that they have close values. Figure 2 shows the exact and numerical values of the solution with $h = 0.01$, $t = 0.000001$. It has been observed that the errors decrease when the $N$ value increases.

### *3.2 Example 2*

We consider the following parabolic equation representing the heat phenomena

$$\frac{\partial u(x,t)}{\partial t} = a\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t)$$

with the following conditions [2];

$$h(x) = x^2$$

$$g_0(t) = -t^2, \;\; g_1(t) = 1 - t^2$$

$$f(x,t) = -2t - 2a$$

The exact solution is given by $u_{exact} = x^2 - t^2$. The following Table 3 show the absolute error using the technique presented with different $N$ and $dt$ values. Absolute errors at different a values are given in Table 4 for $h = \frac{1}{100}$, $dt = 0.00001$.

| $dt$ | $N = 10$ | $N = 20$ | $N = 30$ |
|---|---|---|---|
| 0.001 | 4.2687e-6 | 1.6181e-6 | 3.0868e-6 |
| 0.0005 | 6.6789e-6 | 6.9043e-5 | 5.8113e-5 |

**Table 3** Absolute errors for $u(0.2, 0.001)$ with $a = 1$ and different values of $N$ and $dt$.

| $i$ | $a = 1$ | $a = 100$ |
|---|---|---|
| 1 | 4.0722e-8 | 5.0409e-6 |
| 2 | 1.6955e-8 | 9.8380e-6 |
| 3 | 2.9441e-8 | 1.4159e-5 |
| 4 | 1.0040e-9 | 1.7796e-5 |
| 5 | 1.7178e-8 | 2.0573e-5 |
| 6 | 6.5250e-9 | 2.2355e-5 |
| 7 | 1.3103e-8 | 2.3056e-5 |
| 8 | 8.4600e-9 | 2.2644e-5 |
| 9 | 1.2178e-8 | 2.1137e-5 |

**Table 4** Absolute errors with $N = 100$, $dt = 0.00001$ at different space level for different value $a$.

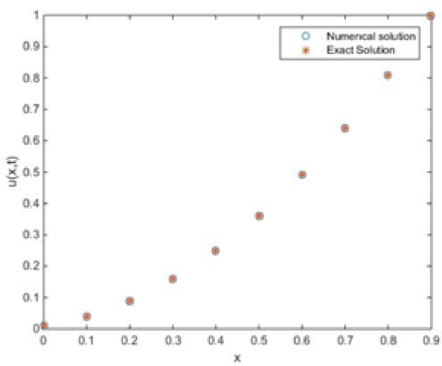


**Fig. 5**: Numerical solutions for $a = 1$ with $N = 10$,$dt = 0.00001$

Figure 5, shows the numerical results with $N = 10$, $dt = 0.00001$. Numerical and exact values are given in Figure 6 for $N = 100$, $dt = 0.00001$.

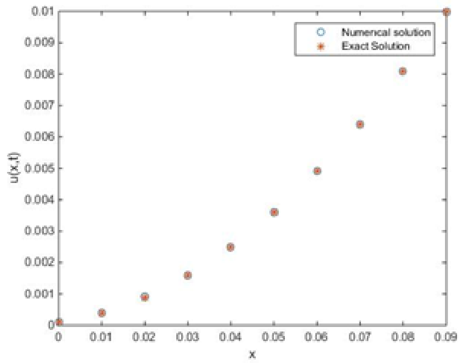


**Fig. 6**: Numerical and Exact solutions for $a = 1$ with $N = 100$, $dt = 0.00001$

**Algorithm**

1. Input the time step $(dt)$ , and space step $(dx)$.
2. Compute grid points "$x_i$" for $i = 0, ..., N$ and "$t_j$" for $j = 0, ..., T$
3. Compute $u_i^1$ for all mesh points by Eq.17.
4. Step1: Compute the right hand side of Eq.17 for all mesh points.

- For each fixed $i = 0, , N$ and $j = 0, , T$ compute $u_i^j$ and $u_i^{j-1}$ by using Eqs. 15-17.
- For each fixed $i = 0, , N$ and $j = 0, , T$ compute $f_i^j$.

5. Step2: Compute the boundary conditions $u_i^j$ for $i = 0, , N$ and $j = 0, , T$.
6. Step3: Calculate $u_i^{j+1}$ by using given scheme Eq. 15-17.
7. Evaluate absolute errors.

## 4 Conclusion

In this study, finite differences, Lagrange polynomials are developed with interpolation forms and applied to the heat equation. The equation was made discrete with these formulas and solutions in each time step were obtained. Implicit method is used in each step. This method is unconditionally stable according to Von-Neumann stability analysis. Since the heat equation is very important in the literature, its solutions must be very close to the exact solution. The solutions obtained are sufficient and suitable errors in some steps are shown in the tables.

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# A Chebyshev Collocation Method for Solving Nonlinear Blasius Equations

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**Abstract:** In this work, we give approximate solutions for the nonlinear Blasius equations. To solve Blasius equation, the truncated shifted Chebyshev polynomials and collocation matrix-vector method are considered. The proposed method converts the Blasius equation into a nonlinear system equation with unknown Chebyshev coefficients. Some examples are presented to approve the given method.

**Keywords:** Blasius equation, Collocation method, Shifted Chebyshev polynomials.

## 1 Introduction

The Blasius equation is a third order nonlinear differential equation which describes as a laminar viscous flow over a semi-infinite flat plate [1, 2]

$$y'''(x) + \frac{1}{2}y(x)y''(x) = 0 \tag{1}$$

with conditions

$$y(0) = y'(0) = 0 \tag{2}$$

$$y'(M) = 1 \tag{3}$$

where $y$ is unknown function and $M$ is constant and sufficiently large. The Blasius equation is characterized by the value of $\alpha = y''(0)$ [1, 2]. The existence and the uniqueness of the solution have been studied in [3]. But, the analytical solution of Blasius can not be obtained. Since some numerical schemes can not be applied Eqs. (1)-(3), some transformations are arose to obtain numerical solution of Eqs. (1)-(3). In literature, some authors investigate some feature and approximate solutions such as the case of two paralel streams [3, 4], the lower stream is at rest as well as when it is in motion [5], two fluids of different viscosities and densities [6], Adomian decomposition method [7], homotopy perturbation method [8], totally analytic method [9], Pade approximation [10], and others [11]-[18]. For solving this problem, we assume that the following truncated shifted Chebyshev series is a solution of the Eq.(1)

$$y_N(t) = \sum_{n=0}^{N} a_n T_n^*(t), T_n^*(t) = cos(n\theta), 2t-1 = cos\theta, t \in [0,1] \tag{4}$$

where $T_n^*(t)$ is a polynomial in $t$ of degree $n$, is called the shifted Chebyshev polynomial of the first kind and $a_n$ are unknown Chebyshev coefficients, $N \geq 3$.

## 2 The shifted Chebyshev polynomials of the first kind and some their properties

The shifted Chebyshev polynomials $T_n^*(t)$ of the first kind is defined by the following relation relation [19, 20]

$$T_n^*(t) = 2(2t-1)T_{n-1}^*(t) - T_{n-2}^*(t), n \geq 1$$

with $T_0^*(t) = 1, T_1^*(t) = 2t - 1$.

The leading coefficient of $t^n$ in $T_n^*(t)$ to be $2^{2n-1}$.

$$t_{n,i} = \frac{1}{2}[1 + cos\frac{(2(n-i)+1)\pi}{2(n+1)}], i = 0, 1, ..., n \tag{5}$$

are zeroes on the interval $[0, 1]$ of $T^*_{n+1}(t)$. The relation between the powers $t^n$ and the second kind Chebyshev polynomials $T^*_n(t)$ is

$$t^n = 2^{-2n+1} \sum_{k=0}^{n} \binom{2n}{k} T^*_{n-k}(t) \tag{6}$$

where $\sum'$ denotes a sum whose first term is halved. For more details see [19, 20].

## 3 Fundamental relations

In this analysis, we are using the collocation method with the truncated shifted Chebyshev series. We want to find a truncated shifted Chebyshev series which is represented the solution of the Eq.(1) with the conditions Eqs.(2)-(3). Initially, we want to write a matrix-vector form of $y_N(t)$ and its derivative $y_N^{(k)}(t)$ . Those forms can be written as the following form

$$y_N(t) = T^*_N(t)A,\ y_N^{(k)}(t) = {T^*_N}^{(k)}(t)A \tag{7}$$

where

$$T^*(t) = \left[\begin{array}{cccc} T^*_0(t) & T^*_1(t) & \cdots & T^*_N(t) \end{array}\right]$$

$$T^{*(k)}(t) = \left[\begin{array}{cccc} T_0^{*(k}(t) & T_1^{*(k)}(t) & \cdots & {T_N^*}^{(k)}(t) \end{array}\right]$$

$$A = \left[\begin{array}{cccc} a_0 & a_1 & \cdots & a_N \end{array}\right]^T$$

If we want to find a relation between $X(t)$ and $T^*(t)$. , using the expression (6), the following relation can be written

$$X^T(t) = D{T^*}^T(t) \text{ and so } X(t) = T^*(t)D^T \tag{8}$$

where

$$X(x) = \left[\begin{array}{cccc} 1 & t & ... & t^N \end{array}\right].$$

and

$$D = \left[\begin{array}{ccccc} 2^0\binom{0}{0} & 0 & 0 & \cdots & 0 \\ 2^{-2}\binom{2}{1} & 0 & 2^{-1}\binom{2}{2} & \cdots & 0 \\ 2^{-4}\binom{4}{2} & 2^{-3}\binom{4}{3} & 2^{-3}\binom{4}{4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2^{-2N}\binom{2N}{N} & 2^{-2N+1}\binom{2N}{N+1} & 2^{-2N+1}\binom{2N}{N+2} & \cdots & 2^{-2N+1}\binom{2N}{2N} \end{array}\right]$$

Since the $D$ has an inverse, Eq.(8) can be rewritten as

$$T^*(t) = X(t)(D^T)^{-1} \tag{9}$$

The $B$ matrix can be found is satisfied the following relation:

$$X^{(k)}(t) = X(t)B^k \tag{10}$$

where

$$B = \left[\begin{array}{ccccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{array}\right].$$

To obtain the matrix form of the approximate solution of and its derivatives $y_N^{(k)}(t)$ , if Eqs.(9)-(10) are put into Eq.(7), we have the matrix-vector form

$$y_N^{(k)}(t) = X(t)B^k(D^T)^{-1}A,\ k = 0, 1, 2, 3 \tag{11}$$

Also, the matrix-vector form of the nonlinear part of Eq.(1) can be written from [21, 22]

 

$$Y^m(t) = (\overline{Y(t)})^{m-1} Y(t) \tag{12}$$

where

$$Y(t)^m = \begin{bmatrix} y^m(t) \\ y^m(t) \\ \vdots \\ y^m(t) \end{bmatrix} \overline{Y(t)} = \begin{bmatrix} y(t) & 0 & \cdots & 0 \\ 0 & y(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y(t) \end{bmatrix}$$

and

$$\overline{Y(t)} = \overline{T}(t)\overline{A} \tag{13}$$

where

$$\overline{T(t)} = \begin{bmatrix} T(t) & 0 & \cdots & 0 \\ 0 & T(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T(t) \end{bmatrix} \overline{A} = \begin{bmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{bmatrix}$$

Then, we construct the following relation

$$y_N(t_i) y_N^{''}(t_i) = (\overline{T(t)}\overline{A}) X(t) (B^T)^2 (D^T)^{-1} A \tag{14}$$

From Eq.(11), the matrix form for conditions Eqs. (2)-(3) are obtained by the following relations

$$y_N(0) = X(0)(D^T)^{-1} A = \begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0N} \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \tag{15}$$

$$y_N^{'}(0) = X(0) B^T (D^T)^{-1} A = \begin{bmatrix} u_{10} & u_{11} & \cdots & u_{1N} \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \tag{16}$$

$$y_N^{'}(M) = X(M) B^T (D^T)^{-1} A = \begin{bmatrix} u_{20} & u_{21} & \cdots & u_{2N} \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \tag{17}$$

## 4 Method of solution

In this section, we convert the (1) into a matrix-vector equation. The numerical solutions of (1) with conditions (2)-(3) try to find as a Chebyshev series using collocation method. If (11)-(14) put into (1), we get the system of matrix equations

$$X(t_i)(B^T)^3(D^T)^{-1}A + 0.5(\overline{T(t_i)}\overline{A})X(t_i)(B^T)^2(D^T)^{-1}A = 0 \tag{18}$$

where $t_i$, $0 \le i \le N$ are the zeroes of the $T^*_{n+1}(t)$. WE can briefly write the (18)

$$(X(B^T)^3(D^T)^{-1}A + P(\overline{T}\overline{A})X(B^T)^2(D^T)^{-1}A) = \mathbf{0} \tag{19}$$

where

$$P = \begin{bmatrix} 1/2 & 0 & 0 & \cdots & 0 \\ 0 & 1/2 & 0 & \cdots & 0 \\ 0 & 0 & 1/2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1/2 \end{bmatrix} X = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^N \\ 1 & x_1 & x_1^2 & \cdots & x_1^N \\ 1 & x_2 & x_2^2 & \cdots & x_2^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^N \end{bmatrix} \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \overline{T} = \begin{bmatrix} T(t_0) & 0 & 0 & \cdots & 0 \\ 0 & T(t_1) & 0 & \cdots & 0 \\ 0 & 0 & T(t_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & T(t_N) \end{bmatrix}$$

To obtain the solution of Eq.(1) under the conditions (2), by replacing the rows matrices (15)-(16)-(17) by the last 3 rows of the matrix (19). Hence, (19) is described a system of $(N+1)$ nonlinear algebraic equation with unknown coefficients. If those nonlinear systems are solved by Maple program, we find the unknown coefficients $\widehat{A}$. $\widehat{A}$ is the approximate value of $A$. Those coefficients are put into (4) , the approximate solution

$$\widehat{y}_N(t) = \sum_{n=0}^{N} \widehat{a_n} T_n^*(t)$$

is obtained. The accuracy of the method can be investigated with the following relation which is called the error estimation function [19] :

$$E_N(t) = |\widehat{y}_N^{'''} - \frac{1}{2}\widehat{y}_N(t)\widehat{y}_N^{''}(t)| \cong 0 \tag{20}$$

## 5 Example

In this section, we shall test accuracy of the proposed method. In Table 1, we give the numerical results for various $N$ and $M = 2$. Those numerical results are plotted in Figure 1. Moreover, we have obtained the value of $\alpha$, $0.527317$, $527687$, $0.503774$ for $N = 4, 6, 8$, respectively.

| $t$ | $E_4$ | $E_6$ | $E_8$ |
|---|---|---|---|
| 0.0 | 0.118E-2 | 0.495E-5 | 0.134E-5 |
| 0.1 | 0.268E-2 | 0.180E-5 | 0.177E-5 |
| 0.2 | 0.468E-3 | 0.161E-4 | 0.148E-5 |
| 0.3 | 0.139E-1 | 0.886E-5 | 0.277E-5 |
| 0.4 | 0.465E-1 | 0.790E-4 | 0.907E-5 |
| 0.5 | 0.103E-1 | 0.100E-4 | 0.121E-5 |
| 0.6 | 0.190E-0 | 0.866E-3 | 0.450E-4 |
| 0.7 | 0.313E-0 | 0.399E-2 | 0.492E-4 |
| 0.8 | 0.478E-0 | 0.123E-1 | 0.122E-3 |
| 0.9 | 0.690E-0 | 0.308E-1 | 0.536E-2 |
| 1.0 | 0.956E-0 | 0.680E-1 | 0.163E-1 |

**Table 1** Comparison of estimation error functions.

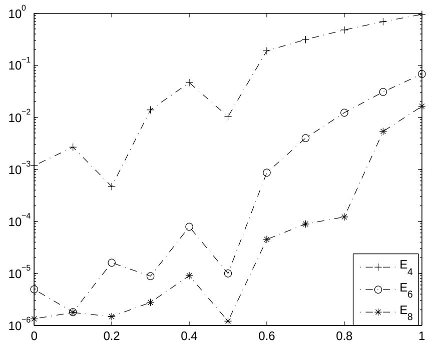


**Fig. 1**: Comparison of the estimation error functions for various $N$.

## 6 Conclusion

A collocation matrix-vector method has been proposed to obtained the numerical solution of Blasius equation. The solution algorithm of the Blasius equation is written by Maple computer program and so, the unknown coefficients are found very easily. Numerical results arise the effectiveness of the method.

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# An Interactive Illustration of Ideal Flow Field

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**Abstract:** Being one of the recently emerged modern teaching and learning tools, GeoGebra has become a widely utilized interactive mathematical application that can be employed for educational purposes for various branches of science and engineering. In this particular study, the implementation of this software for an ideal two-dimensional flow field is presented. The developed applet aims to enable the user to grasp a visual understanding of the relation between the velocity vector field, streamlines and potential lines in a flow field. Besides, the applet portrays the pressure distribution given by the Euler equations which govern the ideal fluid flow and it allows the user to examine the spatial variation of the terms in the Bernoulli equation.

**Keywords:** Bernoulli equation, Euler equations, GeoGebra, Potential lines, Streamlines

## 1 Introduction

GeoGebra is an interactive, easy-to-use mathematical tool developed for educational purposes from primary school up to college level [1]. It enables the visualization of mathematical concepts by bridging the gap between algebra and geometry. As an open source software, GeoGebra can be freely downloaded or it can be launched directly at www.geogebra.org. GeoGebra is a Java based software, thus it is executable on every operating system (Windows, macOS, Linux, Android, iOS) . A cloud service is also supported enabling the users to upload and share their GeoGebra applets. Various studies can be found in the literature focusing on the implementation of GeoGebra for picturing engineering problems, e.g. [2]-[5].

Ideal flow theory constitutes an important part of fluid dynamics and the principles of ideal flow provides a basis for the analyses of more complex flow types. Most of the undergraduate science and engineering students gain an understanding of fluid motion by first assimilating ideal flow which is governed by a set of differential equations. This work aims to portray the solution of these differential equations with the use of GeoGebra. The developed GeoGebra applet allows the user to observe the influence of various dynamically editable parameters on flow behavior.

## 2 Mathematical background

In fluid dynamics, ideal flow is defined as flow of a fluid that is incompressible and inviscid. Incompressible flow refers to a flow where the fluid density is constant everywhere in the flow field. For incompressible flow, the continuity equation, also known as the principle of conservation of mass, can be formulated as;

$$\nabla \cdot \vec{V} = 0 \tag{1}$$

where $\vec{V}$ is the velocity vector describing the flow field. For a two dimensional flow field the continuity equation can be satisfied by introducing stream function $\psi$, such that the curl of the stream function yields the velocity vector;

$$\vec{V} = \nabla \times \psi \tag{2}$$

A streamline is a contour line obtained for the stream function. In the flow field, instantaneous velocity vector is tangential to the streamlines, therefore, these curves are practical graphical constructs to portray the direction of fluid flow.

Inviscid flow designates the flow of a fluid with zero viscosity. Inviscid flow assumption is employed when viscous normal and shear stresses are negligible in the flow field. Applying Newton's second law of motion for an inviscid flow field gives the Euler equations, as follows;

$$\frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \nabla\right) \vec{V} = -\frac{1}{\rho} \nabla P + \vec{g} \tag{3}$$

where $P$ denotes pressure, $\rho$ is the density of the fluid and $\vec{g}$ is the gravity vector, that is $\vec{g} = -g\hat{k}$, where $\hat{k}$ is the unit vector in vertical direction and $g$ is the magnitude of gravitational acceleration.

Analysis of incompressible, invisicid flow problems are further simplified with the assumption of irrotational flow. The angular velocity of a fluid particle in an irrotational flow field is zero, so that the curl of the velocity vector becomes zero;

$$\nabla \times \vec{V} = 0 \tag{4}$$

(4) can be satisfied by introducing a velocity potential function, $\phi$, such that,

$$\vec{V} = \nabla \phi \tag{5}$$

The value of the potential function drops in flow direction. Potential lines are provided by the potential function whose value is constant along a potential line. The potential lines are perpendicular to the streamlines and a finite set of lines selected from these mutually orthogonal families of lines form a flow net in the flow field.

For a steady (time-independent), incompressible, inviscid and irrotataional flow field, substituting (5) into (3) yields the well-known Bernoulli relation, that is;

$$\frac{P}{\rho g} + \frac{|\vec{V}|^2}{2g} + z = \text{constant} \tag{6}$$

where z stands for the elevation of a point in the flow field and $|\vec{V}|$ is the magnitude of the velocity vector at that point. The Bernoulli equation can be applied for points that are positioned along the same streamline, however, for irrotational flow, this relation is applicable to any selected two points in the flow field.

## 3 GeoGebra application

To exhibit the use of GeoGebra as a tool for picturing ideal flow, a hypothetical, two-dimensional velocity field in Cartesian coordinate system is taken into consideration, which has the following form;

$$\vec{V} = (ax + by + c)\,\hat{i} + (bx - ay + d)\,\hat{j} \tag{7}$$

where the steady-state velocity field $\vec{V}(m/s)$ is introduced as a function of the spatial variables $x\ (m)$ and $y\ (m)$ together with the user-specified parameters; $a\ (1/s)$, $b\ (1/s)$, $c\ (m/s)$ and $d\ (m/s)$. $\hat{i}$ and $\hat{j}$ stand for the unit vectors in $x$ and $y$ directions, respectively. Apart from these parameters, the density of the fluid, $\rho\ (kg/m^3)$, the elevation with respect to a reference plane, $z\ (m)$ and the reference pressure, $P_0\ (N/m^2)$, which is assumed to be known at $x = 0,\ y = 0,\ z = 0$, can be interactively specified by the user as well. The interface of the developed GeoGebra applet, which can be reached at https://www.geogebra.org/m/ueqdfywy, is presented in Fig. 1.

For an incompressible, irrotational flow field described in x-y plane, based on (2) and (5) the functional relationship between the velocity components, stream function and potential function can be established as;

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \tag{8}$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} \tag{9}$$

where

$$\vec{V} = u(x,y)\hat{i} + v(x,y)\hat{j} \tag{10}$$

The stream and potential functions characterizing this flow field can be attained by solving the partial differential equations introduced in (8) and (9), respectively, so to get;

$$\psi = -\frac{b}{2}x^2 + \frac{b}{2}y^2 + axy - dx + cy \tag{11}$$

$$\phi = \frac{a}{2}x^2 - \frac{a}{2}y^2 + bxy + cx + dy \tag{12}$$

The produced stream and potential functions can be solved for $y$, as depicted in (13) and (14), respectively, to obtain two families of streamlines and potential lines, constructing a flow net.

 

$$y = \frac{-c - ax \mp \sqrt{(a^2+b^2)\,x^2 + 2\,(ac+bd)\,x + c^2 + 2b\psi}}{b} \tag{13}$$

$$y = \frac{d + bx \mp \sqrt{(a^2+b^2)\,x^2 + 2\,(ac+bd)\,x + d^2 - 2a\phi}}{a} \tag{14}$$

Solving the three $(x,\ y,\ z)$ components of the differential equation (Euler equations) given in (3) yields the spatial variation of the pressure field, as follows;

$$P = -\rho\left[\frac{a^2+b^2}{2}\left(x^2+y^2\right) + (ac+bd)x + (bc-ad)y + gz\right] + P_0 \tag{15}$$

To visualize the pressure field, (15) can be solved for $y$ to construct isobars (lines of constant pressure), as;

$$y = \frac{\rho(ad-bc) \mp (-\rho\beta)^{1/2}}{\rho(a^2+b^2)} \tag{16}$$

where

$$\beta = \rho(a^2+b^2)\left[x^2(a^2+b^2) + 2x(ac+bd) + 2gz\right] - \rho(ad-bc)^2 + 2(P-P_0)(a^2+b^2) \tag{17}$$

The developed GeoGebra applet dynamically calculates and shows the spatially varying stream, potential and pressure functions with the use of (11), (12) and (15), respectively. To visualize the flow field, the applet plots representative vectors describing the velocity field given by (7). In addition, the application plots streamlines, potential lines and isobars by substituting representative values of $\psi$, $\phi$ and $P$ into (13), (14) and (16), respectively.

The applet also involves an animation depicting the motion of a fluid particle in the flow field. Midpoint Method (Modified Euler approach) is employed to calculate the particle position varying as a function of time, $(x(t),\ y(t))$, as follows;

$$x(t+\Delta t) = x(t) + u\left(x(t) + \frac{1}{2}u\left(x(t),y(t)\right)\Delta t, y(t) + \frac{1}{2}v\left(x(t),y(t)\right)\Delta t\right)\Delta t \tag{18}$$

$$y(t+\Delta t) = y(t) + v\left(x(t) + \frac{1}{2}u\left(x(t),y(t)\right)\Delta t, y(t) + \frac{1}{2}v\left(x(t),y(t)\right)\Delta t\right)\Delta t \tag{19}$$

where $\Delta t$ denotes the time step increment which is taken as 0.01 s for this particular study. As the particle moves, the user will be able to monitor the variation of the components of the Bernoulli equation introduced in (6).

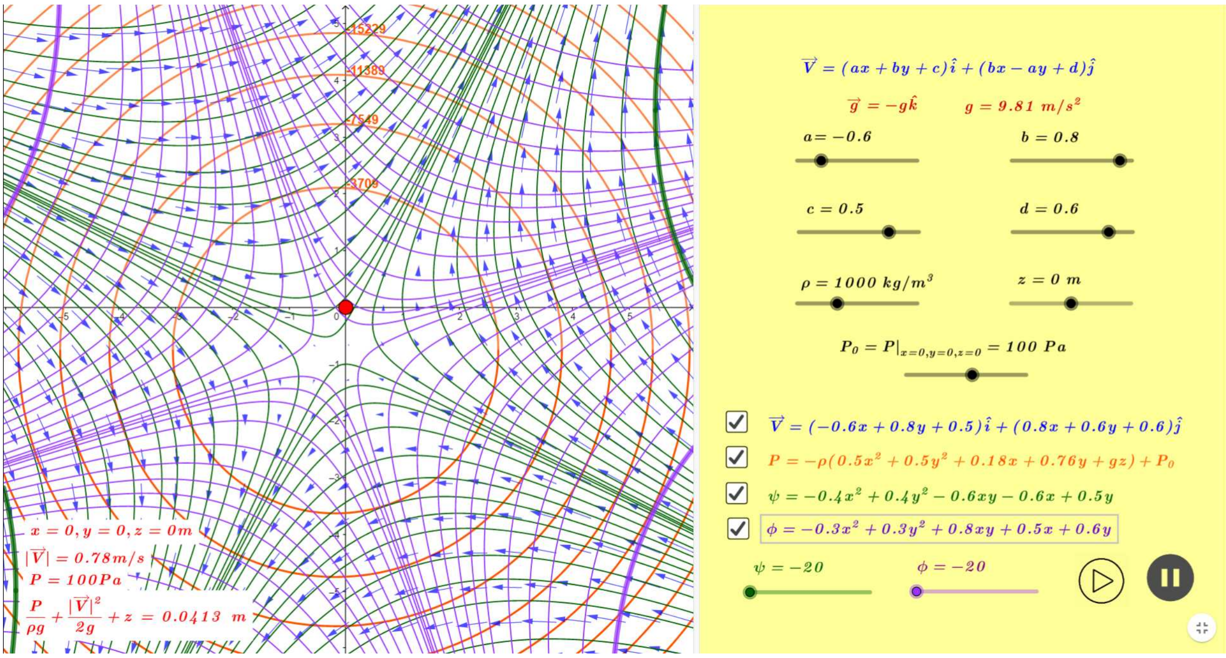


**Fig. 1**: The interface of the developed GeoGebra Applet

## 4 Conclusion

Analysis of ideal fluid flow is of crucial importance as it is the very first type of fluid flow that the engineering students study in hydrodynamics and the theory of ideal flow establishes a basis for more complex flows. In order to enable the students to visualize the fundamental concepts of ideal flow, this study proposes the use of an applet developed with GeoGebra. Employing this application the users will be able to observe the impact of a set of dynamically editable parameters on the main constructs of an ideal flow field, such as streamlines, potential lines or equal pressure lines. The developed application provides a picture of a flow net formed by a set of mutually perpendicular streamlines and potential lines. At every point in the flow field, the streamlines are clearly seen to be tangent to the velocity vector, whereas flow always orthogonally crosses the potential lines. The shape of the equal pressure lines provides a perception of the nature of the pressure field. In addition, the relation between the magnitudes of flow velocity and pressure can be grasped quantitatively by following the representative fluid particle in the flow field.

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# Invariant and Invariant Statistical Equivalence of Order $\beta$ for Double Set Sequences

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**Abstract:** In this paper, for double set sequences, the concepts of asymptotical invariant equivalence and asymptotical invariant statistical equivalence of order $\beta$ $(0 < \beta \leq 1)$ in the Wijsman sense were introduced. Also, some properties of these new equivalence concepts and the relations between them were investigated.

**Keywords:** Asymptotic equivalence, Convergence in the Wijsman sense, Double set sequences, Invariant mean, Order $\beta$, Statistical convergence.

## 1 Introduction

After Pringsheim [1] defined the concept of convergence for double sequences, this concept has attracted the attention of many authors. And as a result, some new convergence concepts for double sequences were presented ([2, 3]). Recently, Çolak and Altın [4] introduced the concept of statistical convergence of order $\alpha$ for double sequences in addition to these new concepts. Also, the concepts of asymptotical and asymptotical statistical equivalence for double sequences were respectively defined by Patterson [5] and Patterson-Savaş [6].

There are many convergence concepts on set sequences. The concept of convergence in the Wijsman sense is based on this study (see, [7, 8]). Using the concepts of statistical convergence, invariant mean etc., new convergence concepts in the Wijsman sense for double set sequences were introduced by some authors ([9]-[11]). In [10], Nuray and Ulusu studied on the concepts of invariant summability and invariant statistical convergence in the Wijsman sense for double set sequences. Also, for double set sequences, the concepts of asymptotical equivalence in the Wijsman sense were introduced by Nuray et al. [12] and then Ulusu et al. [13] presented the concepts of asymptotical invariant and asymptotical invariant statistical equivalence for double set sequences.

In this paper, using the concepts of invariant mean and order $\beta$, we studied on new asymptotical equivalence concepts in the Wijsman sense for double set sequences.

More information on the concepts of convergence and asymptotic equivalence for real or set sequences can be found in [3, 9, 14, 16], [18]-[21].

## 2 Definitions and notations

Before to continue main result, we now give some preliminaries necessary to better understand of our paper (see, [11, 12, 17]).

For a metric space $(Y, \rho)$, $d(y, C)$ denote the distance from $y$ to $C$ where

$$d(y, C) := d_y(C) = \inf_{c \in C} \rho(y, c)$$

for any $y \in Y$ and any non-empty set $C \subseteq Y$.

For a non-empty set $Y$, let a function $h : \mathbb{N} \to 2^Y$ (the power set of $Y$) is defined by $h(i) = C_i \in 2^Y$ for each $i \in \mathbb{N}$. Then, the sequence $\{C_i\} = \{C_1, C_2, \ldots\}$ is called set sequences.

Throughout this study, $(Y, \rho)$ will be considered as a metric space and $C, C_{ij}, D_{ij}$ will be considered as any non-empty closed subsets of $Y$.

The double set sequence $\{C_{ij}\}$ is called convergent to the set $C$ in the Wijsman sense if each $y \in Y$,

$$\lim_{i,j \to \infty} d_y(C_{ij}) = d_y(C).$$

The double set sequence $\{C_{ij}\}$ is called statistically convergent to the set $C$ in the Wijsman sense if every $\varepsilon > 0$ and each $y \in Y$,

$$\lim_{m,n \to \infty} \frac{1}{mn} \left| \left\{ (i,j) : i \leq m, j \leq n, \ \left| d_y(C_{ij}) - d_y(C) \right| \geq \varepsilon \right\} \right| = 0.$$

The term $d_y\Big(\dfrac{C_{ij}}{D_{ij}}\Big)$ is defined as follows:

$$d_y\Big(\frac{C_{ij}}{D_{ij}}\Big)=\begin{cases}\dfrac{d(y,C_{ij})}{d(y,D_{ij})} & , \quad y\notin C_{ij}\cup D_{ij}\\ \lambda & , \quad y\in C_{ij}\cup D_{ij}.\end{cases}$$

Double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are called asymptotically equivalent in the Wijsman sense if each $y\in Y$,

$$\lim_{i,j\to\infty} d_y\Big(\frac{C_{ij}}{D_{ij}}\Big)=1$$

and denoted by $C_{ij}\overset{W}{\sim}D_{ij}$.

Let $\sigma$ be a mapping such that $\sigma:\mathbb{N}^+\to\mathbb{N}^+$ (the set of positive integers). A continuous linear functional $\psi$ on $\ell_\infty$ is called an invariant mean (or a $\sigma$-mean) if it satisfies the following conditions:

1. $\psi(x_s)\geq 0$, when the sequence $(x_s)$ has $x_s\geq 0$ for all $s$,
2. $\psi(e)=1$, where $e=(1,1,1,\ldots)$ and
3. $\psi(x_{\sigma(s)})=\psi(x_s)$ for all $(x_s)\in\ell_\infty$.

The mappings $\sigma$ are assumed to be one to one and such that $\sigma^i(s)\neq s$ for all $i,s\in\mathbb{N}^+$, where $\sigma^i(s)$ denotes the $i$ th iterate of the mapping $\sigma$ at $s$. Thus $\psi$ extends the limit functional on $c$, in the sense that $\psi(x_s)=\lim x_s$ for all $(x_s)\in c$.

Double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are called asymptotically invariant equivalent to multiple $\lambda$ in the Wijsman sense if each $y\in Y$,

$$\lim_{m,n\to\infty}\frac{1}{mn}\sum_{i,j=1,1}^{m,n} d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}}\Big)=\lambda$$

uniformly in $s,t$.

Double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are called asymptotically strongly invariant equivalent to multiple $\lambda$ in the Wijsman sense if each $y\in Y$,

$$\lim_{m,n\to\infty}\frac{1}{mn}\sum_{i,j=1,1}^{m,n}\left|d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}}\Big)-\lambda\right|=0$$

uniformly in $s,t$.

## 3 Main results

In this section, for double set sequences, the concepts of asymptotical invariant equivalence and asymptotical invariant statistical equivalence of order $\beta$ $(0<\beta\leq 1)$ in the Wijsman sense were introduced. Also, some properties of these new equivalence concepts and the relations between them were investigated.

**Definition 1.** *Double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are asymptotically invariant statistical equivalent to multiple $\lambda$ of order $\beta$ in the Wijsman sense if every $\varepsilon>0$ and each $y\in Y$,*

$$\lim_{m,n\to\infty}\frac{1}{(mn)^\beta}\left|\left\{(i,j):i\leq m, j\leq n,\left|d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}}\Big)-\lambda\right|\geq\varepsilon\right\}\right|=0$$

*uniformly in $s,t$ where $0<\beta\leq 1$ and this type of equivalence is denoted by $C_{ij}\overset{W_2^\lambda(S_\sigma^\beta)}{\sim}D_{ij}$, and simply called asymptotically invariant statistical equivalent of order $\beta$ in the Wijsman sense if $\lambda=1$.*

**Example 1.** *Let $Y=\mathbb{R}^2$ and the double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ be defined as following:*

$$C_{ij}:=\begin{cases}\{(a,b)\in\mathbb{R}^2:a^2+b^2+ijb=0\} & , \quad \text{if } ij \text{ is a square integer,}\\ \{(1,0)\} & , \quad \text{otherwise.}\end{cases}$$

*and*

$$D_{ij}:=\begin{cases}\{(a,b)\in\mathbb{R}^2:a^2+b^2-ijb=0\} & , \quad \text{if } ij \text{ is a square integer,}\\ \{(1,0)\} & , \quad \text{otherwise.}\end{cases}$$

*In this case, the double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are asymptotically invariant statistical equivalent of order $\beta\,(0<\beta\leq 1)$ in the Wijsman sense.*

**Remark 1.** *For $\beta=1$, the concept of $W_2^\lambda(S_\sigma^\beta)$-equivalence coincides with the concept of asymptotically invariant statistical equivalence in the Wijsman sense for double set sequences in [13].*

**Theorem 1.** *If $0 < \beta \leq \gamma \leq 1$, then*

$$C_{ij} \overset{W_2^\lambda(S_\sigma^\beta)}{\sim} D_{ij} \Rightarrow C_{ij} \overset{W_2^\lambda(S_\sigma^\gamma)}{\sim} D_{ij}.$$

*Proof:* Let $0 < \beta \leq \gamma \leq 1$ and suppose that $C_{ij} \overset{W_2^\lambda(S_\sigma^\beta)}{\sim} D_{ij}$. For every $\varepsilon > 0$ and each $y \in Y$, we have

$$\frac{1}{(mn)^\gamma}\left|\left\{(i,j) : i \leq m, j \leq n, \left|d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}}\Big) - \lambda\right| \geq \varepsilon\right\}\right| \leq \frac{1}{(mn)^\beta}\left|\left\{(i,j) : i \leq m, j \leq n, \left|d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}}\Big) - \lambda\right| \geq \varepsilon\right\}\right|$$

for all $s, t$. Hence, by our assumption, we get $C_{ij} \overset{W_2^\lambda(S_\sigma^\gamma)}{\sim} D_{ij}$. □

If we take $\gamma = 1$ in Theorem 1, then we obtain the following corollary.

**Corollary 1.** *Let $\beta \in (0, 1]$. If double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are asymptotically invariant statistical equivalent to multiple $\lambda$ of order $\beta$ in the Wijsman sense, then the double set sequences are asymptotically statistical equivalent of multiple $\lambda$ in the Wijsman sense, i.e.,*

$$C_{ij} \overset{W_2^\lambda(S_\sigma^\beta)}{\sim} D_{ij} \Rightarrow C_{ij} \overset{W_2^\lambda(S_\sigma)}{\sim} D_{ij}.$$

**Definition 2.** *Double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are asymptotically invariant equivalent to multiple $\lambda$ of order $\beta$ in the Wijsman sense if each $y \in Y$,*

$$\lim_{m,n\to\infty} \frac{1}{(mn)^\beta} \sum_{i,j=1,1}^{m,n} d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}}\Big) = \lambda$$

*uniformly in $s, t$ where $0 < \beta \leq 1$ and this type of equivalence is denoted by $C_{ij} \overset{W_2^\lambda(V_\sigma^\beta)}{\sim} D_{ij}$, and simply called asymptotically invariant equivalent of order $\beta$ in the Wijsman sense if $\lambda = 1$.*

**Definition 3.** *Double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are asymptotically strong q-invariant equivalent to multiple $\lambda$ of order $\beta$ in the Wijsman sense if each $y \in Y$,*

$$\lim_{m,n\to\infty} \frac{1}{(mn)^\beta} \sum_{i,j=1,1}^{m,n} \left|d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}}\Big) - \lambda\right|^q = 0$$

*uniformly in $s, t$ where $0 < \beta \leq 1$ and $0 < q < \infty$, and this type of equivalence is denoted by $C_{ij} \overset{W_2^\lambda[V_\sigma^\beta]^q}{\sim} D_{ij}$, and simply called asymptotically strong q-invariant equivalent of order $\beta$ in the Wijsman sense if $\lambda = 1$.*

*If $q = 1$, the double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are simply called asymptotically strong invariant equivalent to multiple $\lambda$ of order $\beta$ in the Wijsman sense and this type of equivalence is denoted by $C_{ij} \overset{W_2^\lambda[V_\sigma^\beta]}{\sim} D_{ij}$.*

**Example 2.** *Let $Y = \mathbb{R}^2$ and the double sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ be defined as following:*

$$C_{ij} := \begin{cases} \{(a,b) \in \mathbb{R}^2 : (a-1)^2 + b^2 = \frac{1}{ij}\} & , \text{ if } ij \text{ is a square integer,} \\ \{(0,1)\} & , \text{ otherwise.} \end{cases}$$

*and*

$$D_{ij} := \begin{cases} \{(a,b) \in \mathbb{R}^2 : (a+1)^2 + b^2 = \frac{1}{ij}\} & , \text{ if } ij \text{ is a square integer,} \\ \{(0,1)\} & , \text{ otherwise.} \end{cases}$$

*In this case, the double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are asymptotically strong invariant equivalent of order $\beta$ $(0 < \beta \leq 1)$ in the Wijsman sense.*

**Remark 2.** *For $\beta = 1$, the concept of $W_2^\lambda[V_\sigma^\beta]^q$-equivalence coincides with the concept of asymptotically strong q-invariant equivalence in the Wijsman sense for double set sequences in [13].*

**Theorem 2.** *If $0 < \beta \leq \gamma \leq 1$, then*

$$C_{ij} \overset{W_2^\lambda[V_\sigma^\beta]^q}{\sim} D_{ij} \Rightarrow C_{ij} \overset{W_2^\lambda[V_\sigma^\gamma]^q}{\sim} D_{ij}.$$

*Proof:* Let $0 < \beta \leq \gamma \leq 1$ and suppose that $C_{ij} \overset{W_2^\lambda[V_\sigma^\beta]^q}{\sim} D_{ij}$. For each $y \in Y$, we have

$$\frac{1}{(mn)^\gamma} \sum_{i,j=1,1}^{m,n} \left|d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}}\Big) - \lambda\right|^q \leq \frac{1}{(mn)^\beta} \sum_{i,j=1,1}^{m,n} \left|d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}}\Big) - \lambda\right|^q$$

for all $s, t$. Hence, by our assumption, we get $C_{ij} \overset{W_2^\lambda[V_\sigma^\gamma]^q}{\sim} D_{ij}$. □

If we take $\gamma = 1$ in Theorem 2, then we obtain the following corollary.

**Corollary 2.** *Let $\beta \in (0,1]$ and $0 < q < \infty$. If double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are asymptotically strong q-invariant equivalent to multiple $\lambda$ of order $\beta$ in the Wijsman sense, then the double set sequences are asymptotically strong q-invariant equivalent to multiple $\lambda$ in the Wijsman sense, i.e.,*

$$C_{ij} \overset{W_2^\lambda [V_\sigma^\beta]^q}{\sim} D_{ij} \Rightarrow C_{ij} \overset{W_2^\lambda [V_\sigma]^q}{\sim} D_{ij}.$$

Now, a theorem that gives a relation between $W_2^\lambda [V_\sigma^\beta]^q$-equivalence and $W_2^\lambda [V_\sigma^\beta]^p$-equivalence, where $0 < \beta \leq 1$ and $0 < q < p < \infty$, will be stated.

**Theorem 3.** *Let $0 < \beta \leq 1$ and $0 < q < p < \infty$. Then,*

$$C_{ij} \overset{W_2^\lambda [V_\sigma^\beta]^p}{\sim} D_{ij} \Rightarrow C_{ij} \overset{W_2^\lambda [V_\sigma^\beta]^q}{\sim} D_{ij}.$$

*Proof:* Let $0 < \beta \leq 1$ and $0 < q < p < \infty$. Also, we suppose that $C_{ij} \overset{W_2^\lambda [V_\sigma^\beta]^p}{\sim} D_{ij}$. By Hölder inequality, we have for each $y \in Y$

$$\frac{1}{(mn)^\beta} \sum_{i,j=1,1}^{m,n} \left| d_y \Big( \frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}} \Big) - \lambda \right|^q < \frac{1}{(mn)^\beta} \sum_{i,j=1,1}^{m,n} \left| d_y \Big( \frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}} \Big) - \lambda \right|^p$$

for all $s, t$. Hence, by our assumption, we get $C_{ij} \overset{W_2^\lambda [V_\sigma^\beta]^q}{\sim} D_{ij}$. □

**Theorem 4.** *If double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are asymptotically strong q-invariant equivalent to multiple $\lambda$ of order $\beta$ in the Wijsman sense, then the double set sequences are asymptotically invariant statistical equivalent to multiple $\lambda$ of order $\gamma$ in the Wijsman sense, where $0 < \beta \leq \gamma \leq 1$ and $0 < q < \infty$.*

*Proof:* Let $0 < \beta \leq \gamma \leq 1$ and $0 < q < \infty$. Also, we suppose that the double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are asymptotically strong $q$-invariant equivalent to multiple $\lambda$ of order $\beta$ in the Wijsman sense. For every $\varepsilon > 0$ and each $y \in Y$, we have

$$\begin{aligned}
\sum_{i,j=1,1}^{m,n} \left| d_y \Big( \frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}} \Big) - \lambda \right|^q &= \sum_{\substack{i,j=1,1 \\ \left| d_y \left( \frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}} \right) - \lambda \right| \geq \varepsilon}}^{m,n} \left| d_y \Big( \frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}} \Big) - \lambda \right|^q \\
&\quad + \sum_{\substack{i,j=1,1 \\ \left| d_y \left( \frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}} \right) - \lambda \right| < \varepsilon}}^{m,n} \left| d_y \Big( \frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}} \Big) - \lambda \right|^q \\
&\geq \sum_{\substack{i,j=1,1 \\ \left| d_y \left( \frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}} \right) - \lambda \right| \geq \varepsilon}}^{m,n} \left| d_y \Big( \frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}} \Big) - \lambda \right|^q \\
&\geq \varepsilon^q \left| \left\{ (i,j) : i \leq m, j \leq n, \left| d_y \Big( \frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}} \Big) - \lambda \right| \geq \varepsilon \right\} \right|
\end{aligned}$$

and so

$$\begin{aligned}
\frac{1}{(mn)^\beta} \sum_{i,j=1,1}^{m,n} \left| d_y \Big( \frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}} \Big) - \lambda \right|^q &\geq \frac{\varepsilon^q}{(mn)^\beta} \left| \left\{ (i,j) : i \leq m, j \leq n, \left| d_y \Big( \frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}} \Big) - \lambda \right| \geq \varepsilon \right\} \right| \\
&\geq \frac{\varepsilon^q}{(mn)^\gamma} \left| \left\{ (i,j) : i \leq m, j \leq n, \left| d_y \Big( \frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}} \Big) - \lambda \right| \geq \varepsilon \right\} \right|
\end{aligned}$$

for all $s, t$. Hence, by our assumption, we get that the double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are asymptotically invariant statistical equivalent to multiple $\lambda$ of order $\gamma$ in the Wijsman sense. □

If we take $\gamma = \beta$ in Theorem 4, then we obtain the following corollary.

**Corollary 3.** *Let $\beta \in (0,1]$ and $0 < q < \infty$. If double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are asymptotically strong q-invariant equivalent to multiple $\lambda$ of order $\beta$ in the Wijsman sense, then the double set sequences are asymptotically invariant statistical equivalent to multiple $\lambda$ of order $\beta$ in the Wijsman sense.*

## Conclusion

We gave the definitions of asymptotical invariant and asymptotical invariant statistical equivalence of order $\beta$ in the Wijsman sense for double set sequences where $0 < \beta \leq 1$. Also, we investigated some properties of these new equivalence concepts and the relations between them.

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# Lacunary Invariant Summability and Lacunary Invariant Statistical Convergence of Order $\eta$ for Double Set Sequences

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**Abstract:** In this study, for double set sequences, we introduce the notions of lacunary invariant summability and lacunary invariant statistical convergence of order $\eta$ $(0 < \eta \leq 1)$ in the Wijsman sense. Also, we investigate some properties of these new notions and the relations between them.

**Keywords:** Convergence in the Wijsman sense, Double lacunary sequence, Double set sequences, Invariant statistical convergence, Order $\eta$.

## 1 Introduction

Long after the notion of convergence for double sequences was introduced by Pringsheim [1], using the notions of statistical convergence, double lacunary sequence, invariant mean etc., this notion was extended to new convergence notions for double sequences by some authors [2]-[4]. Recently, for double sequences, on two new convergence notions called double almost statistical and double almost lacunary statistical convergence of order $\alpha$ were studied by Savaş [5, 6].

Over the years, on the various convergence notions for set sequences have been studied by many authors. One of them, discussed in this study, is the notion of convergence in the Wijsman sense [7]-[9]. Using the notions of statistical convergence, double lacunary sequence, invariant mean etc., this notion was extended to new convergence notions for double set sequences by some authors [10]-[13]. In [12], Nuray and Ulusu studied on the notions of lacunary invariant summability and lacunary invariant statistical convergence in the Wijsman sense for double set sequences. Recently, on the notions of lacunary statistical convergence of order $\alpha$ and strong $p$-lacunary summability of order $\alpha$ for double set sequences were studied by Gülle and Ulusu [14].

In this paper, using the notions of invariant mean and order $\eta$, we studied on new convergence notions in the Wijsman sense for double set sequences.

More information on the notions of convergence for real or set sequences can be found in [15]-[21].

## 2 Definitions and notations

In this section, let us remind the basic notions that need for a better understanding of our study (see, [3, 8, 18], [10]-[14]).

For a metric space $(Y, d)$, $\mu(y, C)$ denote the distance from $y$ to $C$ where

$$\mu(y, C) := \mu_y(C) = \inf_{c \in C} d(y, c)$$

for any $y \in Y$ and any non-empty set $C \subseteq Y$.

For a non-empty set $Y$, let a function $g : \mathbb{N} \to P_Y$ (the power set of $Y$) is defined by $g(m) = C_m \in P_Y$ for each $m \in \mathbb{N}$. Then, the sequence $\{C_m\} = \{C_1, C_2, \ldots\}$, which is the codomain elements of $g$, is called set sequences.

Throughout this study, $(Y, d)$ will be considered as a metric space and $C, C_{mn}$ will be considered as any non empty closed subsets of $Y$.

The double set sequence $\{C_{mn}\}$ is called convergent to the set $C$ in the Wijsman sense if each $y \in Y$,

$$\lim_{m,n \to \infty} \mu_y(C_{mn}) = \mu_y(C).$$

The double set sequence $\{C_{mn}\}$ is called statistical convergent to the set $C$ in the Wijsman sense if every $\varepsilon > 0$ and each $y \in Y$,

$$\lim_{i,j \to \infty} \frac{1}{ij} \left| \left\{ (m, n) : \ m \leq i, n \leq j, \ \left| \mu_y(C_{mn}) - \mu_y(C) \right| \geq \varepsilon \right\} \right| = 0.$$

A double sequence $\theta_2 = \{(j_u, k_v)\}$ is called a double lacunary sequence if there exist increasing sequences $(j_u)$ and $(k_v)$ of the integers such that

$$j_0 = 0,\ h_u = j_u - j_{u-1} \to \infty \text{ and } k_0 = 0,\ \bar{h}_v = k_v - k_{v-1} \to \infty \text{ as } u, v \to \infty.$$

In general, the following notations are used for any double lacunary sequence:

$$h_{uv} = h_u \bar{h}_v,\ \ I_{uv} = \big\{(m,n) : j_{u-1} < m \leq j_u \text{ and } k_{v-1} < n \leq k_v\big\}.$$

Throughout this study, $\theta_2 = \{(j_u, k_v)\}$ will be considered as a double lacunary sequence.

The double set sequence $\{C_{mn}\}$ is called lacunary statistically convergent to the set $C$ in the Wijsman sense if every $\varepsilon > 0$ and each $y \in Y$,

$$\lim_{u,v\to\infty} \frac{1}{h_{uv}} \left| \Big\{ (m,n) \in I_{uv} : \big|\mu_y(C_{mn}) - \mu_y(C)\big| \geq \varepsilon \Big\} \right| = 0.$$

Let $\sigma$ be a mapping such that $\sigma : \mathbb{N}^+ \to \mathbb{N}^+$ (the set of positive integers). A continuous linear functional $\psi$ on $\ell_\infty$ is called an invariant mean (or a $\sigma$-mean) if it satisfies the following conditions:

1. $\psi(x_s) \geq 0$, when the sequence $(x_s)$ has $x_s \geq 0$ for all $s$,
2. $\psi(e) = 1$, where $e = (1, 1, 1, \ldots)$ and
3. $\psi(x_{\sigma(s)}) = \psi(x_s)$ for all $(x_s) \in \ell_\infty$.

The mappings $\sigma$ are assumed to be one to one and such that $\sigma^m(s) \neq s$ for all $m, s \in \mathbb{N}^+$, where $\sigma^m(s)$ denotes the $m$ th iterate of the mapping $\sigma$ at $s$. Thus $\psi$ extends the limit functional on $c$, in the sense that $\psi(x_s) = \lim x_s$ for all $(x_s) \in c$.

The double set sequence $\{C_{mn}\}$ is called lacunary invariant summable to the set $C$ in the Wijsman sense if each $y \in Y$,

$$\lim_{u,v\to\infty} \frac{1}{h_{uv}} \sum_{(m,n)\in I_{uv}} \mu_y(C_{\sigma^m(s)\sigma^n(t)}) = \mu_y(C).$$

The double set sequence $\{C_{mn}\}$ is called strong lacunary summable to the set $C$ in the Wijsman sense if each $y \in Y$,

$$\lim_{u,v\to\infty} \frac{1}{h_{uv}} \sum_{(m,n)\in I_{uv}} \big|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\big| = 0.$$

## 3 Main results

In this section, for double set sequences, we introduce the notions of lacunary invariant summability and lacunary invariant statistical convergence of order $\eta$ $(0 < \eta \leq 1)$ in the Wijsman sense. Also, we investigate some properties of these new notions and the relations between them.

**Definition 1.** *The double set sequence $\{C_{mn}\}$ is lacunary invariant summable of order $\eta$ to the set $C$ in the Wijsman sense if each $y \in Y$,*

$$\lim_{u,v\to\infty} \frac{1}{h_{uv}^\eta} \sum_{(m,n)\in I_{uv}} \mu_y(C_{\sigma^m(s)\sigma^n(t)}) = \mu_y(C)$$

*uniformly in $s, t$ where $0 < \eta \leq 1$ and we denote this in $C_{mn} \overset{W_2^\theta(N_\sigma^\eta)}{\longrightarrow} C$ format.*

**Definition 2.** *The double set sequence $\{C_{mn}\}$ is strong lacunary q-invariant summable of order $\eta$ to the set $C$ in the Wijsman sense if each $y \in Y$,*

$$\lim_{u,v\to\infty} \frac{1}{h_{uv}^\eta} \sum_{(m,n)\in I_{uv}} \big|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\big|^q = 0$$

*uniformly in $s, t$ where $0 < \eta \leq 1$ and we denote this in $C_{mn} \overset{W_2^\theta[N_\sigma^\eta]^q}{\longrightarrow} C$ format.*

*If $q = 1$, then the double set sequence $\{C_{mn}\}$ is simply called strong lacunary invariant summable of order $\eta$ to the set $C$ and we denote this in $C_{mn} \overset{W_2^\theta[N_\sigma^\eta]}{\longrightarrow} C$ format.*

**Example 1.** *Let $Y = \mathbb{R}^2$ and a double set sequence $\{C_{mn}\}$ be defined as following:*

$$C_{mn} := \begin{cases} \Big\{(a,b) \in \mathbb{R}^2 : (a+m)^2 + (y-n)^2 = 1\Big\} & ; \text{ if } (m,n) \in I_{uv},\ m \text{ and } n \text{ are square integers}, \\ \{(1,-1)\} & ; \text{ otherwise}. \end{cases}$$

*In this case, the double set sequence $\{C_{mn}\}$ is strong lacunary invariant summable of order $\eta$ to the set $C = \{(1,-1)\}$ in the Wijsman sense.*

**Remark 1.** *For $\eta = 1$, respectively, the notions of $W_2^\theta(N_\sigma^\eta)$-summability and $W_2^\theta[N_\sigma^\eta]$-summability coincide with the notions of lacunary invariant summability and strong lacunary invariant summability in the Wijsman sense for double set sequences in [12].*

**Theorem 1.** *If $0 < \eta \leq \vartheta \leq 1$, then*

$$C_{mn} \stackrel{W_2^\theta[N_\sigma^\eta]^q}{\longrightarrow} C \Rightarrow C_{mn} \stackrel{W_2^\theta[N_\sigma^\vartheta]^q}{\longrightarrow} C$$

*for every double lacunary sequence $\theta_2 = \{(j_u, k_v)\}$.*

*Proof:* Let $0 < \eta \leq \vartheta \leq 1$ and suppose that $C_{mn} \stackrel{W_2^\theta[N_\sigma^\eta]^q}{\longrightarrow} C$. For each $y \in Y$, we have

$$\frac{1}{h_{uv}^\vartheta} \sum_{(m,n)\in I_{uv}} \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right|^q \leq \frac{1}{h_{uv}^\eta} \sum_{(m,n)\in I_{uv}} \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right|^q$$

for all $s, t$. Hence, by our assumption, we get $C_{mn} \stackrel{W_2^\theta[N_\sigma^\vartheta]^q}{\longrightarrow} C$. □

If $\vartheta = 1$ is taken in Theorem 1, then following corollary is obtained.

**Corollary 1.** *Let $\eta \in (0, 1]$ and $0 < q < \infty$. If a double set sequence $\{C_{mn}\}$ is strong lacunary q-invariant summable of order $\eta$ to a set $C$ in the Wijsman sense, then the double set sequence is strong lacunary q-invariant summable to the set $C$ in the Wijsman sense, i.e.,*

$$C_{mn} \stackrel{W_2^\theta[N_\sigma^\eta]^q}{\longrightarrow} C \Rightarrow C_{mn} \stackrel{W_2^\theta[N_\sigma]^q}{\longrightarrow} C.$$

Now, we can state a theorem giving the relationship between $W_2^\theta[N_\sigma^\eta]^q$-summability and $W_2^\theta[N_\sigma^\eta]^p$-summability, where $0 < \eta \leq 1$ and $0 < q < p < \infty$.

**Theorem 2.** *Let $0 < \eta \leq 1$ and $0 < q < p < \infty$. Then,*

$$C_{mn} \stackrel{W_2^\theta[N_\sigma^\eta]^p}{\longrightarrow} C \Rightarrow C_{mn} \stackrel{W_2^\theta[N_\sigma^\eta]^q}{\longrightarrow} C$$

*for every double lacunary sequence $\theta_2 = \{(j_u, k_v)\}$.*

*Proof:* Let $0 < \eta \leq 1$ and $0 < q < p < \infty$. Also, we suppose that $C_{mn} \stackrel{W_2^\theta[N_\sigma^\eta]^p}{\longrightarrow} C$. For every $y \in Y$, by Hölder inequality, we have

$$\frac{1}{h_{uv}^\eta} \sum_{(m,n)\in I_{uv}} \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right|^q < \frac{1}{h_{uv}^\eta} \sum_{(m,n)\in I_{uv}} \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right|^p$$

for all $s, t$. Hence, by our assumption, we get $C_{mn} \stackrel{W_2^\theta[N_\sigma^\eta]^q}{\longrightarrow} C$. □

**Definition 3.** *The double set sequence $\{C_{mn}\}$ is lacunary invariant statistically convergent of order $\eta$ to the set $C$ in the Wijsman sense if every $\varepsilon > 0$ and each $y \in Y$,*

$$\lim_{u,v\to\infty} \frac{1}{h_{uv}^\eta} \left| \left\{ (m,n) \in I_{uv} : \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \varepsilon \right\} \right| = 0$$

*uniformly in $s, t$ where $0 < \eta \leq 1$ and we denote this in $C_{mn} \stackrel{W_2^\theta(S_\sigma^\eta)}{\longrightarrow} C$ format.*

**Example 2.** *Let $Y = \mathbb{R}^2$ and a double set sequence $\{C_{mn}\}$ be defined as following:*

$$C_{mn} := \begin{cases} \left\{(a,b) \in \mathbb{R}^2 : (a-1)^2 + b^2 = \frac{1}{mn}\right\} & ; \text{ if } (m,n) \in I_{uv},\ m \text{ and } n \text{ are square integers} \\ \{(0,1)\} & ; \text{ otherwise.} \end{cases}$$

*In this case, the double set sequence $\{C_{mn}\}$ is lacunary invariant statistically convergent of order $\eta$ to the set $C = \{(0,1)\}$ in the Wijsman sense.*

**Remark 2.** *For $\eta = 1$, the notion of $W_2^\theta(S_\sigma^\eta)$-convergence coincides with the notion of lacunary invariant statistical convergence in the Wijsman sense for double set sequences in [12].*

 

**Theorem 3.** *If $0 < \eta \leq \vartheta \leq 1$, then*

$$C_{mn} \overset{W_2^{\theta}(S_{\sigma}^{\eta})}{\longrightarrow} C \Rightarrow C_{mn} \overset{W_2^{\theta}(S_{\sigma}^{\vartheta})}{\longrightarrow} C$$

*for every double lacunary sequence $\theta_2 = \{(j_u, k_v)\}$.*

*Proof:* Let $0 < \eta \leq \vartheta \leq 1$ and suppose that $C_{mn} \overset{W_2^{\theta}(S_{\sigma}^{\eta})}{\longrightarrow} C$. For every $\varepsilon > 0$ and each $y \in Y$, we have

$$\frac{1}{h_{uv}^{\vartheta}}\left|\left\{(m,n) \in I_{uv} : \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \varepsilon\right\}\right| \leq \frac{1}{h_{uv}^{\eta}}\left|\left\{(m,n) \in I_{uv} : \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \varepsilon\right\}\right|$$

for all $s, t$. Hence, by our assumption, we get $C_{mn} \overset{W_2^{\theta}(S_{\sigma}^{\vartheta})}{\longrightarrow} C$. □

If $\vartheta = 1$ is taken in Theorem 3, then the following corollary is obtained.

**Corollary 2.** *Let $\eta \in (0, 1]$. If a double set sequence $\{C_{mn}\}$ is lacunary invariant statistically convergent of order $\eta$ to a set $C$ in the Wijsman sense, then the double set sequence is lacunary invariant statistically convergent to the set $C$ in the Wijsman sense, i.e.,*

$$C_{mn} \overset{W_2^{\theta}(S_{\sigma}^{\eta})}{\longrightarrow} C \Rightarrow C_{mn} \overset{W_2^{\theta}(S_{\sigma})}{\longrightarrow} C.$$

**Theorem 4.** *If a double set sequence $\{C_{mn}\}$ is strong lacunary q-invariant summable of order $\eta$ to a set $C$ in the Wijsman sense, then the double set sequence is lacunary invariant statistically convergent of order $\vartheta$ to the set $C$ in the Wijsman sense, where $0 < \eta \leq \vartheta \leq 1$ and $0 < q < \infty$.*

*Proof:* Let $0 < \eta \leq \vartheta \leq 1$ and $0 < q < \infty$. Also, we suppose that a double set sequence $\{C_{mn}\}$ is strong lacunary $q$-invariant summable of order $\eta$ to a set $C$. For every $\varepsilon > 0$ and each $y \in Y$, we have

$$\begin{aligned}
\sum_{(m,n)\in I_{uv}} \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right|^q &= \sum_{\substack{(m,n)\in I_{uv} \\ \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \varepsilon}} \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right|^q \\
&\quad + \sum_{\substack{(m,n)\in I_{uv} \\ \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| < \varepsilon}} \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right|^q \\
&\geq \sum_{\substack{(m,n)\in I_{uv} \\ \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \varepsilon}} \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right|^q \\
&\geq \varepsilon^q \left|\left\{(m,n) \in I_{uv} : \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \varepsilon\right\}\right|
\end{aligned}$$

and so

$$\begin{aligned}
\frac{1}{h_{uv}^{\eta}} \sum_{(m,n)\in I_{uv}} \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right|^q &\geq \frac{\varepsilon^q}{h_{uv}^{\eta}}\left|\left\{(m,n) \in I_{uv} : \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \varepsilon\right\}\right| \\
&\geq \frac{\varepsilon^q}{h_{uv}^{\vartheta}}\left|\left\{(m,n) \in I_{uv} : \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \varepsilon\right\}\right|.
\end{aligned}$$

for all $s, t$. Hence, by our assumption, we get that the double set sequence $\{C_{mn}\}$ is lacunary invariant statistically convergent of order $\vartheta$ to the set $C$ in the Wijsman sense. □

If $\vartheta = \eta$ is taken in Theorem 4, then the following corollary is obtained.

**Corollary 3.** *Let $\eta \in (0, 1]$ and $0 < q < \infty$. If a double set sequence $\{C_{mn}\}$ is strong lacunary q-invariant summable of order $\eta$ to a set $C$ in the Wijsman sense, then the double set sequence is lacunary invariant statistically convergent of order $\eta$ to the set $C$ in the Wijsman sense.*

## Conclusion

We gave the definitions of lacunary invariant summability and lacunary invariant statistical convergence of order $\eta$ $(0 < \eta \leq 1)$ in the Wijsman sense for double set sequences. Our results included the relations between these new notions.

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# Construction of a Sequence of Eigenelements of a Two-Parameter Problem with Compact Self-Adjoint Operators

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**Abstract:** In the classical theory of functional analysis, a variational method is used to construct a sequence of eigenvalues and corresponding eigenelements of a compact self-adjoint linear operator, given in some Hilbert space.
The number $|\lambda_1 \lambda_2|$ is called the modulus of a pair of numbers $(\lambda_1, \lambda_2)$ in the hyperbolic sense. There is a similar variational method for finding the eigenvalue with the minimum modulus in the hyperbolic sense and the corresponding eigenelements for the two-parameter problem

$$\begin{cases} \lambda_1 K_{i,1}\varphi_i + \lambda_2 K_{i,2}\varphi_i = \varphi_i, \quad \varphi_i \in H_i \\ i = 1;\ 2 \end{cases}$$

with compact self-adjoint operators $K_{i,1}$ , $K_{i,2}$ , $i = 1; 2$ in the Hilbert space $H_i$ , $i = 1; 2$. Moreover, the minimizing element is found as an element of the weight space $H_{\Delta_0} = H_1 \otimes H_2$, which gives a minimum value to the functional $F(\varphi) = \frac{(\Delta_1\varphi,\varphi)(\Delta_2\varphi,\varphi)}{(\Delta_0\varphi,\varphi)^2}$ and this element, generally speaking, is not a decomposable tensor and therefore, cannot be called an eigenelement of the given problem. In this article, under the condition of right definiteness, we study a similar problem of constructing a sequence of eigenelements of a given two-parameter problem. Moreover, a) all elements of this sequence are eigenelements of this problem, b) all elements of this sequence are decomposable tensors and c) the sequence of eigenelements is a complete orthonormal basis for the space $H_{\Delta_0} = H_1 \otimes H_2$.

**Keywords:** Multiparameter eigenvalue problems, Spectrum, Variational principles, Definiteness conditions.

## 1 Introduction

It is known that [1], a variational method can be used to construct a sequence of orthonormal systems of eigenelements of a compact, self-adjoint operator given in some Hilbert space.

The modulus of a pair of numbers $(\lambda_1 , \lambda_2)$ in the hyperbolic sense, we call the number $|\lambda_1 \lambda_2|$. In [2], [6], using a family of operators separating spectral parameters (see [4], [5]), a variational method was studied for finding of the eigenvalue with minimal modulus, in the hyperbolic sense, and the corresponding eigenelement for the two-parameter problem

$$\begin{cases} \lambda_1 K_{i,1}\varphi_i + \lambda_2 K_{i,2}\varphi_i = \varphi_i, \quad \varphi_i \in H_i \\ i = 1;\ 2 \end{cases} \tag{1}$$

with compact self-adjoint operators $K_{i,1}$ , $K_{i,2}$ , $i = 1; 2$ in the space $H_i$ , $i = 1; 2$ , where $\lambda_1, \lambda_2$ are the spectral parameters. Moreover, the minimizing element is found as an element of the weight space $H_{\Delta_0} = H_1 \otimes H_2$ that gives the minimum value to the functional $F(\varphi) = \frac{(\Delta_1\varphi,\varphi)(\Delta_2\varphi,\varphi)}{(\Delta_0\varphi,\varphi)^2}$ and this element, generally speaking, is not a decomposable tensor. Therefore, that element cannot be called an eigenelement of the given problem (1). Here

$$\Delta_0 = K_{1,1} \otimes K_{2,2} - K_{1,2} \otimes K_{2,1},\ \Delta_1 = I_1 \otimes K_{2,2} - K_{1,2} \otimes I_2,\ \Delta_2 = K_{1,1} \otimes I_2 - I_1 \otimes K_{2,1}$$

In [3], the numerical range of problem (1) was defined as a subset of points of a two-dimensional plane in the form $M = \left\{ \left( \frac{(\Delta_1\varphi,\varphi)}{(\Delta_0\varphi,\varphi)}; \frac{(\Delta_2\varphi,\varphi)}{(\Delta_0\varphi,\varphi)} \right) : \forall\varphi \in H = H_1 \otimes H_2 \right\}$.

The structure of the numerical range $M$ suggests the form of the functional $F(\varphi)$ . The structure of the numerical range of problem (1) was studied in [3]. Note that if instead of the functional $F(\varphi)$ we take the functional $F_1(\varphi) = \frac{(\Delta_1\varphi,\varphi)}{(\Delta_0\varphi,\varphi)} + \frac{(\Delta_2\varphi,\varphi)}{(\Delta_0\varphi,\varphi)}$ or functional $F_2(\varphi) =$

 

$\frac{(\Delta_1\varphi,\varphi)^2}{(\Delta_0\varphi,\varphi)^2}+\frac{(\Delta_2\varphi,\varphi)^2}{(\Delta_0\varphi,\varphi)^2}$, then the minimizing element, generally speaking, does not correspond to any eigenvalue of problem (1) or of problem

$$\begin{cases} \lambda_i\,\Delta_0\varphi = \Delta_i\varphi\ , \quad \varphi\in\ H=\ H_1\otimes H_2 \\ i=1;\ 2 \end{cases} \tag{2}$$

which is obtained from problem (1) by separating the spectral parameters. When studying the spectrum of problem (1), we also have to consider a similar problem

$$\begin{cases} \lambda_1\,K_{i,1}^t\varphi+\lambda_2\,K_{i,2}^t\varphi=\varphi, \quad \varphi\in\ H=\ H_1\otimes H_2 \\ i=1;\ 2 \end{cases} \tag{3}$$

in the tensor product $H=H_1\otimes H_2$. Here $K_{1,i}^t=K_{1,i}\otimes I_2$ and $K_{2,i}^t=I_1\otimes K_{2,i}$ are operators in the space $H=H_1\otimes H_2$ and $I_i$ is the unit operator in the space $H_i$.

## 2 Comparison of the spectra of problems (1), (2), and (3).

Let $\sigma(K)$ , $\sigma(K^t)$ and $\sigma(\Delta)$ be the spectral set of problems (1), (3) and (2), respectively. It is known that $\sigma(K)$ consists only of eigenvalues. This is obtained from the compactness and self-adjointness of the operators $K_{i,j}, i,j=1;2$. The sets $\sigma(K)$ and $\sigma(K^t)$ are equal to each other. Since

$$\begin{aligned}\sigma(K^t)&=\sigma(K_{11}^t,K_{12}^t)\cap\sigma(K_{21}^t,K_{22}^t)=\{\sigma(K_{11},K_{12})\sigma(I_2)\}\cap\{\sigma(I_1)\sigma(K_{21},K_{22})\}=\\&=\sigma(K_{11},K_{12})\cap\sigma(K_{21},K_{22})=\sigma(K)\end{aligned}$$

where the sets $\sigma(K_{11}^t,K_{12}^t)$ and $\sigma(K_{21}^t,K_{22}^t)$ are the spectral sets of the first and second equations of problem (3), respectively. The sets $\sigma(K_{11},K_{12})$ and $\sigma(K_{21},K_{22})$ are the spectral sets of the first and second equations of problem (1), respectively. Therefore, the spectral set $\sigma(K^t)$ also consists of only eigenvalues. Let us prove the following theorem.

**Theorem 1.** *Suppose that in problem* (1) *the operators* $K_{i,1}$ , $K_{i,2}$ , $i=1;\ 2$ *are compact self-adjoint operators in the space* $H_i$ , $i=1;\ 2$ *and the operators* $\Delta_1$ *and* $\Delta_2$ *are positive definite, i.e.,* $(\Delta_i\varphi,\ \varphi)>0,\ \ \forall\varphi\in H$. *Then the equality* $\sigma(K)=\sigma(K^t)=\sigma(\Delta)$ *holds, where* $\sigma(K)$, $\sigma(K^t)$ *and* $\sigma(\Delta)$ *are the spectral set of problems (1), (3) and (2) respectively.*

*Proof:* Under the condition of the theorem, the spectral set of problem (1) and together with it, the spectral set $\sigma(K^t)$ of problem (3), consist of only eigenvalues and the inclusion $\sigma(K)=\sigma(K^t)\subseteq\sigma(\Delta)$ is true. Let us now prove the reverse inclusion $\sigma(\Delta)\subseteq\sigma(K^t)$. Let $(\lambda_1,\ \lambda_2)\in\sigma(\Delta)$, i.e. there is such a sequence $\{\varphi_n\}_{n=1,2,3,\ldots}\subset H$ that $\|\varphi_n\|=1,\ \ n=1,2,3,\ \ldots$, and

$$\begin{cases} \|\lambda_r\,\Delta_0\varphi_n-\Delta_r\varphi_n\|\to\ 0 \\ r=1,\,2 \end{cases} \qquad at \quad n\to\infty.$$

Let us introduce the notation $\lambda_r\,\Delta_0\varphi_n-\Delta_r\varphi_n=\upsilon_{r,\,n}$. Then sequence $\{\upsilon_{r,\,n}\}$ converges to 0 strongly, i.e.

$$\begin{cases} \lambda_r\,\Delta_0\varphi_n-\Delta_r\varphi_n=\upsilon_{r,\,n}\to 0 \\ r=1,\,2 \end{cases} \quad \text{at} \quad n\to\infty$$

strongly.

Multiplying both sides of the first equation of the last system by $(-\lambda_2)$ and the second equation by $\lambda_1$ and summing, we obtain $\lambda_1\,\Delta_2\varphi_n-\lambda_2\,\Delta_1\varphi_n=\lambda_1\,\upsilon_{2\,n}-\lambda_2\,\upsilon_{1\,n}\ \to\ 0$ at $n\ \to\infty$. Let us introduce the notation $\lambda_1\,\upsilon_{2\,n}-\lambda_2\,\upsilon_{1\,n}=x_n$, then considering $\upsilon_{r\,n}\to 0$ strongly, the sequence $x_{\,n}\to 0$ strongly. Therefore, $\lambda_1\,K_{11}^t\varphi_n-\lambda_1\,K_{21}^t\varphi_n-\lambda_2\,K_{22}^t\varphi_n+\lambda_2\,K_{12}^t\varphi_n=x_n\ \to\ 0$ strongly, or $(\lambda_1\,K_{11}^t+\lambda_2\,K_{12}^t)\varphi_n-(\lambda_1\,K_{21}^t+\lambda_2\,K_{22}^t)\,\varphi_n=x_n\ \to\ 0$ strongly. We also introduce the notation $K_r=\lambda_1\,K_{r1}+\lambda_2\,K_{r2}$ and $f_n=K_1^t\varphi_n-\frac{x_n}{2}$. Then $f_n=K_2^t\varphi_n+\frac{x_n}{2}$. Hence $f_n=\frac{1}{2}(K_1^t\varphi_n+K_2^t\varphi_n)$ and $x_n=K_1^t\varphi_n-K_2^t\varphi_n$. Let the sequence $\{e_{r\,n}\}_{n=1}^\infty$ , $r=1;\,2$ be an orthonormal system of eigenelements of the operator $K_r$ corresponding to the eigenvalues $\{\nu_{r\,n}\}_{n=1}^\infty$ , $r=1;2$, respectively. Due to the selfadjointness and compactness of the operator $K_r$ , the set $\{e_{r\,n}\}_{n=1}^\infty$ , $r=1;2$ will be a complete and orthonormal basis in the space $H_r$. Then the system $\{e_{1\,n}\otimes e_{2\,m}\}_{n,m=1}^\infty$ will be a complete orthonormal basis in the space $H=\ H_1\otimes H_2$ and, therefore, for the element $\varphi_n\in H=\ H_1\otimes H_2$ occurs a decomposition $\varphi_n=\sum\limits_{i,\,j=1}^{\infty}a_{n\,i\,j}e_{1\,i}\otimes e_{2\,j}$. Hence,

$$f_n=\sum_{i,\,j=1}^{\infty}a_{n\,i\,j}\frac{\nu_{1\,i}+\nu_{2\,j}}{2}e_{1\,i}\otimes e_{2\,j} \tag{4}$$

$$x_n=\sum_{i,\,j=1}^{\infty}a_{n\,i\,j}(\nu_{1\,i}-\nu_{2\,j})\,e_{1\,i}\otimes e_{2\,j} \tag{5}$$

The norm of the element $\varphi_n$ is equal to one, $\|\varphi_n\|=1$. Therefore $\sum\limits_{i,\,j=1}^{\infty}a_{n\,i\,j}^2=1$. Due to the boundedness of the operators $K_r$ , $r=1;\,2$, the set $\{f_n\}$ is bounded. Therefore, a weakly converging subsequence can be distinguished from it. For the sake of simplicity, we will assume that $f_n\to f_0$ (weakly). Let $\varphi_n\to\varphi_0$ also converge weakly. For each fixed value of the index $n$, series (4) converges uniformly with respect to the parameter $n$. We will prove that this series converges uniformly. $x_n\to 0$ (strongly). Since, by a compact operator, any weakly converging sequence is mapped to a strongly converging sequence. Therefore, series (5) converges uniformly, i.e. for any $\varepsilon > 0$ there are numbers $N_1$ and $N_2$ such that

$$\left\| x_n - \sum_{i,=1}^{N_1} \sum_{j=1}^{N_2} a_{n\,i\,j} (\nu_{1\,i} - \nu_{2\,j})\, e_{1\,i} \otimes e_{2\,j} \right\|^2 \leq \frac{\varepsilon}{4}$$

for any $n \in N$. I.e.

$$\sum_{i,=1}^{N_1} \sum_{j=N_2+1}^{\infty} a_{n\,i\,j}^2 (\nu_{1\,i} - \nu_{2\,j})^2 + \sum_{i,=N_1+1}^{\infty} \sum_{j=1}^{N_2} a_{n\,i\,j}^2 (\nu_{1\,i} - \nu_{2\,j})^2 + \sum_{i,=N_1+1}^{\infty} \sum_{j=N_2+1}^{\infty} a_{n\,i\,j}^2 (\nu_{1\,i} - \nu_{2\,j})^2 \leq \frac{\varepsilon}{4} \tag{6}$$

For elements $f_n$, the corresponding norm will be in the form

$$\begin{aligned} \left\| f_n - \sum_{i,=1}^{N_1} \sum_{j=1}^{N_2} a_{n\,i\,j} \tfrac{\nu_{1\,i} + \nu_{2\,j}}{2}\, e_{1\,i} \otimes e_{2\,j} \right\|^2 = \sum_{i,=1}^{N_1} \sum_{j=N_2+1}^{\infty} a_{n\,i\,j}^2 \tfrac{(\nu_{1\,i} + \nu_{2\,j})^2}{4} + \\ + \sum_{i,=N_1+1}^{\infty} \sum_{j=1}^{N_2} a_{n\,i\,j}^2 \tfrac{(\nu_{1\,i} + \nu_{2\,j})^2}{4} + \sum_{i,=N_1+1}^{\infty} \sum_{j=N_2+1}^{\infty} a_{n\,i\,j}^2 \tfrac{(\nu_{1\,i} + \nu_{2\,j})^2}{4} \end{aligned} \tag{7}$$

The sequence $\{e_{r\,n}\}_{n=1}^{\infty}$, $r = 1;\ 2$ is an orthonormal system of eigenelements of a compact self-adjoint operator $K_r$, corresponding to the eigenvalues of $\{\nu_{r\,n}\}_{n=1}^{\infty}$, $r = 1;\ 2$. Therefore, for any $\varepsilon > 0$, there are such numbers $N_1^0$ and $N_2^0$ that, for $i > N_1^0,\ \ j > N_2^0$, the inequality $\frac{(\nu_{1\,i} + \nu_{2\,j})^2}{4} \leq \frac{\varepsilon}{2}$ holds, and for the third sum in equality (7), the inequality $\sum\limits_{i,=N_1+1}^{\infty} \sum\limits_{j=N_2+1}^{\infty} a_{n\,i\,j}^2 \frac{(\nu_{1\,i} + \nu_{2\,j})^2}{4} \leq \frac{\varepsilon}{2}$ is satisfied.
Let us now estimate the first two sums in equality (7). Using inequality (6), we write

$$\sum_{i,=1}^{N_1} \sum_{j=N_2+1}^{\infty} a_{n\,i\,j}^2 \frac{(\nu_{1\,i} + \nu_{2\,j})^2}{4} = \frac{1}{4} \sum_{i,=1}^{N_1} \sum_{j=N_2+1}^{\infty} a_{n\,i\,j}^2 (\nu_{1\,i} - \nu_{2\,j})^2 + \sum_{i,=1}^{N_1} \sum_{j=N_2+1}^{\infty} a_{n\,i\,j}^2 \nu_{1\,i} \nu_{2\,j} \leq$$

$$\leq \frac{\varepsilon}{16} + \max_{1 \leq i \leq N_1} |\nu_{1\,i}| \max_{j \geq N_2+1} |\nu_{2\,j}| \sum_{i,=1}^{N_1} \sum_{j=N_2+1}^{\infty} a_{n\,i\,j}^2 \leq \frac{\varepsilon}{16} + \max_{1 \leq i \leq N_1} |\nu_{1\,i}| \max_{j \geq N_2+1} |\nu_{2\,j}| \leq \frac{\varepsilon}{16} + \|K_1\| \max_{j \geq N_2+1} |\nu_{2\,j}|,$$

where $\|K_1\| = \max\limits_{1 \leq i < \infty} |\nu_{1\,i}|$. We choose the number $N_2^0$ so that, the inequality $\max\limits_{j \geq N_2+1} |\nu_{2\,j}| \leq \frac{3\varepsilon}{16\|K_1\|}$ is satisfied. Then, for a natural number $\bar{N}_2 = \max\{N_2,\ N_2^0\}$, the inequality $\sum\limits_{i,=1}^{N_1} \sum\limits_{j=\bar{N}_2+1}^{\infty} a_{n\,i\,j}^2 \frac{(\nu_{1\,i} + \nu_{2\,j})^2}{4} \leq \frac{\varepsilon}{4}, \quad \forall n \in N$ is satisfied. Similarly, we can prove that there are such numbers $\tilde{N}_1,\ \tilde{N}_2$, that the inequality $\sum\limits_{i,=\tilde{N}_1+1}^{\infty} \sum\limits_{j=1}^{\tilde{N}_2} a_{n\,i\,j}^2 \frac{(\nu_{1\,i} + \nu_{2\,j})^2}{4} \leq \frac{\varepsilon}{4}$ holds. Then, in equality (7), choosing the maximum of the numbers $N_r^0,\ \bar{N}_r,\ \tilde{N}_r$ as the value of the number $N_r,\ \ r = 1;\ 2$, respectively, we can write that

$$\begin{aligned} \left\| f_n - \sum_{i,=1}^{N_1} \sum_{j=1}^{N_2} a_{n\,i\,j} \tfrac{\nu_{1\,i} + \nu_{2\,j}}{2}\, e_{1\,i} \otimes e_{2\,j} \right\|^2 = \sum_{i,=1}^{N_1} \sum_{j=N_2+1}^{\infty} a_{n\,i\,j}^2 \tfrac{(\nu_{1\,i} + \nu_{2\,j})^2}{4} + \\ + \sum_{i,=N_1+1}^{\infty} \sum_{j=1}^{N_2} a_{n\,i\,j}^2 \tfrac{(\nu_{1\,i} + \nu_{2\,j})^2}{4} + \sum_{i,=N_1+1}^{\infty} \sum_{j=N_2+1}^{\infty} a_{n\,i\,j}^2 \tfrac{(\nu_{1\,i} + \nu_{2\,j})^2}{4} \leq \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for $\forall n \in N$. This means that series (7) converges uniformly and in it one can pass to the limit term by term , i.e. $\|f_n\| \to \|f_0\|$ at $n \to \infty$.Then $\|f_n - f_0\|^2 = \|f_n\|^2 - (f_n,\ f_0) - (f_0,\ f_n) + \|f_0\|^2 = 2\,\|f_0\|^2 + \varepsilon_n - Re(f_0,\ f_n)$, where $\varepsilon_n \to 0$.
The sequence $\{f_n\}$ converges weakly to the element $f_0$. Therefore, $(f_n,\ f_0) \to \|f_0\|^2$and $\|f_n - f_0\|^2 \to 0$ for $n \to \infty$. So $K_1^t \varphi_n = f_n - \frac{x_n}{2} \to f_0$, at $n \to \infty$ (strongly).
The sequence $\{\varphi_n\}$ converges weakly to the element $\varphi_0$ at $n \to \infty$. Therefore, for $\forall h \in H$ we can write $(f_0 - K_1^t \varphi_0,\ h) = \lim\limits_{n \to \infty} ((f_n - K_1^t \varphi_0,\ h) = \lim\limits_{n \to \infty} (K_1^t \varphi_n - K_1^t \varphi_0,\ h) = \lim\limits_{n \to \infty} (\varphi_n - \varphi_0,\ K_1^t\, h) = 0,\ f_0 - K_1^t \varphi_0 = 0$ or $K_1^t \varphi_0 = f_0$. Similarly, we can prove that $K_2^t \varphi_0 = f_0$. I.e.

$$\begin{cases} \lambda_1\, K_{i,1}^t \varphi_0 + \lambda_2\, K_{i,2}^t \varphi_0 = f_0, \\ i = 1;\ 2 \end{cases} \tag{8}$$

or

$$\begin{cases} \lambda_i\, \Delta_0 \varphi_0 = \Delta_i f_0 \\ i = 1;\ 2 \end{cases}$$

It remains only to show that $\varphi_0 = f_0$. Equalities $f_n = K_1^t \varphi_n - \frac{x_n}{2}$ and $f_n = K_2^t \varphi_n + \frac{x_n}{2}$ will be written in the form

$$\begin{cases} \lambda_1 K^t_{1,1}\varphi_n + \lambda_2 K^t_{1,2}\varphi_n = f_n - \frac{x_n}{2} \\ \lambda_1 K^t_{2,1}\varphi_n + \lambda_2 K^t_{2,2}\varphi_n = f_n + \frac{x_n}{2} \end{cases}$$

or

$$\begin{cases} \lambda_i \Delta_0\varphi_n = \Delta_i f_n + \frac{(-1)^i}{2}(K^t_{3-i,\,3-i} + K^t_{i,\,3-i})x_n \\ i = 1;\ 2 \end{cases}$$

$x_n \to 0$ strongly. Therefore, $\|\Delta_i f_n - \lambda_i\Delta_0\varphi_n\| = \left\|(K^t_{3-i,3-i} + K^t_{i,3-i})x_n\right\| \to 0$ and $\|\Delta_i\varphi_n - \Delta_i f_n\| \le \|\Delta_i\varphi_n - \lambda_i\Delta_0\varphi_n\| + \|\lambda_i\Delta_0\varphi_n - \Delta_i f_n\| \to 0$. Then $\|\Delta_i\varphi_n - \Delta_i f_0\| \le \|\Delta_i\varphi_n - \Delta_i f_n\| + \|\Delta_i\varphi_n - \Delta_i f_0\| \to 0$, i.e. $\Delta_i\varphi_n \to \Delta_i f_0$ (strongly).
But on the other hand, from the weak convergence of the sequence $\{\varphi_n\}$ to the element $\varphi_0$, we write $\Delta_i\varphi_n \to \Delta_i\varphi_0$ weakly. By virtue of the uniqueness of the weak limit, we obtain, $\Delta_i\varphi_0 = \Delta_i f_0$ or $\Delta_i(\varphi_0 - f_0) = 0$. By the condition $(\Delta_i\varphi, \varphi) > 0, i = 1; 2, \forall\varphi \in H$ the kernel of operators $\Delta_i$ is trivial, i.e, ker $\Delta_i = \{0\}$. Then $\varphi_0 - f_0 = 0$ either $\varphi_0 = f_0$, and the system of equalities (8) shows that $\varphi_0$ is an eigenelement of problem (3), i.e. $\sigma(\Delta) \subseteq \sigma(K^t) = \sigma(K)$. Combining this embedding with the relation $\sigma(K) = \sigma(K^t\Delta) \subseteq \sigma(\Delta)$, we obtain $\sigma(K) = \sigma(K^t\Delta) = \sigma(\Delta)$. The theorem is proved. □

## 3 Constructing a sequence of eigenelements

The variational method for finding the eigenvalue with minimal modulus in the hyperbolic sense and the corresponding eigenelement for the two-parameter problem (1), studied in [2], [6], makes it possible to find the first eigenvalue and the corresponding eigenelement. Here under the right definiteness condition in the form

$$\Delta_0 > 0; \Delta_1 > 0; \Delta_2 > 0, \tag{9}$$

is solved the problem of constructing a sequence of eigenelements for the two-parameter problem (1) in the form of decomposable tensors in space $H_{\Delta_0} = H_1 \otimes H_2$.

**Theorem 2.** *Under the of right definiteness condition* (9) *there is a sequence of decomposable tensors*

$$\varphi_n = \varphi_{1n} \otimes \varphi_{2n}\,,\quad \varphi_n \in H_{\Delta_0} = H_1 \otimes H_2,\ \ \varphi_{1n} \in H_1,\quad \varphi_{2n} \in H_2\,,\ \ n = 1,\,2,\,3,\,.....$$

*for which*
*a) all elements of this sequence are eigenelements of the problem* (1)*,*
*b) all elements of this sequence are decomposable tensors*
*c) this sequence of eigenelements is a complete orthonormal basis in space* $H_{\Delta_0} = H_1 \otimes H_2$ *, i.e.*

$$[\varphi_n, \varphi_m] \equiv (\Delta_0\varphi_n, \varphi_m) = \begin{cases} 0, & \textit{if}\, n \neq m \\ 1, & \textit{if}\, n = m \end{cases}$$

*Proof:* By Theorem 1, the spectra of problems (1) and (2) coincide. Each solution $(\varphi_{10},\ \varphi_{20})$ of the problem (1) corresponds to a solution $\varphi_{10} \otimes \varphi_{20}$ of the problem (2) in the form of a decomposable tensor. This means that among the eigenelements corresponding to each eigenvalue of problem (2) there is always a decomposable tensor. Therefore, when finding the eigenelements of problem (1) by the variational method in the weight space $H_\Delta = H_1 \otimes H_2$, the extremum can be found on the subset of decomposable tensors, i.e., minimization can be carried out in the subset $H' \subset H_\Delta$ of decomposable tensors. Using the variational method, by minimizing the functional $F(\varphi) = \frac{(\Delta_1\varphi,\varphi)(\Delta_2\varphi,\varphi)}{(\Delta_0\varphi,\varphi)^2}$ on the set of decomposable tensors, the first eigenelement of $\varphi_1 = \varphi_{11} \otimes \varphi_{21}$, $\varphi_1 \in H' \subset H_{\Delta_0} = H_1 \otimes H_2$, $\varphi_{11} \in H_1$, $\varphi_{22} \in H_2$ the corresponding eigenvalue of problem (2), and hence problem (1) can be found. By theorem 1, the second eigenelement $\varphi_2 = \varphi_{12} \otimes \varphi_{22}$ of the problem (1) can also be found by minimizing the functional $F(\varphi)$ on the subset $H'' \subset H_\Delta = H_1 \otimes H_2$, where $H''$ is the set of all decomposable tensors of the space $H_\Delta$ and orthogonal to the element $\varphi_1 = \varphi_{11} \otimes \varphi_{21}$ in this space.

The third eigenelement $\varphi_3 = \varphi_{13} \otimes \varphi_{23}$ of the problem (1) is also found by minimizing the functional $F(\varphi)$ on the subset of the decomposable tensors $H''' \subset H_\Delta = H_1 \otimes H_2$, where $H'''$ is the set of all decomposable tensors of the space $H_\Delta$ which is orthogonal to the elements of $\varphi_1 = \varphi_{11} \otimes \varphi_{21}$ and $\varphi_2 = \varphi_{12} \otimes \varphi_{22}$ simultaneously. All other eigenelements of problem (1) can be found in a similar way. This process shows that there is a sequence of decomposable tensors, all elements of which are eigenelements of the problem (1). And this is the proof of items a) and c) of theorem 2. Let us now prove item c). Let $L$ be the linear hull of the sequence $\{\varphi_n\}_{n=1,2,...} = \{\varphi_{1n} \otimes \varphi_{2n}\}_{n=1,2,...}$. We must show that the set $\bar{L}$ which is the closure of the linear manifold $L$, coincides with the space $H_\Delta$. Suppose the contrary, $\bar{L} \neq H_\Delta$. The manifold $\bar{L}$ is a subspace in the space $H_\Delta$. We denote the orthogonal complement of this subspace in the space $H_\Delta$ by $\tilde{L}$, i.e. $H_\Delta = \bar{L} \oplus \tilde{L}$. The subset $\bar{L} \subset H_\Delta$ is an invariant subspace for the operators $\Gamma_1 = \Delta_0^{-1}\Delta_1$ and $\Gamma_2 = \Delta_0^{-1}\Delta_2$ simultaneously. All elements of the sequence $\{\varphi_n\}_{n=1,2,...} = \{\varphi_{1n} \otimes \varphi_{2n}\}_{n=1,2,...}$ are eigenelements of the $\Gamma_1 = \Delta_0^{-1}\Delta_1$ and $\Gamma_2 = \Delta_0^{-1}\Delta_2$ operators simultaneously. Then it is easy to prove that $\tilde{L} \subset H_\Delta$ is also an invariant subspace for the operators $\Gamma_1 = \Delta_0^{-1}\Delta_1$ and $\Gamma_2 = \Delta_0^{-1}\Delta_2$ simultaneously. By virtue of the closedness of the subspace $\tilde{L} \subset H_\Delta$, by minimizing the functional $F(\varphi) = \frac{(\Delta_1\varphi,\varphi)(\Delta_2\varphi,\varphi)}{(\Delta_0\varphi,\varphi)^2}$ on the subspace $\tilde{L} \subset H_\Delta$, we can prove that there exists an element $\exists\tilde{\varphi} \in \tilde{L} \subset H_\Delta$ which is an eigenelement of problem (2) corresponding to the eigenvalue $(\tilde{\lambda}_1,\ \tilde{\lambda}_2) = \left(\frac{(\Delta_1\tilde{\varphi},\tilde{\varphi})}{(\Delta_0\tilde{\varphi},\tilde{\varphi})},\ \frac{(\Delta_2\tilde{\varphi},\tilde{\varphi})}{(\Delta_0\tilde{\varphi},\tilde{\varphi})}\right)$, i.e. $(\tilde{\lambda}_1,\ \tilde{\lambda}_2) \in \sigma(\Delta)$. Since $\sigma(K) = \sigma(K^t\Delta) = \sigma(\Delta)$, then $(\tilde{\lambda}_1,\ \tilde{\lambda}_2) \in \sigma(K)$. This means that there is an $\exists\hat{\varphi} = \hat{\varphi}_1 \otimes \hat{\varphi}_2 \in \tilde{L} \subset H_\Delta$ element in the form of a decomposable tensor, which is an eigenelement of problem (1). However, this is impossible. Since, all the eigenelements that have the form of a decomposable tensor are in the subspace. So, $\tilde{L} = \{0\}$ and $\bar{L} = H_\Delta$. This proves the point c) of theorem 2. □

## 4 Conclusion

In this paper, by proving Theorem 2, a method for finding and ordering the remaining elements of sequence eigenvalues and corresponding eigenelements of problem (1) is suggested.

 

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# On the Adaptive Control Synchronization and Anti-Synchronization of Chaotic Systems in 3D

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**Abstract:** In this work, we are interested in the synchronization and anti-synchronization of chaotic systems in 3d according to the adaptive control method. Firstly, Complete synchronization is achieved between tow non identical 3d novel chaotic systems. Next, Anti-synchronization between tow identical 3d chaotic systems is achieved via adaptive control method. Finally, illustrative figures are obtained using numerical simulation in Matlab to validate the results.

**Keywords:** 3-D chaotic system, Lyapunov exponent, Adaptive control, Synchronization, Anti-synchronization.

## 1 Introduction

Finding methods to synchronize chaotic dynamic systems is required because of its important applications, especially in the fields of secure communication and encryption. Chaos synchronization was introduced in the work of Pecora and Carroll [1], then several powerful different methodologies have been developed for chaos control and synchronization of chaotic systems such as, active control method [3], sliding mode control [4], backstepping control [5], Projective synchronization [6],... etc. Another powerful method is the adaptive control technique which is deferent from others control methods since it is used when parameters are unknown or initially uncertain and it does not need a priori information about the bounds on these uncertain or time-varying parameters because this method of control is concerned with control law changing them-selves. Recently, many papers are available on Synchronization of chaotic systems using this method of control [7]-[13]. In This work, we consider, the synchronization and anti-synchronization between tow non identical and identical chaotic systems in 3D according to the adaptive control method and Lyapunov theory of stability. Firstly, Complete synchronization is achieved between tow non identical 3d novel chaotic systems. Next, Anti-synchronization between tow identical 3D chaotic systems is achieved via adaptive control method. Finally, illustrative figures are obtained using numerical simulation in Matlab to validate the results.

## 2 Preliminaries

### 2.1 *Description of the new chaotic system*

In this work, we consider a new 3D chaotic system with two nonlinear quadratic forms, which is given by:

$$\begin{cases} \frac{dx_1}{dt} = a(x_2 - x_1) \\ \frac{dx_2}{dt} = cx_1 + x_1x_3 \\ \frac{dx_3}{dt} = -x_1x_2 + b(x_1 - x_3) \end{cases} \tag{1}$$

were $a, b, c$ are positive real parameters. The system (1) is chaotic when the parameters have the following values:

$$a = 13, b = 2,5, c = 50. \tag{2}$$

For the values of the parameters (2), according to Matlab the Lyapunov exponents of the system (1) are given by:

$$L_1 = 1,4375,\ L_2 = -0,000166417,\ L_3 = -16,9373. \tag{3}$$

**Definition 1.** *Complete synchronization occur between master-slave systems* $(4)$ *and* $(5)$ *when there exist controllers* $u_i, i = 1, 2, ..., n$*, such that the synchronization errors*

$$e_i\left(t\right) = y_i\left(t\right) - x_j\left(t\right), i = 1, .., n \tag{4}$$

*satisfy that* $\lim_{t\to+\infty} e_i\left(t\right) = 0$.

*For anti-synchronization the synchronization errors is defined as:*

$$e_i\left(t\right) = y_i\left(t\right) + x_j\left(t\right), i = 1, .., n \tag{5}$$

## 3 Complete synchronization of non-identical chaotic systems

As master system, we take the system:

$$\begin{cases} \frac{dx_1}{dt} = a(x_2 - x_1) \\ \frac{dx_2}{dt} = cx_1 + x_1x_3 \\ \frac{dx_3}{dt} = -x_1x_2 + b(x_1 - x_3) \end{cases} \tag{6}$$

where $x_1, x_2, x_3$ are the state variables and $a, b, c$ are undefined real constant parameters of the system.

As slave system, we choose the controlled system of Lü as follows:

$$\begin{cases} \frac{dy_1}{dt} = \alpha(y_2 - y_1) + u_1 \\ \frac{dy_2}{dt} = \gamma y_2 - y_1y_3 + u_2 \\ \frac{dy_3}{dt} = -\beta y_3 + y_1y_2 + u_3 \end{cases} \tag{7}$$

such that $y_1, y_2, y_3$ are state variables and $\alpha, \beta, \gamma$ are real constant parameters unknown to the system and $u_1, u_2, u_3$ are nonlinear controllers.

This system is chaotic for the given values: $\alpha = 36$, $\beta = 3$, $\gamma = 20$.

we have the error of this type defined by:

$$e_i = y_i - x_i,\ i = 1, 2, 3. \tag{8}$$

it implies that:

$$\dot{e_i} = \dot{y_i} - \dot{x_i},\ i = 1, 2, 3. \tag{9}$$

we get the error as:

$$\begin{cases} \dot{e}_1 = \alpha(y_2 - y_1) + u_1 - a(x_2 - x_1) \\ \dot{e}_2 = \gamma y_2 - y_1y_3 + u_2 - cx_1 - x_1x_3 \\ \dot{e}_3 = -\beta y_3 + y_1y_2 + u_3 + x_1x_2 - b(x_1 - x_3) \end{cases} \tag{10}$$

therefore the controllers are:

$$\begin{cases} u_1 = -\alpha_1\left(t\right)\left(y_2 - y_1\right) + a_1\left(t\right)\left(x_2 - x_1\right) - k_1e_1 \\ u_2 = -\gamma_1\left(t\right)y_2 + y_1y_3 + c_1\left(t\right)x_1 + x_1x_3 - k_2e_2 \\ u_3 = \beta_1\left(t\right)y_3 - y_1y_2 - x_1x_2 + b_1\left(t\right)\left(x_1 - x_3\right) - k_3e_3 \end{cases} \tag{11}$$

By substituting (13) into (12), the error dynamics is obtained as

$$\begin{cases} \dot{e}_1 = (\alpha - \alpha_1\left(t\right))(y_2 - y_1) - \left(a - a_1\left(t\right)\right)\left(x_2 - x_1\right) - k_1e_1 \\ \dot{e}_2 = (\gamma - \gamma_1\left(t\right))y_2 - (c - c_1\left(t\right))x_1 - k_2e_2 \\ \dot{e}_3 = -\left(\beta - \beta_1\left(t\right)\right)y_3 - \left(b - b_1\left(t\right)\right)\left(x_1 - x_3\right) - k_3e_3 \end{cases} \tag{12}$$

we define the estimation of the errors of the parameters as follows:

$$e_a = a - a_1(t), e_b = b - b_1\left(t\right), e_c = c - c_1(t), \tag{13}$$

$$e_\alpha = \alpha - \alpha_1\left(t\right), e_\beta = \beta - \beta_1(t), e_\gamma = \gamma - \gamma_1\left(t\right). \tag{14}$$

Replacing (13-14) in (12), we get:

$$\begin{cases} \dot{e}_1 = e_\alpha(y_2 - y_1) - e_a(x_2 - x_1) - k_1e_1 \\ \dot{e}_2 = e_\gamma y_2 - e_cx_1 - k_2e_2 \\ \dot{e}_3 = -e_\beta y_3 - e_b(x_1 - x_3) - k_3e_3 \end{cases} \tag{15}$$

using Lyapunov's function:

$$V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_a^2 + e_b^2 + e_c^2 + e_\alpha^2 + e_\beta^2 + e_\gamma^2) \tag{16}$$

that is a positive definite function on $\mathbb{R}^9$.

and we have:

$$\left\{\begin{array}{l} \frac{de_a(t)}{dt} = -\frac{da_1(t)}{dt}, \frac{de_b(t)}{dt} = -\frac{db_1(t)}{dt}, \frac{de_c(t)}{dt} = -\frac{dc_1(t)}{dt}, \\ \frac{de_\alpha(t)}{dt} = -\frac{d\alpha_1(t)}{dt}, \frac{de_\beta(t)}{dt} = -\frac{d\beta_1(t)}{dt}, \frac{de_\gamma(t)}{dt} = -\frac{d\gamma_1(t)}{dt} \end{array}\right. \tag{17}$$

Drifting $V$ along the trajectories of the systems (17) and (15), we obtain:

$$\dot{V} = \dot{e}_1 e_1 + \dot{e}_2 e_2 + \dot{e}_3 e_3 + \dot{e}_a e_a + \dot{e}_b e_b + \dot{e}_c e_c + \dot{e}_\alpha e_\alpha + \dot{e}_\beta e_\beta + \dot{e}_\gamma e_\gamma \tag{18}$$

so:

$$\begin{array}{l} \dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 + e_a(t)\left[-e_1(x_2 - x_1) - \frac{da_1(t)}{dt}\right] + \\ e_c(t)\left[-e_2 x_1 - \frac{dc_1(t)}{dt}\right] + e_b(t)\left[-(x_1 - x_3)e_3 - \frac{db_1(t)}{dt}\right] + \\ e_\alpha(t)\left[(y_2 - y_1)e_1 - \frac{d\alpha_1(t)}{dt}\right] + \\ e_\beta(t)\left[-y_3 e_3 - \frac{d\beta_1(t)}{dt}\right] + e_\gamma(t)\left[y_2 e_2 - \frac{d\gamma_1(t)}{dt}\right]. \end{array} \tag{19}$$

for

$$\left\{\begin{array}{l} \frac{da_1(t)}{dt} = -e_1(x_2 - x_1) + k_4 e_a \\ \frac{db_1(t)}{dt} = (x_1 - x_3)e_3 + k_5 e_b \\ \frac{dc_1(t)}{dt} = -e_2 x_1 + k_6 e_c \\ \frac{d\alpha_1(t)}{dt} = (y_2 - y_1)e_1 + k_7 e_\alpha \\ \frac{d\beta_1(t)}{dt} = -y_3 e_3 + k_8 e_\beta \\ \frac{d\gamma_1(t)}{dt} = y_2 e_2 + k_9 e_\gamma \end{array}\right. \tag{20}$$

such as $k_i, (i = 4, 5, 6, .., 9)$ positive constants.
Replacing (20) in (19) we obtain:

$$\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_a^2 - k_5 e_b^2 - k_6 e_c^2 - k_7 e_\alpha^2 - k_8 e_\beta^2 - k_9 e_\gamma^2 < 0 \tag{21}$$

that is a negative definite function on $\mathbb{R}^9$.

**Theorem 1.** *The non-identical master system (6) and slave system (7) with undefined parameters are globally synchronized under the law of adaptive control (11) and the update parameter estimates law (20) .*

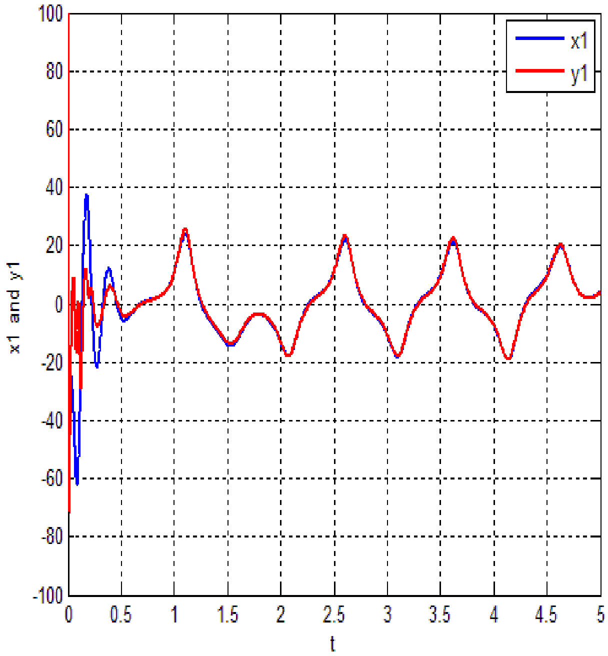


**Fig. 1**: The synchronization of states $x_1$ and $y_1$

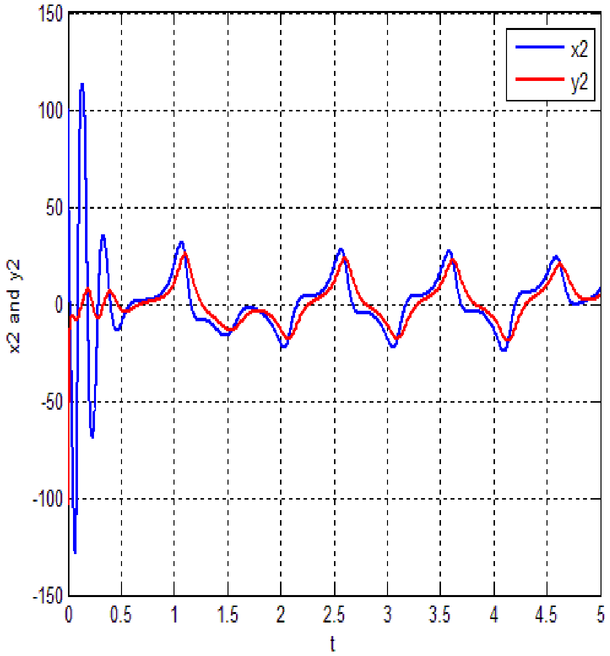


**Fig. 2**: The synchronization of states $x_2$ and $y_2$

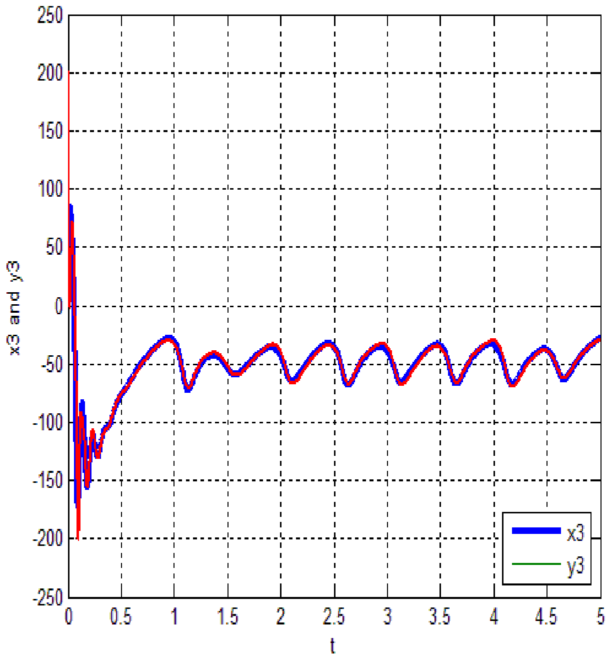


**Fig. 3**: The synchronization of states $x_3$ and $y_3$

## 4 Anti-synchronisation

We use the same master system (1) and as slave system the controled of the system (1) given by:

$$\begin{cases} \frac{dy_1}{dt} = a(y_2 - y_1) + u_1 \\ \frac{dy_2}{dt} = cy_1 + y_1y_3 + u_2 \\ \frac{dy_3}{dt} = -y_1y_2 + b(y_1 - y_3) + u_3 \end{cases} \tag{22}$$

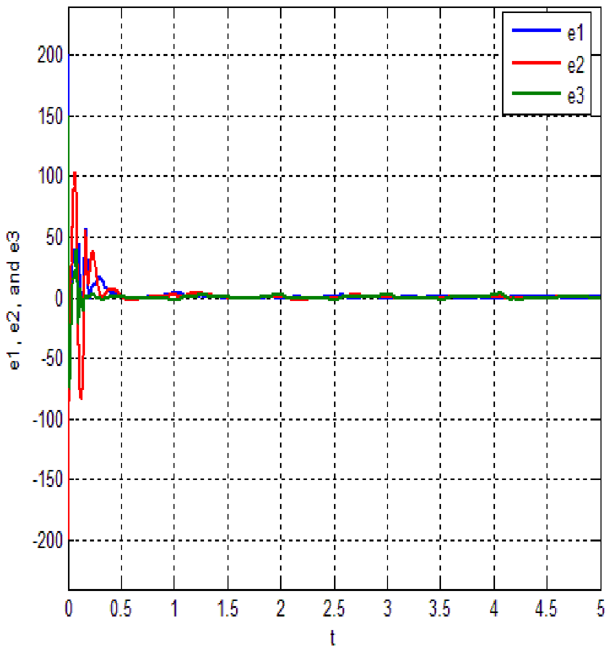


**Fig. 4**: Time history of the synchronization error.

The anti-synchronization error between the chaotic systems (1) and (22) is defined as:

$$e_i = y_i + x_i,\ i = 1, 2, 3. \tag{23}$$

it implies that:

$$\dot{e_i} = \dot{y_i} + \dot{x_i},\ i = 1, 2, 3. \tag{24}$$

We obtain:

$$\begin{cases} \dot{e}_1 = a(e_2 - e_1) + u_1 \\ \dot{e}_2 = ce_1 + x_1x_3 + y_1y_3 + u_2 \\ \dot{e}_3 = b(e_1 - e_3) - x_1x_2 - y_1y_2 + u_3 \end{cases} \tag{25}$$

The law of adaptive control is:

$$\begin{cases} u_1 = -a_1(t)(e_2 - e_1) - k_1e_1 \\ u_2 = -c_1(t)e_1 - x_1x_3 - y_1y_3 - k_2e_2 \\ u_3 = -b_1(t)(e_1 - e_3) + x_1x_2 + y_1y_2 - k_3e_3 \end{cases} \tag{26}$$

with $k_1$, $k_2$, $k_3$ are positive constants.
Introducing (26 in (25), we obtain:

$$\begin{cases} \dot{e}_1 = (a - a_1(t))(e_2 - e_1) - k_1e_1 \\ \dot{e}_2 = (c - c_1(t))e_1 - k_2e_2 \\ \dot{e}_3 = (b - b_1(t))(e_1 - e_3) - k_3e_3 \end{cases} \tag{27}$$

then;

$$\begin{cases} \dot{e}_1 = e_a(t)(e_2 - e_1) - k_1e_1 \\ \dot{e}_2 = e_c(t)e_1 - k_2e_2 \\ \dot{e}_3 = e_b(t)(e_1 - e_3) - k_3e_3 \end{cases} \tag{28}$$

We consider a quadratic Lyapunov function given by:

$$V(e_1, e_2, e_3, e_a, e_b, e_c) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_a^2 + e_b^2 + e_c^2) \tag{29}$$

with (29) is a positive definite function in $\mathbb{R}^6$ .

Differentiating $V$ along the trajectories of the systems (17) and (28), we obtain:

$$\begin{aligned}\dot{V} &= -\sum_{i=1}^{3} k_i e_i^2 + e_a(t)(e_1 e_2 - e_1^2 - \frac{da_1(t)}{dt}) + \\ & e_b(t)(e_1 e_3 - e_3^2 - \frac{db_1(t)}{dt}) + e_c(t)(e_1 e_2 - \frac{dc_1(t)}{dt}).\end{aligned} \quad (30)$$

From (30), let us set:

$$\begin{cases} \frac{da_1(t)}{dt} = e_1 e_2 - e_1^2 \\ \frac{db_1(t)}{dt} = e_1 e_3 - e_3^2 \\ \frac{dc_1(t)}{dt} = e_1 e_2 \end{cases} \quad (31)$$

introducing (31) into (30), we obtain:

$$\dot{V} = -\sum_{i=1}^{3} k_i e_i^2 < 0 \quad (32)$$

where (32) is a negative definite function in $\mathbb{R}^3$, hence by Lyapunov's stability theory it follows that $e_i(t) \longrightarrow 0$ when $t \longrightarrow \infty$ for $i = 1, 2, 3$.

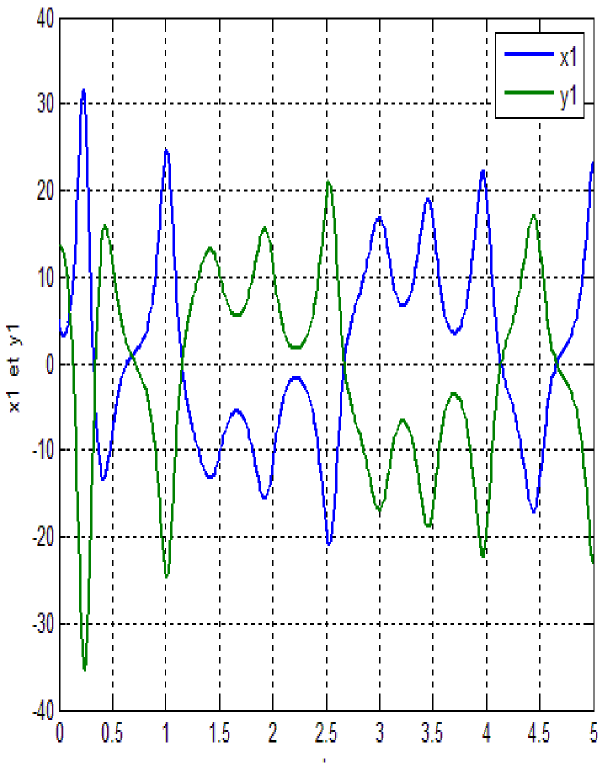


**Fig. 5**: Anti-synchronization of states $x_1$ and $y_1$

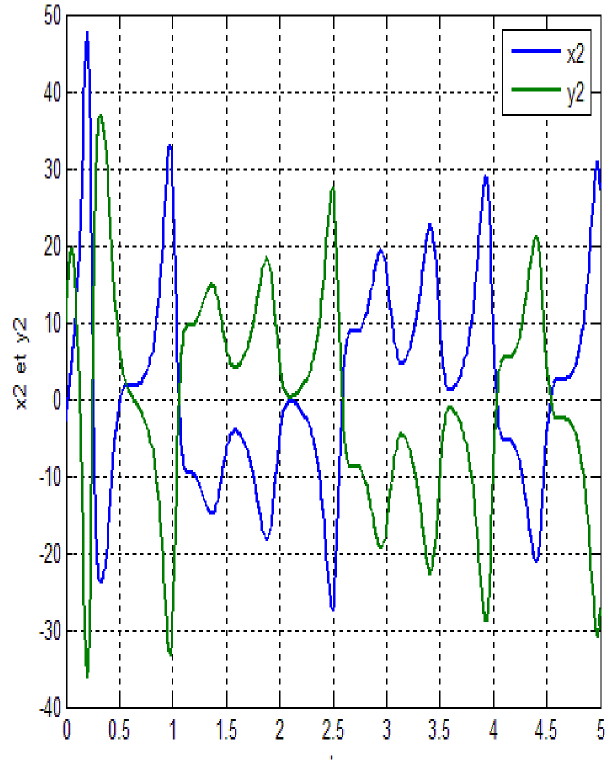


**Fig. 6**: Anti-synchronization of states $x_2$ and $y_2$

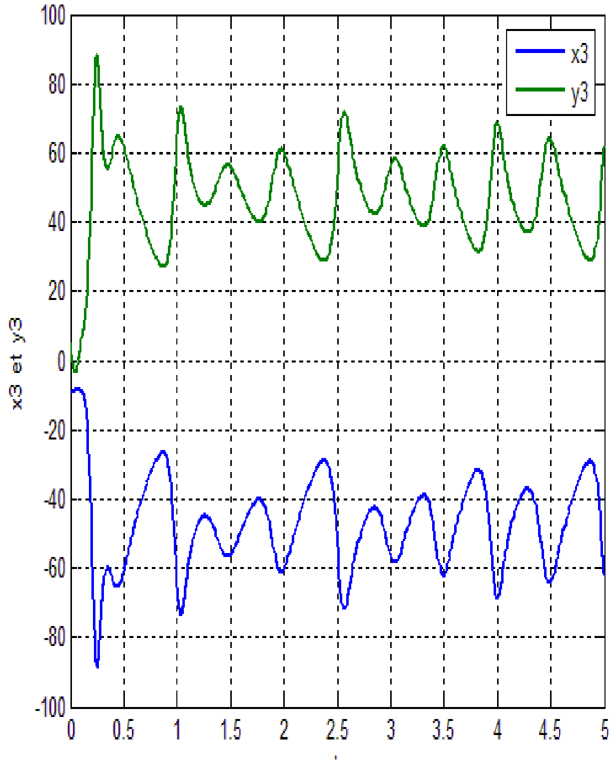


**Fig. 7**: Anti-synchronization of states $x_3$ and $y_3$

## 5 Conclusion

In this work, the synchronization and anti-synchronization of 3D chaotic systems were introduced via adaptive control method. First, full synchronization is achieved between two non-identical three-dimensional chaotic systems. Then, the anti-synchronization between two identical 3D chaotic systems is achieved via the adaptive control method. The results were validated by numerical simulation in Matlab.

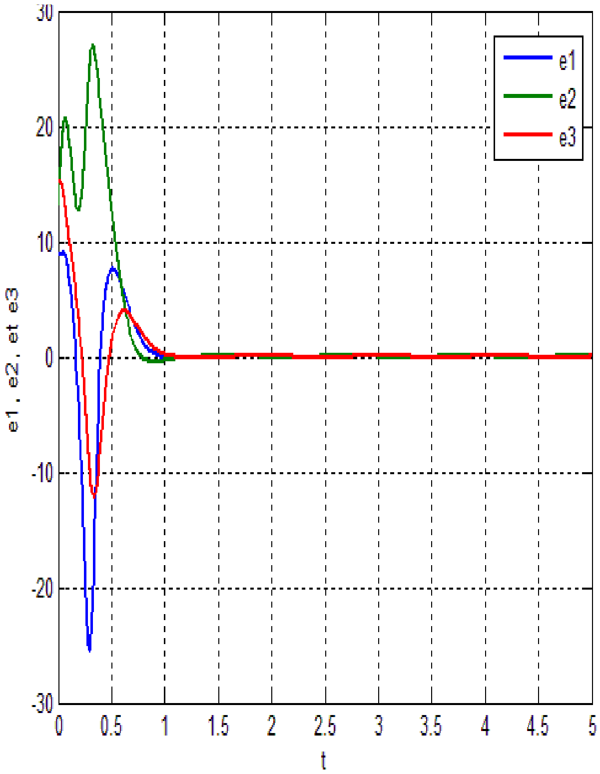


**Fig. 8**: Time history of the synchronization error.

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# Nature Inspired Algorithm to Find the Current Expression -Series RC Electric Circuit Case-

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**Abstract:** The differential equations that describe many attractive natural phenomena are one of the most motivating fields of mathematics. Solving Initial Value Problems (IVPs) in Ordinary Differential Equations (ODEs) via fundamental mathematical methods gives in general insuitable results especially face to difficult problems. Hence the solution of this lacks is found by using a Meta-heuristic Algorithms (MAs). In this paper we propose by means of the Flower Pollination Algorithm (FPA) an approximative solution of IVPs arising from a circuit consisting of a resistor and a capacitor in both constant voltage and variable voltage cases. The conducted comparison between the exact solution and the algorithm outcomes in the investigated examples showed that FPA yields satisfactorily precise approximation of the solutions.

**Keywords:** Flower pollination algorithm (FPA), Initial value problems (IVP), Metaheuristic algorithms (MAs), Ordinary differential equations (ODE), Series RC circuit)

## 1 Introduction

Optimization is a process that locates a best, or optimal values of the variables that minimize or maximize the objective function while satisfying the constraints, it arises in various disciplines. Engineering problems under growing dimensions, moment complexity, variables, and space complications are becoming more and more difficult and hard to optimize; consequently optimization algorithms are then used to overcome this situation but traditional optimization techniques, including many heuristic approaches still insufficient. To cope up with such situation, many researchers focus on nature by creating a large collection of MAs which is characterized by its convergence speed and augmentation searched variables number. These methods generate a simpler procedure to solve an optimization problem to find good solutions with less computational effort than simple algorithms or traditional heuristics [3, 5]. MAs take dissimilar forms according to the inspired process of the natural systems like Genetic algorithm [9, 11], Ant colony optimization algorithm [7], Bee algorithm [6, 16], Particle Swarm Optimization [14], Bat algorithm [26], Fractional Lévy Flight Bat Algorithm (FLFBA) [4] and modified Salp Swarm Algorithm (SSA) [18]...etc. All these algorithms have several advantages illustrated via a wide range of applications. The Flower Pollination Algorithm (FPA) [25] is a recent bio-inspired optimization algorithm that takes off the real life processes of the fertilization (pollination) process of flowers. In FPA, abiotic pollination is considered for local pollination while biotic pollination is considered for the global pollination between the flower plants. The algorithm maintains a balance between local and global pollination. It took an interesting place between the more recent nature inspired algorithms kept by its nice performance against several classical MAs. This is behind the vast utilization of FPA in various domains such as chemical engineering, civil engineering, communication engineering, medical field, computer science. . . etc. FPA was hybridized with other nature-inspired MAs. In order to overcome its limitations and to benefit from their strength e.g. PSO [27], frog leaping local search [15] and simulated annealing [1], Bat algorithm [28] . . . etc. In electronics and electric engineering a first order RC circuit (RC filter or RC network) [13]. It is an electric circuit composed of resistors and capacitors, either in series driven by a voltage source or in parallel driven by a current source [12]. The importance of this study is to consider the ODEs arising from a series RC circuit in both constant voltage and variable voltage cases as an IVPs then they are formulated as an optimization problem [17], when the FPA [25] is used as a tool to find numerical solutions for this problem. The remainder paper is organized as follows: The formulation of the problem is revealed in section 2; section 3 provides basics on FPA and its main steps for finding an approximate solution of IVP. The Section 4 gives essential formulae with brief explanation of series RC circuit ODEs. The Section 5 exposes examples of series RC circuit IVPs to show how the FPA can lead to a satisfactory result for solving IVP. The comments and conclusion are made in section 6.

## 2 Problem formulation

Let $f = f(x, y)$ be a real-valued function of two real variables defined for $a \leq x \leq b$, where $a$ and $b$ are finite, and for all real values of $y$. The equations

$$\begin{cases} y^{'} = f(x,y) \\ y(a) = y_0 \end{cases}, \tag{1}$$

are called IVP; they symbolize the following problem: To find a function $y(x)$, continuous and differentiable for $x \in [a, b]$ such that $y' = f(x, y)$ from $y(a) = y_0$ for all $x \in [a, b]$ [10].
This problem possesses unique solutions when: $f$ is continuous on $[a, b] \times \mathbb{R}$, and satisfies the Lipschitz condition; it exists a real constant $k > 0$, as $\mid f(x, \theta_1) - f(x, \theta_2) \mid \leq k \mid \theta_1 - \theta_2 \mid$, for all $x \in [a, b]$ and all couple $(\theta_1, \theta_2) \in \mathbb{R} \times \mathbb{R}$.
Finding the optimal solutions numerically of an IVP is gotten with approximations: $y(x_0 + h), \ldots, y(x_0 + nh)$ where $a = x_0$ and $h = (b - a)/n$. For more precision of the solution, we must use a very small step size $h$ that includes a larger number of steps, thus more computing time which not available in the useful numerical methods like Euler and Runge-Kutta methods [10], which may approximate solutions of IVP and perhaps yield useful information, often sufficing in the absence of exact, analytic solutions.

### 2.1 Objective function

The main idea behind the algorithm is to use the finite difference formula for the derivative and equation (1) we obtain,

$$\frac{y(x_j) - y(x_{j-1})}{h} \approx f(x_{j-1}, y(x_{j-1})),$$

Thus,

$$\frac{y_j - y_{j-1}}{h} \approx f\Big(x_{j-1}, y_{j-1}\Big).$$

Consequently, we have to consider the error formula:

$$\left[\frac{y_j - y_{j-1}}{h} - f\Big(x_{j-1}, y_{j-1})\Big)\right]^2.$$

The objective function associated to $Y = (y_1, y_2, ..., y_d)$ will be:

$$F(Y) = \sum_{j=1}^{d} \left[\frac{y_j - y_{j-1}}{h} - f\Big(x_{j-1}, y_{j-1}\Big)\right]^2. \tag{2}$$

### 2.2 Consistency

We are interested in the calculation of $Y = (y_1, y_2, ..., y_d)$ which minimizes the objective function equation ((2)). We have from Taylor's Formula order 1;

$$y_j = y_{j-1} + hy'_{j-1} + O\left(h^2\right), j = 1, \cdots d.$$

So,

$$\frac{y_j - y_{j-1}}{h} = y'_{j-1} + O(h)$$

If we subtract $f(x_{j-1}, y_{j-1})$ from both sides of last equation, we obtain

$$\frac{y_j - y_{j-1}}{h} - f(x_{j-1}, y_{j-1}) = y'_{j-1} - f(x_{j-1}, y_{j-1}) + O(h), j = 1, \cdots d$$

The last relation shows that the final value $Y = (y_1, y_2, \cdots, y_d)$ is an approximate solution of IVP, for small value of $h$.

## 3 Flower pollination algorithm (FPA)

### 3.1 Flower pollination description

Pollination is very important. It leads to the creation of new seeds that grow into new plants. It begins in the flower. Flowering plants have several different parts that are important in pollination. Flowers have male parts called **stamens** that produce a sticky powder called pollen. Flowers also have a female part called the pistil. The top of the pistil is called the **stigma**, and is often sticky. Seeds are made at the base of the pistil, in the ovule. To be pollinated, pollen must be moved from a stamen to the stigma [24]. There are two types of pollination:

• Self Pollination (Abiotic pollination): Only about 10% of plants fall in this category, it's the fertilization of one flower, when the pollen from a flower pollinates the same flower or flowers of the same plant, it does not require any pollinators. It occurs when a flower contains both the male and the female gametes is a process where the pollination happens without involvement of external agents [23].
• Cross Pollination (biotic pollination): Is typically associated when pollen from a plant's stamen is transferred to a different plant's stigma (of the same species), and such transfer is often linked with pollinators (see Figure 1). Pollination occurs in several ways:
◦ People: They can transfer pollen from one flower to another, but most plants are pollinated without any help from people.
◦ Animals: such as bees, butterflies, moths, flies pollinate plants by an accidental way when they are at the plant to get food. The pollinators can fly a long distance, thus they can be considered as the global pollination. In addition, bees and birds may behave as Lévy flight behavior


(a) Self pollination (b) Cross pollination

**Fig. 1**: Flower pollination.

[19], with jump or fly distance steps obey a Lévy distribution. Furthermore, flower constancy can be used an increment step using the similarity or difference of two flowers [8, 22].
○ Wind and Diffusion in water: it picks up pollen from one plant and blows it into another.

### *3.2 FPA formulation*

The four rules given below are used to summarize the above characteristics of pollination process, flower constancy and pollinator behavior [25] .

1. Biotic and cross-pollination is considered as global pollination process and pollinators carrying pollen move in a way that confirms to Lévy flights.
2. For local pollination, abiotic pollination and self-pollination are used.
3. Flower constancy can be considered as the reproduction probability is proportional to the similarity of two flowers involved.
4. Local pollination and global pollination is controlled by a switch probability $p \in [0, 1]$.

To formulate the updating formulas, these rules have to be changed into correct updating equations. In principle, flower pollination process can happen at both local and global levels. But in reality, flowers in the neighborhood have higher chances of getting pollinated by pollen from local flowers than those which are far away. To simulate this feature, a proximity probability (Rule 4) can be commendably used to switch between intensive local pollination to common global pollination. To start with, a raw value of $p = 0.5$ may be used as an initial value. A preliminary parametric study indicated that $p = 0.8$ may work better for most applications. The main steps of FPA, or simply the flower algorithm [24] are illustrated below:

Pseudo code of the proposed Flower Pollination Algorithm (FPA).

Objective $\min$ or $\max f(x), x = (x_1, x_2, ..., x_d)$
Initialize a population of $n$ flowers/pollen gametes with random solutions
Find the best solution $g_*$in the initial population
Define a switch probability $p \in [0, 1]$
while (t <MaxGeneration)
  for $i = 1 : n$ (all $n$ flowers in the population)
    if $rand < p$,
      Draw a (d-dimensional) step vector $L$ which obeys a Lévy distribution
      Global pollination via $x_i^{t+1} = x_i^t + L(g_* - x_i^t)$
    else
      Draw $\epsilon$ from a uniform distribution in $[0, 1]$
      Randomly choose $j$ and $k$ among all the solutions
      Do local pollination via $x_i^{t+1} = x_i^t + \epsilon(x_j^t - x_k^t)$
    end if
    Evaluate new solutions
    If new solutions are better, update them in the population
  end for
    Find the current best solution $g_*$
end while

## 4 Case study: Solving IVP for a series RC Circuit

The RC circuit (RC filter or RC network) is an electric circuit composed of resistors and capacitors driven by a voltage or current source, when the circuit is composed of one resistor and one capacitor then it's called a first order RC circuit which is the simplest type of RC circuit [12] (Figure 2). RC circuits have a several utilities, its may be used to filter a signal by blocking certain frequencies and passing others and charge transport behavior in various complex systems described using models of many-element RC networks like the battery anodes and fuel cells [13]. The two most common RC filters are the high-pass filters and low-pass filters; band-pass filters and band-stop filters usually require RLC filters, though crude ones can be made with RC filters [2].

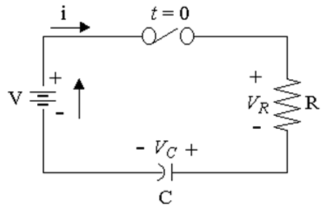



**Fig. 2**: The RC circuit diagram.

### *4.1 Case 1: Constant voltage*

The voltage across the resistor and capacitor are as follows: $V_R = Ri$ and $V_C = \frac{1}{C}\int idt$, Kirchhoff's voltage law says the total voltages must be zero. So applying this law to a series RC circuit results in the equation:

$$Ri + \frac{1}{C}\int idt = V$$

One way to solve this equation is to turn it into a differential equation, by differentiating throughout with respect to $t$:

$$R\frac{di}{dt} + \frac{i}{C} = 0$$

Solving the equation gives us:

$$i = \frac{V}{R}\exp(\frac{-t}{RC}) \tag{3}$$

The time constant in the case of an RC circuit is:

$$\tau = RC \tag{4}$$

### *4.2 Case 2: Variable voltage and 2-mesh circuits*

We need to solve variable voltage cases in $q$, rather than in $i$, since we have an integral to deal with if we use $i$. So we will make the substitutions: $i = \dfrac{dq}{dt}$ and $q = \int idt$. So the equation in $i$ involving an integral: $Ri + \dfrac{1}{C}\int idt = V$ becomes the differential equation in $q$:

$$R\frac{dq}{dt} + \frac{1}{C}q = V \tag{5}$$

## 5 Numerical experiments

### *5.1 Parameters settings*

To illustrate the treated method and to demonstrate its computationally efficiency, the courant expression problem in both constant and variable voltage cases is considered by taking a uniform step size $h$. In Table (1) the parameters settings to generate the FPA and RK4 are presented. For convenience, the numerical results are exposed in graphical and tabular form. Table (4) offers the FPA results vs. the exact and RK4 results for the studied cases of the problem, as well as the absolute error that is summarized in Table (5). For a better analysis of the results, each optimization procedure was repeated 50 times overall the objective function for the dimension $D = 10$, and the population size in all algorithms is set to 30. The search space in all algorithms is restricted to the interval $[-5;5]^D$. The maximum number of iterations was set to 50 times the dimension such as for $D = 10$ is 1000. All computations were performed on an MSWindow 2007 professional operating system in the Matlab environment version $R2013a$ compiler on Intel Duo Core $2.20Ghz$ PC. The problem treatment demanding, two types of parameters, the first are related to FPA and the second are connected to IVP. These parameters are described as follows:

| Parameter | Value |
|---|---|
| Dimension of the search variables $(d)$ | 10 |
| Total number of iterations $(N)$ | 2000 |
| Population size $(n)$ | 20 |
| Probabibility switch $(p)$ | 0.8 |

**Table 1** Parameters adopted for the FPA.

In the experimental study we introduce two IVP that arising from a RC circuit in both constant voltage and variable voltage cases. The objective function defined as:

$$F\left(y_1, y_2, ..., y_{10}\right) = \sum_{j=1}^{10} \left(\frac{y_j - y_{j-1}}{h} - f\left(x_{j-1}, y_{j-1}\right)\right)^2 = \sum_{j=1}^{10} \left(\frac{y_j - y_{j-1}}{h} - y_{j-1}\right)^2 \quad (6)$$

When then, the essential differential equation is converting into discretization form. The backward difference formula is used to convert differential equation into discretizations form when the derivative term is replaced by a difference quotient for approximations. The interval of the IVP is equally partitioned into $(n+1)$ equidistant subintervals with the length $h = (b-a)/n+1$. Where $n = 9$ is a number of interior nodes. The IVP related parameters are as follows:

1. The step size $h = 0.5, h = 0.02$ for case 1 and case 2 respectively.
2. The initial condition in the first case is $i = 0$ for $t = 0$ and for the second case we assume that the charge on the capacitor is $-0.05C$ for $t = 0$. The interval between which the differential equation is solved is varying from case to case.

### 5.2 Computational time results

In Table ( 2), the average computational time (in seconds) by FPA algorithm of 50 different trials for the studied example computed by using the dimensional space, $D = 10$, are presented. It is clear that FPA algorithm kept a competitive computational time compared to the RK4 algorithm which constitutes an important advantage and a direct result of simple population update procedures.

| $Algorithm$ | $Time$ | |
|---|---|---|
| | **Case 1** | **Case 2** |
| $FPA$ | 4.687 | 4.563 |
| $RK4$ | 5.039 | 4.897 |

**Table 2** Average computational time by the used algorithms using 50 trials for the examples.

### 5.3 Statistical analysis

After establishing the superiority of FPA with respect to computational times, this part of the article deals with the statistical analysis of results obtained by proposed FPA and compared to RK4. These analyses should provide sufficient insight of how FPA outperforms RK4. Table (3) reports the Mean and STD of the difference between true optimum and computed optimum values achieved in the dimension $D = 10$. The results indicate that FPA was the best performing algorithm.

| $Algorithm$ | $Mean$ | | $STD$ | |
|---|---|---|---|---|
| | **Case 1** | **Case 2** | **Case 1** | **Case 2** |
| $FPA$ | 0.8334 | 0.8226 | 0.9570 | 0.8762 |
| $RK4$ | 0.8339 | 0.8198 | 0.9573 | 0.8956 |

**Table 3** Statistical results obtained for the studied examples over Dim=10.

### 5.4 Application examples

**Example 1 (Case 1: Constant Voltage):** We want to Find the current in the RC circuit for $t > 0$ that has an emf of $100V$, a resistance $R = 50W, C = 0.02F$ and no initial current. The time constant in this case is calculated via equation (4) gives $\tau = 1$ Seconds.
**Example 2 (Case 2: Variable Voltage and 2-mesh Circuits):** We want to Find the charge and the current for $t > 0$ in a series RC circuit where $R = 10W, C = 4 \times 10 - 3F$ and $E = 85 \cos 150tV$. Assume that when the switch is closed at $t = 0$, the charge on the capacitor is $-0.05C$. Since the voltage source is not constant, we cannot use the formulae in Eq.(3), and from the formula of Eq.(4) we have:

$$\frac{dq}{dt} + 25q = 8.5 \cos 150t$$

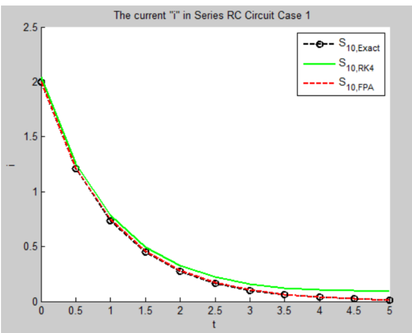
Now, we can solve this differential equation in $q$ using the linear ODE process so this gives us:

$$q(t) = 00092 \cos 150t + 0.055 \sin 150t - 0.059 \exp(-25t)$$

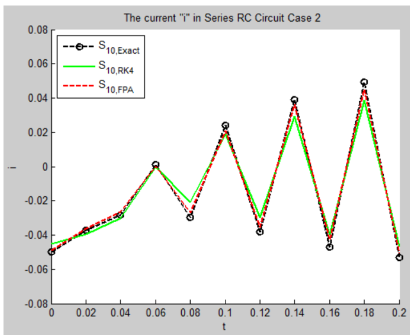
| $j$ | $t_j$ | Exact Results | FPA Results | RK4 Results |
|---|---|---|---|---|
| | | **Case 1** | | |
| 0 | 0.0000 | 2.0000 | 2.0002 | 2.0402 |
| 1 | 0.5000 | 1.2131 | 1.2163 | 1.2569 |
| 2 | 1.0000 | 0.7358 | 0.7465 | 0.7845 |
| 3 | 1.5000 | 0.4463 | 0.4583 | 0.4964 |
| 4 | 2.0000 | 0.2707 | 0.2803 | 0.3253 |
| 5 | 2.5000 | 0.1642 | 0.1696 | 0.2223 |
| 6 | 3.0000 | 0.0996 | 0.1041 | 0.1598 |
| 7 | 3.5000 | 0.0604 | 0.0637 | 0.1228 |
| 8 | 4.0000 | 0.0366 | 0.0387 | 0.1084 |
| 9 | 4.5000 | 0.0222 | 0.0240 | 0.0978 |
| 10 | 5.0000 | 0.0135 | 0.0145 | 0.0912 |
| | | **Case 2** | | |
| 0 | 00000 | -0.0498 | -0.0491 | -0.0451 |
| 1 | 0.0200 | -0.0371 | -0.0363 | -0.0395 |
| 2 | 0.0400 | -0.0282 | -0.0264 | -0.0304 |
| 3 | 0.0600 | 0.0011 | 0.0002 | 0.0000 |
| 4 | 0.0800 | -0.0297 | -0.0267 | -0.0207 |
| 5 | 0.1000 | 0.0239 | 0.0209 | 0.0189 |
| 6 | 0.1200 | -0.0382 | -0.0352 | -0.0297 |
| 7 | 0.1400 | 0.0392 | 0.0362 | 0.0292 |
| 8 | 0.1600 | -0.0470 | -0.0429 | -0.0399 |
| 9 | 0.1800 | 0.0493 | 0.0453 | 0.0388 |
| 10 | 0.2000 | -0.0533 | -0.0509 | -0.0469 |

**Table 4** Numerical results of the case 1 and case 2 examples for d=10.

The graphical representation of Table (4) results are visualized via Figure (3) that shows an exponential decay shape which means the current stops flowing as the capacitor becomes fully charged. A straightforward remark detection of difference between FPA performance and RK4's performance is very clear; hence FPA is better than RK4 because its results curve is very close to the exact results curve. We note that the graph can be very smooth by augmenting the number of steps $h$. The absolute error between exact, FPA and RK4 method results are made in Table (5) as well as their graphical representations which is given through Figure (4). In both representations of the absolute error, FPA method provides a very minimal absolute error compared to RK4 method.




(a) Case 1

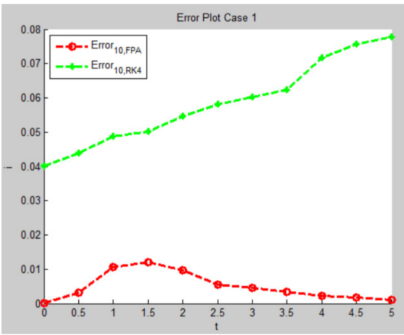



(b) Case 2

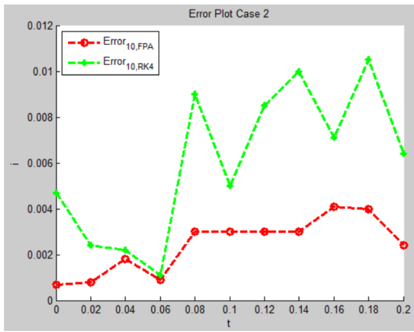
**Fig. 3**: Numerical Results

| $j$ | $t_j$ | FPA | RK4 |
|---|---|---|---|
| | **Case 1** | | |
| 0 | 0.0000 | 0.0002 | 0.0402 |
| 1 | 0.5000 | 0.0032 | 0.0438 |
| 2 | 1.0000 | 0.0107 | 0.0487 |
| 3 | 1.5000 | 0.0120 | 0.0501 |
| 4 | 2.0000 | 0.0096 | 0.0546 |
| 5 | 2.5000 | 0.0054 | 0.0581 |
| 6 | 3.0000 | 0.0045 | 0.0602 |
| 7 | 3.5000 | 0.0033 | 0.0624 |
| 8 | 4.0000 | 0.0021 | 0.0718 |
| 9 | 4.5000 | 0.0018 | 0.0756 |
| 10 | 5.0000 | 0.0010 | 0.0777 |
| | **Case 2** | | |
| 0 | 00000 | 0.0007 | 0.0047 |
| 1 | 0.0200 | 0.0008 | 0.0024 |
| 2 | 0.0400 | 0.0018 | 0.0022 |
| 3 | 0.0600 | 0.0009 | 0.0011 |
| 4 | 0.0800 | 0.0030 | 0.0090 |
| 5 | 0.1000 | 0.0030 | 0.0050 |
| 6 | 0.1200 | 0.0030 | 0.0085 |
| 7 | 0.1400 | 0.0030 | 0.0100 |
| 8 | 0.1600 | 0.0041 | 0.0071 |
| 9 | 0.1800 | 0.0040 | 0.0105 |
| 10 | 0.2000 | 0.0024 | 0.0064 |

**Table 5** Absolute Error.



(a) Case 1



(b) Case 2

**Fig. 4**: Absolute Error.

## 6 Conclusion

In this study, we applied the FPA to solve approximately the IVPs arises in electronic engineering field that is ODEs of the series RC circuit via a chosen examples in both voltage constant and voltage variable cases. After a comparison between the exact solutions and the algorithm outcomes with RK4 method results; FPA conduct to a precise solution with least errors compared to the RK4's. That is another argument given by the MAs in demonstrating such good proprieties. Behind the evaluations of various research papers, FPA was found as an algorithm having fabulous aptitude to solve a variety of optimization problems. As a future research, there are profound studies on FPA that will give hopeful results such as the use of more diverse parameters, more extensive comparison studies with more open sort of algorithms; for this reason these comparisons will enhance the qualities and back up the limitations of all the algorithms. Also, FPA should be looked into in several applications of engineering and industrial optimization problems.

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# Some Approximation Results About Positive Linear Operators

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**Abstract:** Positive linear operators are used to approximate functions. Recently, certain new bases polynomials have been introduced to extend Bernstein operators and obtain more accurate and sensitive numerical results. In this study, we focus and compare some recent positive linear operators to approximate functions. We also provide graphics to see convergence of these operators.

**Keywords:** Bersntein operators, Positive linear operators, Numerical analysis.

## 1 Introduction

For any function $f$ on $C[0,1]$ and $m \in \mathbb{N}$ (the set of natural numbers), the Bernstein operators constructed by

$$B_m(h;z) = \sum_{j=0}^{m} h\left(\frac{j}{m}\right) b_{m,j}(z) \qquad (z \in [0,1]),$$

where the Bernstein polynomials $b_{m,j}(z)$ of degree at most $m$ defined as

$$b_{m,j}(z) = \binom{m}{j} z^j (1-z)^{m-j} \qquad (j = 0,1,\cdots,m; z \in [0,1])$$

and

$$b_{m,j}(z) = 0 \qquad (j < 0 \text{ or } j > m).$$

The Bernstein polynomials satisfy the following recursive formula

$$b_{m,j}(z) = (1-z)b_{m-1,j}(z) + zb_{m-1,j-1}(z).$$

There are several generalization mentioned regarding Bernstein operators, for example,

(a) $\lambda$-Bernstein operators [17] with $\tilde{b}_{n,i}(\lambda;x)$ Bézier bases and shape parameter $\lambda$ (see [35]):

$$\begin{aligned}
\tilde{b}_{n,0}(\lambda;x) &= b_{n,0}(x) - \frac{\lambda}{n+1} b_{n+1,1}(x), \\
\tilde{b}_{n,i}(\lambda;x) &= b_{n,i}(x) + \frac{n-2i+1}{n^2-1}\lambda b_{n+1,i}(x) - \frac{n-2i-1}{n^2-1}\lambda b_{n+1,i+1}(x), \quad i = 1,2\ldots,n-1, \\
\tilde{b}_{n,n}(\lambda;x) &= b_{n,n}(x) - \frac{\lambda}{n+1} b_{n+1,n}(x). \qquad (1)
\end{aligned}$$

(b) Bernstein type operators by using continuously differentiable $\infty$ times function $\tau$ on $[0,1]$ [18].
(c) New variant of Bernstein operators [24]
(d) $(p,q)$-Bernstein operators [29].
(e) Stancu-type $\lambda$-Bernstein operators [34].
(f) Modified $U_n$ operators [15] and references therein.

(g) $\alpha$-Bernstein operators [19, 22] $p_{m,\gamma,j}^{(\alpha)}(z)$ denotes the $\alpha$-Bernstein-Schurer polynomials defined by

$$p_{1,\gamma,0}^{(\alpha)}(z) = 1 - z, \quad p_{1,\gamma,1}^{(\alpha)}(z) = z$$

and

$$\begin{aligned} p_{m,\gamma,j}^{(\alpha)}(z) &= \left[(1-\alpha)\, z \binom{m+\gamma-2}{j} + (1-\alpha)(1-z)\binom{m+\gamma-2}{j-2} \right. \\ &\left. +\alpha z (1-z)\binom{m+\gamma}{j}\right] z^{j-1}(1-z)^{m+\gamma-(j+1)} \qquad (m \geq 2). \end{aligned} \tag{2}$$

(h) Bivariate extension of $\alpha$-Bernstein-Durrmeyer operators [23].
(i) Kantorovich modifications of $\alpha$-Bernstein operators [26].
(j) $\lambda$-Bernstein-Schurer operators [32].
(k) Bivariate $\lambda$-Bernstein operators [5].
(l) $\lambda$-Bernstein-Kantorovich operators [20].
(m) Univariate and bivariate $\lambda$-Bernstein-Kantorovich operators [7].
(n) Genuine modified Bernstein-Durrmeyer operators [25].

We refer to see [13, 14, 21, 27, 28, 30] for certain further development, related concepts and statistical approximation.

## 2 Some positive linear operators

Assume that $\gamma$ is a non-negative integer. For any $m \in \mathbb{N}$, in the year 1962, Schurer [33] constructed the following linear positive operators

$$S_{m,\gamma} : C[0, 1+\gamma] \to C[0,1]$$

defined for all $h \in [0, 1+\gamma]$ by

$$S_{m,\gamma}(h; z) = \sum_{j=0}^{m+\gamma} h\left(\frac{j}{m}\right) s_{m,\gamma,j}(z) \qquad (z \in [0,1]) \tag{3}$$

where $s_{m,\gamma,j}(z)$ is a fundamental Bernstein-Schurer polynomials given by

$$s_{m,\gamma,j}(z) = \binom{m+\gamma}{j} z^j (1-z)^{m+\gamma-j} \qquad (j = 0, 1, \cdots, m+\gamma). \tag{4}$$

We remark that the choice of $\gamma = 0$ in the operators $S_{m,\gamma}(h; z)$ gives Bernstein operators $B_m(h; z)$ and, in this case, the polynomials $p_{m,\gamma,j}(z)$ reduces to the fundamental polynomials of Bernstein $b_{m,j}(z)$.

Consider a non-negative integer $\gamma$. For any $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ is fixed. Following $\alpha$-Bernstein-Schurer operators $\Phi_{m,\gamma,\alpha} : C[0, 1+\gamma] \to C[0,1]$ were defined for any $h$ in $C[0, 1+\gamma]$ [2] by

$$\Phi_{m,\gamma,\alpha}(h; z) = \sum_{j=0}^{m+\gamma} h_j p_{m,\gamma,j}^{(\alpha)}(z) \qquad (z \in [0,1]), \tag{5}$$

where

$$h_j = h\left(\frac{j}{m}\right).$$

In the last operators (5), Note that our $\alpha$-Bernstein-Schurer operators are a class of linear positive operators for any $\alpha \in [0,1]$. We assume throughout unless stated otherwise that $\alpha$ in $[0,1]$. For some values of $m$ and $j$, we obtain

$$\begin{aligned} p_{2,\gamma,0}^{(\alpha)}(z) &= (1-\alpha z)(1-z)^{\gamma+1}; \\ p_{2,\gamma,1}^{(\alpha)}(z) &= (1-\alpha z)\gamma z(1-z)^{\gamma} + 2\alpha z(1-z)^{\gamma+1}; \\ p_{3,\gamma,0}^{(\alpha)}(z) &= (1-\alpha z)(1-z)^{\gamma+2}. \end{aligned}$$

In the following theorem, monotonicity preserving property of the $\alpha$-Bernstein-Schurer operators is provided [2]:

**Theorem 1.** *Suppose $h \in C[0,1]$. If the function $h(z)$ is monotonically increasing or decreasing on the interval $[0,1]$, so are all of its $\alpha$-Bernstein-Schurer operators for $\alpha \in [0,1]$.*

Convexity preserving property of $\alpha$-Bernstein-Schurer operators is given in the next theorem [2].

**Theorem 2.** *Suppose* $h \in C[0,1]$. *If the function* $h(z)$ *is convex on the interval* $[0,1]$, *so are all of its* $\alpha$*-Bernstein-Schurer operators for* $\alpha \in [0,1]$.

By the following theorem we give uniform convergence of some positive linear operators.

**Theorem 3.** *For any* $\alpha \in [0,1]$, *then* $L_{m,\alpha}(h)$ *converge uniformly to* $h$ *on* $[0,1]$, *that is,*

$$\lim_{m\to\infty} \|L_{m,\alpha}(h) - h\| = 0,$$

*where* $L_{m,\alpha} = \Phi_{m,\gamma,\alpha},\ T_{m,\alpha},\ K^{\beta,\theta}_{m,\alpha},\ K_{m,\alpha}$.

*Proof:* Taking into account moments of Bernstein type operators we have

$$L_{m,\alpha}(e_0) = e_0 \text{ as } m \to \infty, \quad L_{m,\alpha}(e_1;x) = e_1 \text{ as } m \to \infty$$

and similarly $L_{m,\alpha}(e_2) = e_2$ as $m \to \infty$. Hence, by the Korovkin theorem, we obtain

$$\lim_{m\to\infty} \|L_{m,\alpha}(h) - h\| = 0.$$

This comletes the proof. □

Finally, we use *MATHEMATICA* to give an application of Korovkin type theorem, and we consider two functions $g(x) = \cos(\pi x)$ and $g(x) = x^3 \sin(\pi x)$ to demonstrate convergence of some $\alpha$ Berntein operators. In Figure 1 and Figure 2 we provide convergence for operators $\Phi_{m,\gamma,\alpha}$ and $K_{m,\alpha}$, respectively. We also note that, this paper is based on the results in [2, 6, 9], this is why we refere these papers for further literature.

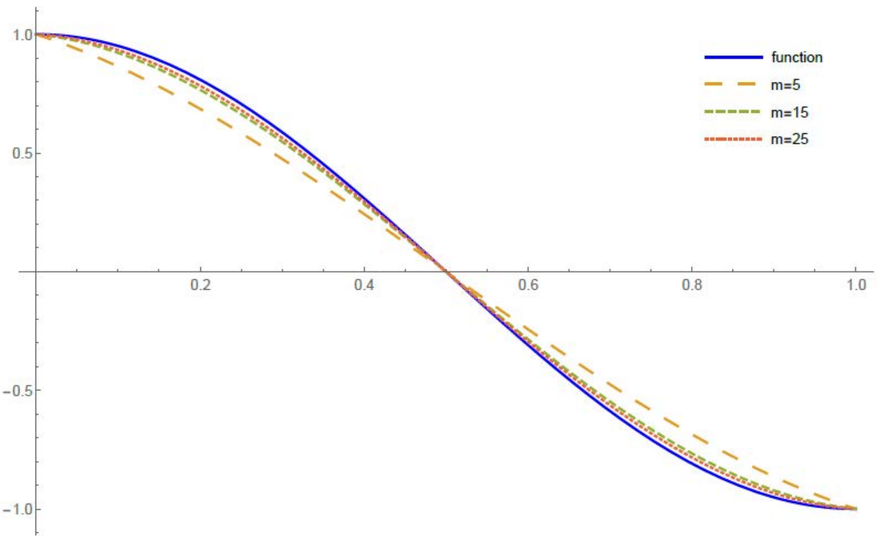


**Fig. 1**: Function 1; Convergence of operators $\Phi_{m,\gamma,\alpha}$ for some $m$ values

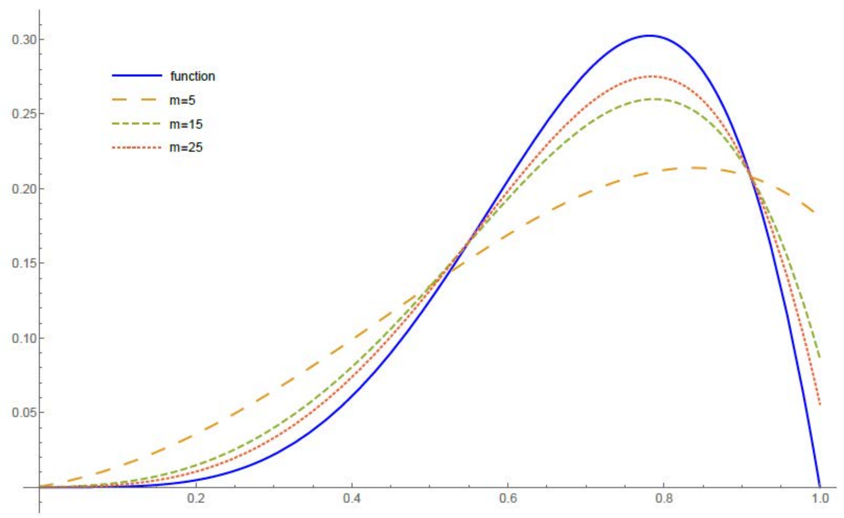


**Fig. 2**: Function 2; Convergence of operators $K_{m,\alpha}$ for some $m$ values

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# Recent Results on $\alpha$-Bernstein Operators

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**Abstract:** Shape parameters have an important role both in Computer Aided Geometric Design and Approximation Theory. The $\alpha$ shape parameter is used to construct many positive linear operators. In this work, we summarize recent $\alpha$-Bernstein type operators in literature. We also use *MATHEMATICA* to give an application of Korovkin type theorems for $\alpha$-Bernstein type operators.

**Keywords:** Bernstein operators, Shape parameter $\alpha$, MATHEMATICA, Approximation properties, Durrmeyer operators, Stancu operators.

## 1 Introduction

Chen et al. [12] constructed the $\alpha$-Bernstein operators as follows:

$$T_{m,\alpha}\left(g;y\right)=\sum_{i=0}^{m} g\left(i/m\right)p_{m,i}^{(\alpha)}\left(y\right) \qquad \left(y\in[0,1]\right), \tag{1}$$

where

$$p_{1,0}^{(\alpha)}\left(y\right)=1-y, \quad p_{1,1}^{(\alpha)}(y)=y$$

and

$$\begin{aligned} p_{m,i}^{(\alpha)}(y) &= \left[(1-\alpha)\,y\binom{m-2}{i}+(1-\alpha)\,(1-y)\binom{m-2}{i-2}\right. \\ &\quad \left.+\alpha y\,(1-y)\binom{m}{i}\right]y^{i-1}\,(1-y)^{m-i-1} \qquad (m\geq 2), \end{aligned}$$

and also or $g\in C[0,1]$, $\alpha\in[0,1]$ is fixed and $m\in\mathbb{N}$.

Recall that $p_{m,i}^{(\alpha)}$ in the relation (1) is called $\alpha$-Bernstein polynomials of order $m$ and the binomial coefficients

$$\binom{a}{b}=\begin{cases}\frac{a!}{b!(a-b)!} & (0\leq b\leq a),\\ 0 & (otherwise).\end{cases}$$

For $\alpha=1$, (1) is reduced to classical Bernstein operators [6].

Chen [5] and Goodman and Sharma [1], independently, introduced the operators $U_m$ (we can also call it genuine Bernstein-Durrmeyer operators) acting from $L_B(0,1)$ into $\Pi_m$, defined by

$$\begin{aligned} U_m(f,y) &= (m-1)\sum_{i=1}^{m-1}\left(\int_0^1 f(t)p_{m-2,i-1}(t)dt\right)p_{m,i}(y)+y^m f(1) \\ &\quad +(1-y)^m f(0) \end{aligned}$$

for all $f\in L_B(0,1)$, where $p_{m,i}(y)$ $(m,i\in\mathbb{N})$ is considered by

$$p_{m,i}(y)=\binom{m}{i}y^i(1-y)^{m-i} \quad (0\leq y\leq 1, 0\leq i\leq m).$$

 

The above operators are limits of the Bernstein-Durrmeyer operators with Jacobi weights, $M_m^{c,d}$ for $c, d > -1$ which was studied by Păltănea [3], that is,

$$U_m(f) = \lim_{c \to -1, d \to -1} M_m^{c,d}(f) \qquad (f \in C[0,1]),$$

where $C[0,1]$ denotes the space of functions which are continuous on $[0,1]$ and

$$M_m^{c,d}(f, y) = \sum_{i=0}^{m} \frac{\int_0^1 f(t) t^c (1-t)^d p_{m,i}(t) dt}{\int_0^1 t^c (1-t)^d p_{m,i}(t) dt} p_{m,i}(y).$$

Păltănea [4] presented a generalization of the operators $U_m$ with the help of $\rho > 0$, namely, genuine $\rho$-Bernstein-Durrmeyer operators and denoted by $U_m^\rho$. For any $f \in C[0,1]$, in the same paper, he showed that the classical Bernstein operators are the limits of the operators $U_m^\rho$ and also obtained a Voronovskaja type result. Gonska and Păltănea [2] proved that the operators $U_m^\rho$ preserves convexity of all orders and also obtained the degree of simultaneous approximation.

For $m \in \mathbb{N}$ and $\rho > 0$, the functional (see [4])

$$F_{m,i}^\rho : C[0,1] \to \mathbb{R}$$

is defined by

$$F_{m,i}^\rho(g) = \int_0^1 \mu_{m,i}^\rho(t) g(t)\, dt \qquad (i = 1, 2, \ldots, m-1), \tag{2}$$

$$F_{m,0}^\rho(g) = g(0), \quad F_{m,m}^\rho(g) = g(1),$$

where $\mu_{m,i}^\rho(t)$ in (2) is given by the formula

$$\mu_{m,i}^\rho(t) = \frac{t^{i\rho-1}(1-t)^{(m-i)\rho-1}}{B(i\rho, (m-i)\rho)}$$

and the Euler's beta function in the last equality is defined by

$$B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt \quad (a, b > 0).$$

Assume that $\theta$ and $\beta$ are two real parameters satisfying $0 \leq \theta \leq \beta$. In view of $\alpha$-Bernstein operators, for $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$ is fixed and given a function $g \in C[0,1]$, we define the operators $U_{m,\alpha}^{\beta,\theta,\rho}$ (or, genuine $(\alpha, \rho)$-Durrmeyer-Stancu operators) by

$$U_{m,\alpha}^{\beta,\theta,\rho}(g; x) = \sum_{i=0}^{m} F_{m,i}^{\beta,\theta,\rho}(g)\, p_{m,i}^{(\alpha)}(x),$$

where

$$F_{m,i}^{\beta,\theta,\rho}(g) = \int_0^1 \mu_{m,i}^\rho(t) g\left(\frac{mt+\theta}{m+\beta}\right) dt$$

for $i = 1, 2, \ldots, m-1$, $F_{m,0}^{\beta,\theta,\rho}(g) = g\left(\frac{\theta}{m+\beta}\right)$ and $F_{m,1}^{\beta,\theta,\rho}(g) = g\left(\frac{m+\theta}{m+\beta}\right)$. Consequently, we can re-write our operators $U_{m,\alpha}^{\beta,\theta,\rho}$ as

$$\begin{aligned} U_{m,\alpha}^{\beta,\theta,\rho}(g; x) &= \sum_{i=1}^{m-1} \int_0^1 \left[\frac{t^{i\rho-1}(1-t)^{(m-i)\rho-1}}{B(i\rho, (m-i)\rho)} g\left(\frac{mt+\theta}{m+\beta}\right) dt\right] p_{m,i}^{(\alpha)}(x) \\ &\quad + g\left(\frac{\theta}{m+\beta}\right) p_{m,0}^{(\alpha)}(x) + g\left(\frac{m+\theta}{m+\beta}\right) p_{m,m}^{(\alpha)}(x). \end{aligned} \tag{3}$$

For the choice of $\theta = 0$ and $\beta = 0$, the operators defined by (3) reduce to the operators $U_{m,\alpha}^\rho(g; x)$ which were studied in [8]. In addition, if $\rho = 1$ then one get the genuine $\alpha$-Bernstein-Durrmeyer operators $U_{m,\alpha}$ defined in [7]. If one take $\rho = 1$, $\alpha = 1$, $\theta = 0$ and $\beta = 0$, then one obtain genuine Bernstein-Durrmeyer operators.

## 2 Recent results for $\alpha$-Bernstein type operators

**Theorem 1.** *If $g$ is continuous on $[0,1]$, for any $\alpha \in [0,1]$, then $T_{m,\alpha}^\rho(g)$ converge uniformly to $g$ on $[0,1]$, that is,*

$$\lim_{m \to \infty} \left\| T_{m,\alpha}^\rho(g) - g \right\| = 0,$$

*where $T_{m,\alpha}^\rho = T_{m,\alpha}, \ M_m^{c,d}, \ U_{m,\alpha}^{\beta,\theta,\rho}, \ U_{m,\alpha}^\rho$.*

*Proof:* Using the moments of mentioned operators we have

$$\lim_{m\to\infty} T^{\rho}_{m,\alpha}(e_0) = e_0, \qquad \lim_{m\to\infty} T^{\rho}_{m,\alpha}(e_1;x) = e_1$$

and similarly $\lim_{m\to\infty} \left\| T^{\rho}_{m,\alpha}(e_2) - e_2 \right\| = 0$. Hence, by the Korovkin theorem, we obtain

$$\lim_{m\to\infty} \left\| T^{\rho}_{m,\alpha}(g) - g \right\| = 0.$$

□

We also want to give a Voronovskaja type theorem for these kinds of operators. Voronovskaja type theorem provides a finite expression for the following

$$\lim_{m\to\infty} m\{T^{\rho}_{m,\alpha}(g;x) - g(x)\},$$

where $T^{\rho}_{m,\alpha} = T_{m,\alpha}, \ \ M^{c,d}_{m}, \ \ U^{\beta,\theta,\rho}_{m,\alpha}, \ \ U^{\rho}_{m,\alpha}$. For instance consider the following Voronovskaja type theorem for operators $U^{\beta,\theta,\rho}_{m,\alpha}$ [17]:

**Theorem 2.** *For every $g \in C_B[0,1]$ such that $g', g'' \in C_B[0,1]$. Then, for each $x \in [0,1]$ and $\rho > 0$, we have*

$$\lim_{m\to\infty} m\{U^{\beta,\theta,\rho}_{m,\alpha}(g;x) - g(x)\} = (\theta - \beta x)g^{'}(x) + \frac{\rho+1}{2\rho}x(1-x)g^{''}(x)$$

*uniformly on $[0,1]$, where $C_B[0,1]$ denotes the set of all real-valued bounded and continuous functions defined on $[0,1]$.*

Finally, we use *MATHEMATICA* to give an application of Korovkin type theorem, and we consider the function $g(x) = x^4\cos(\pi x) + \sin(2\pi x)$ to demonstrate convergence and error of approximation of discussed operators. In Figure 1 and Figure 2 we provide convergence and error of convergence of operators $U^{\beta,\theta,\rho}_{m,\alpha}$, respectively. It is also possible to have convergnce graphics for other $\alpha - \rho$ type Bernstein operators. We also note that, this paper is based on the results in [3, 4, 7, 8, 12, 17], this is why we refere these papers for further literature.

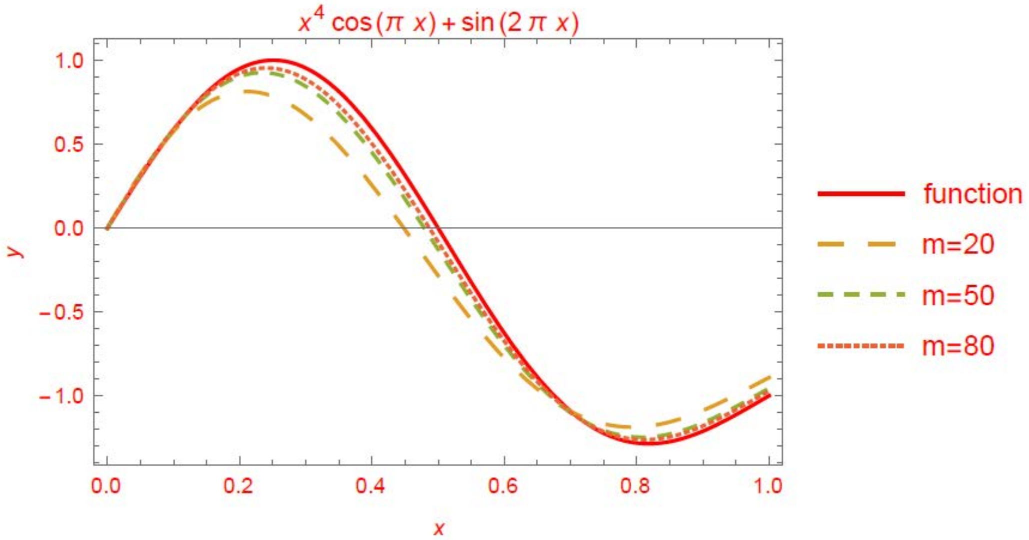


**Fig. 1**: Convergence of operators for some $m$ values

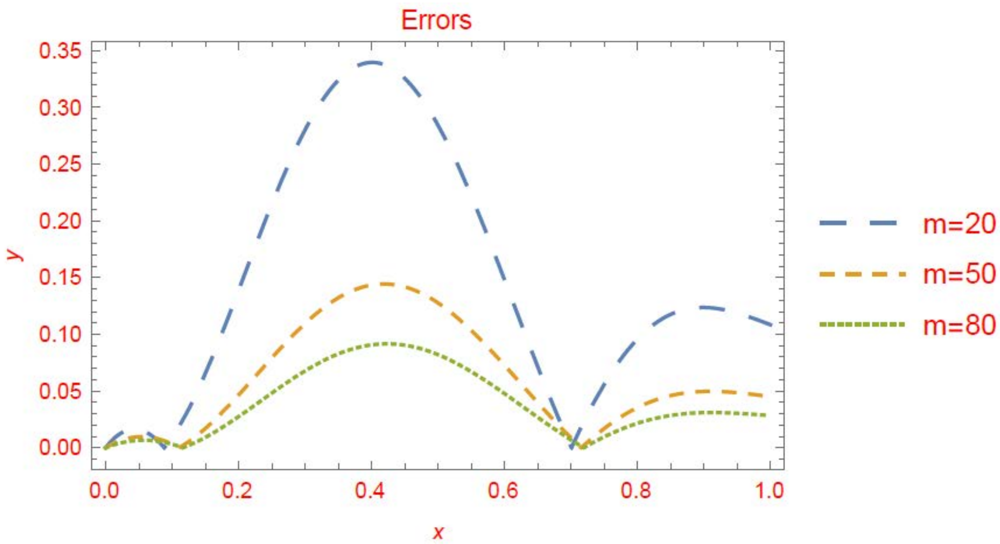


**Fig. 2**: Error of convergence of operators for some $m$ values

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# Compact Multiplication Operators on Semicrossed Products

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**Abstract:** We characterize the compact multiplication operators on a semicrossed product $C_0(X) \times_\phi \mathbb{Z}_+$ in terms of the corresponding dynamical system. We also characterize the compact elements of this algebra and determine the ideal they generate.

**Keywords:** Compact elements, Equicontinuity, Multiplication operators, Non-selfadjoint Operator Algebras, Recurrent points, Semicrossed products, Wandering points.

## 1 Introduction

Let $\mathcal{A}$ be a Banach algebra and $a, b \in \mathcal{A}$. The map $M_{a,b} : \mathcal{A} \to \mathcal{A}$ given by $M_{a,b}(x) = axb$ is called a *multiplication operator*. Properties of compact multiplication operators have been investigated since 1964 when Vala published his work "On compact sets of compact operators" [13]. Let $\mathcal{X}$ be a normed space and $\mathcal{B}(\mathcal{X})$ the algebra of all bounded linear maps from $\mathcal{X}$ into $\mathcal{X}$. Vala proved that a non-zero multiplication operator $M_{a,b} : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ is compact if and only if the operators $a, b \in \mathcal{B}(\mathcal{X})$ are both compact. Also, in [14] Vala defines an element $a$ of a normed algebra to be *compact* if the mapping $x \mapsto axa$ is compact. This concept enabled the study of compactness properties of elements of abstract normed algebras. Ylinen in [15] characterized the compact elements in abstract C*-algebras.

Compactness questions have also been considered in the more general framework of elementary operators by Fong and Sourour [4], Mathieu [7] and Timoney [12].

Akemann and Wright [1] characterized the weakly compact multiplication operators on $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. Saksman and Tylli [10, 11] and Johnson and Schechtman [5] studied weak compactness of multiplication operators in a Banach space setting.

Moreover, strictly singular multiplication operators are studied by Lindström, Saksman and Tylli [6] and Mathieu and Tradacete [8].

Compactness properties of multiplication operators on nest algebras, a class of non selfadjoint operator algebras, are studied by Andreolas and Anoussis in [2].

In this work we study multiplication operators on a semicrossed product $C_0(X) \times_\phi \mathbb{Z}_+$ where $X$ is a locally compact metrizable space, and $\phi : X \to X$ a homeomorhism. We characterize the compact multiplication operators in terms of the corresponding dynamical system. As a consequence, we obtain a characterization of the compact elements of the semicrossed product. We also characterize the ideal generated by the compact elements.

## 2 Semicrossed products

Let $X$ be a locally compact metrizable space and $\phi : X \to X$ a homeomorphism. The pair $(X, \phi)$ is called a dynamical system. An action of $\mathbb{Z}_+$ on $C_0(X)$ by isometric $*$-automorphisms $\alpha_n$, $n \in \mathbb{Z}_+$ is obtained by defining $\alpha_n(f) = f \circ \phi^n$. We write the elements of the Banach space $\ell^1(\mathbb{Z}_+, C_0(X))$ as formal series $A = \sum_{n \in \mathbb{Z}_+} U^n f_n$ with the norm given by $\|A\|_1 = \sum \|f_n\|_{C_0(X)}$. The multiplication on $\ell^1(\mathbb{Z}_+, C_0(X))$ is defined by setting

$$U^n f U^m g = U^{n+m}(\alpha^m(f)g)$$

and extending by linearity and continuity. With this multiplication, $\ell^1(\mathbb{Z}_+, C_0(X))$ is a Banach algebra.

The Banach algebra $\ell^1(\mathbb{Z}_+, C_0(X))$ can be faithfully represented as a (concrete) operator algebra on a Hilbert space. This is achieved by assuming a faithful action of $C_0(X)$ on a Hilbert space $\mathcal{H}_0$. Then, we can define a faithful contractive representation $\pi$ of $\ell_1(\mathbb{Z}_+, C_0(X))$ on the Hilbert space $\mathcal{H} = \mathcal{H}_0 \otimes \ell^2(\mathbb{Z}_+)$ by defining $\pi(U^n f)$ as

$$\pi(U^n f)(\xi \otimes e_k) = \alpha^k(f)\xi \otimes e_{k+n}.$$

The *semicrossed product* $C_0(X) \times_\phi \mathbb{Z}_+$ is the closure of the image of $\ell^1(\mathbb{Z}_+, C_0(X))$ in $\mathcal{B}(\mathcal{H})$ in the representation just defined, where $\mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators on $\mathcal{H}$. We will denote the semicrossed product $C_0(X) \times_\phi \mathbb{Z}_+$ by $\mathcal{A}$ and an element $\pi(U^n f)$ of $\mathcal{A}$ by $U^n f$ to simplify the notation. We refer to [9] and [3], for more information about the semicrossed product.

 

For $A = \sum_{n\in\mathbb{Z}_+} U^n f_n \in \ell^1(\mathbb{Z}_+, C_0(X))$ we call $f_n \equiv E_n(A)$ the $n$th *Fourier coefficient* of $A$. The maps $E_n : \ell^1(\mathbb{N}_+, C_0(X)) \to C_0(X)$ are contractive in the (operator) norm of $\mathcal{A}$, and therefore they extend to contractions $E_n : \mathcal{A} \to C_0(X)$.

## 3 Compact multiplication operators on semicrossed products

Let $(X, \phi)$ be a dynamical system. Then, a point $x \in X$ is called *recurrent* if there exists a strictly increasing sequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$, such that $\lim_{k\to\infty} \phi^{n_k}(x) = x$. The set of the recurrent points of $(X, \phi)$ will be denoted by $X_r$. We will denote by $X_i$ the set of the isolated points of $X$ and by $X_a$ the set of the accumulation points of $X$.

To prove our main result, we first prove the following propositions:

**Proposition 3.1.** *Let $M_{A,B} : \mathcal{A} \to \mathcal{A}$ be a compact multiplication operator, where $A, B \in \mathcal{A}$ and $E_m(A) = f_m$, $E_m(B) = g_m$, for all $m \in \mathbb{Z}_+$. Then,*

1. $(f_m \circ \phi^{n+l} g_n)(X_a) = \{0\}$*, for all $m, n, l \in \mathbb{Z}_+$.*
2. $\lim_{l\to\infty}(f_m \circ \phi^{n+l} g_n)(x) = 0$*, for all $m, n \in \mathbb{Z}_+$ and $x \in X_i$.*
3. *the sequence $\{f_m \circ \phi^{n+l} g_n\}_{l\in\mathbb{Z}_+}$ is pointwise equicontinuous, for all $m, n \in \mathbb{Z}_+$.*

**Proposition 3.2.** *Let $m, n \in \mathbb{Z}_+$ and $A = U^m f$, $B = U^n g \in \mathcal{A}_1$. Then, the multiplication operator $M_{A,B} : \mathcal{A} \to \mathcal{A}$ is compact if the following assertions are valid.*

1. $(f \circ \phi^{n+l} g)(X_a) = \{0\}$*, for all $l \in \mathbb{Z}_+$,*
2. $\lim_{l\to\infty}(f \circ \phi^{n+l} g)(x) = 0$*, for all $x \in X_i$,*
3. *The sequence $\{f \circ \phi^{n+l} g\}_{l\in\mathbb{Z}_+}$ is pointwise equicontinuous.*

Let us decsribe briefly the idea of the proof of Proposition 3.2. The proof is devided in three steps: In the first step we construct an approximation of $M_{A,B}$ by multiplication operators $M_{A_k,B_k}$ where $A_k = U^m f_k$ and $B_k = U^n g_k$ and $f_k, g_k$ are compactly supported. Hence, to prove that $M_{A,B}$ is compact, it suffices to show that there exists a natural number $k_0$ such that $M_{A_k,B_k}$ is compact, for all $k > k_0$.

In the second step we construct finitely supported functions $\tilde{g}_k$ for all $k > k_0$, with the property that

$$f_k \circ \phi^{n+l} g_k = f_k \circ \phi^{n+l} \tilde{g}_k,$$

for all $l \in \mathbb{Z}_+$. It follows that $M_{A_k,B_k} = M_{A_k,\tilde{B}_k}$, where $\tilde{B}_k = U^n \tilde{g}_k$. It the third step we prove that there exists $L_0 \in \mathbb{N}$ such that

$$M_{A_k,\tilde{B}_k}(U^l h) = 0,$$

for all $l \geq L_0$ and $h \in C_0(X)$. Since $\tilde{g}_k$ has finite support, the operator $M_{A_k,\tilde{B}_k}$, is a finite rank operator and hence compact.

We now recall the concept of the $k$th arithmetic mean of an element of the semicrossed product, that we will need in the sequel. Let $A$ be an element of the semicrossed product $\mathcal{A}$. We consider the sequence $\{U^n f_n\}_{n\in\mathbb{Z}_+} \subseteq \mathcal{A}$, where $f_n = E_n(A)$, for $n \in \mathbb{Z}_+$. We note that the series $\sum_{n\in\mathbb{Z}_+} U^n f_n$ does not converge to $A$ in general. The $k$th arithmetic mean of $A$ is defined to be the element $A_k = \frac{1}{k+1}\sum_{l=0}^{k} S_l$, where $S_l = \sum_{n=0}^{l} U^n f_n$. Then, the sequence $\{A_k\}_{k\in\mathbb{Z}_+}$ is norm convergent to $A$ [9, p. 524].

The following is our main result:

**Theorem 3.3.** *Let $A, B \in \mathcal{A}$ and $E_m(A) = f_m \in C_0(X)$, $E_m(B) = g_m \in C_0(X)$, for all $m \in \mathbb{Z}_+$. The following statements are equivalent.*

1. *The multiplication operator $M_{A,B} : \mathcal{A} \to \mathcal{A}$ is compact.*
2. *The following assertions are valid, for all $m, n \in \mathbb{Z}_+$.*

(a) $(f_m \circ \phi^{n+l} g_n)(X_a) = \{0\}$*, for all $l \in \mathbb{Z}_+$.*
(b) $\lim_{l\to\infty}(f_m \circ \phi^{n+l} g_n)(x) = 0$*, for all $x \in X_i$.*
(c) *The sequence $\{f_m \circ \phi^{n+l} g_n\}_{l\in\mathbb{Z}_+}$ is pointwise equicontinuous.*

*Proof:*

The condition (1) implies the condition (2) by Proposition 3.1. We will show the opposite direction.

If $A = \sum_{m=0}^{p} U^m f_m$ and $B = \sum_{n=0}^{q} U^n g_n$, for $p, q \in \mathbb{Z}_+$, we have

$$M_{A,B} = \sum_{m=0}^{p}\sum_{n=0}^{q} M_{U^m f_m, U^n g_n}$$

and the assertion folows from Proposition 3.2.

If $A, B \in \mathcal{A}$ and $k \in \mathbb{Z}_+$, we denote by $A_k$ and $B_k$ the $k$th arithmetic mean of $A$ and $B$ respectively. Since the Fourier coefficients of $A$ and $B$ satisfy the condition (2), the Fourier coefficients of $A_k$ and $B_k$ satisfy the condition (2) as well. Thus, the operator $M_{A_k,B_k}$ is compact, for all $k \in \mathbb{Z}_+$. The operator $M_{A,B}$ is the norm limit of the sequence $\{M_{A_k,B_k}\}_{k\in\mathbb{Z}_+}$ and hence is compact.

□

As a corollary of the above theorem, we obtain the following characterization of the compact elements of the algebra $\mathcal{A}$.

**Corollary 3.4.** *Let $A \in \mathcal{A}$ and $E_m(A) = f_m \in C_0(X)$, for all $m \in Z_+$. Then, $A$ is a compact element of $\mathcal{A}$, if and only if the following conditions are satisfied, for all $m, n \in \mathbb{Z}_+$.*

1. $(f_m \circ \phi^{n+l} f_n)(X_a) = \{0\}$, *for all* $l \in \mathbb{Z}_+$.
2. $f_m(X_r) = \{0\}$.
3. *The sequence* $\{f_m \circ \phi^{n+l} f_n\}_{l \in \mathbb{Z}_+}$ *is pointwise equicontinuous.*

Let us see now how Theorem 3.3 applies to two special cases.

**Corollary 3.5.** *Let $X$ be a discrete space, $A, B \in \mathcal{A}$ and $E_m(A) = f_m$, $E_m(B) = g_m$, for all $m \in \mathbb{Z}_+$. Then, the following are equivalent.*

1. *The multiplication operator* $M_{A,B} : \mathcal{A} \to \mathcal{A}$ *is compact.*
2. $(f_m \circ \phi^{n+l} g_n)(X_r) = \{0\}$, *for all* $m, n, l \in \mathbb{Z}_+$.
3. $\lim_{l\to\infty}(f_m \circ \phi^{n+l} g_n)(x) = 0$, *for all* $x \in X$.

**Corollary 3.6.** *Let $X$ be a space without isolated points, $A, B \in \mathcal{A}$ and $E_m(A) = f_m$, $E_m(B) = g_m$, for all $m \in \mathbb{Z}_+$. Then, the following are equivalent.*

1. *The multiplication operator* $M_{A,B} : \mathcal{A} \to \mathcal{A}$ *is compact.*
2. $(f_m \circ \phi^{n+l} g_n)(X) = \{0\}$ *for all* $l, m, n \in \mathbb{N}$.
3. $M_{A,B} = 0$.

We would like to note that the equicontinuity condition $2c$ of Theorem 3.3 follows from condition $2a$ if $X$ has no isolated points and is automatically satisfied if $X$ is discrete. However, in the general case condition $2c$ cannot be omitted as we show in the following example.

**Example 3.7.** We consider the dynamical system $(X, \phi)$ where

$$X = \{0\} \cup \{x_n\}_{n\in\mathbb{Z}} \cup \{2\}, \qquad x_n = \begin{cases} \frac{1}{|n|+1}, & n < 0 \\ 2 - \frac{1}{n+1}, & n \geq 0 \end{cases}$$

and $\phi$ is the homeomorphism

$$\phi(0) = 0, \quad \phi(x_n) = x_{n-1}, \quad \phi(2) = 2.$$

We define the elements $A = U^1 f$ and $B = U^1 g$ of the semicrossed product $\mathcal{A}$ by the following formulae.

$$f(x) = \begin{cases} 1, & x = 1 \\ 0, & \text{else} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1, & x > 1 \\ 0, & x \leq 1 \end{cases}.$$

We observe that, $(f \circ \phi^{1+l} g)(X_a) = \{0\}$, for all $l \in \mathbb{Z}_+$ and $\lim_{l\to\infty}(f \circ \phi^{1+l} g)(x) = 0$, for all $x \in X_i$. However, the sequence $\{f \circ \phi^{1+l} g\}_{l\in\mathbb{Z}_+}$ is not equicontinuous at $x = 2$ and the multiplication operator $M_{A,B} : \mathcal{A} \to \mathcal{A}$ is not compact.

In the following theorem, we characterize the ideal generated by the set of compact elements of the semicrossed product $\mathcal{A}$. Recall that $Y \subseteq X$ is called wandering if the sets $\phi^{-1}(Y), \phi^{-2}(Y), \ldots$ are pairwise disjoint. A point $x \in X$ is called wandering if it possesses an open wandering neighborhood. We will denote by $X_w$ the set of wandering points of $X$. It is clear that $X_w$ is the the union of all open wandering subsets of $X$.

**Theorem 3.8.** *The ideal generated by the compact elements of $\mathcal{A}$ is the set*

$$\{A \in \mathcal{A} \mid E_n(A)(X \setminus X_w) = \{0\}, \ \forall n \in \mathbb{Z}_+ \text{ and } E_0(A)(X_a) = \{0\}\}.$$

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# A New Subclass of Harmonic Univalent Functions and its Some Properties

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**Abstract:** Harmonic functions are a classic title in the class of geometric functions. Many researchers have studied these function classes from past to present, and since it has a wide range of applications, it is still a popular class. In this study, we will examine harmonic univalent functions, a subclass of harmonic functions.

**Keywords:** Harmonic, Linear Operator, Univalent

## 1 Introduction

Harmonic functions are a classic title in the class of geometric functions. Many researchers have studied these function classes from past to present, and since it has a wide range of applications, it is still a popular class. In this study, we will examine harmonic univalent functions, a subclass of harmonic functions.

Let $H$ denote the family of continuous complex valued harmonic functions which are harmonic in the open unit disk $\mathbb{D} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $A$ be the subclass of $H$ consisting of functions which are analytic in $\mathbb{D}$. A function harmonic in $\mathbb{D}$ may be written as $f = h + \overline{g}$, where $h$ and $g$ are members of $A$. We call $h$ the analytic part and $g$ co-analytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $\mathbb{D}$ is that $\left|h'(z)\right| > \left|g'(z)\right|$ (see Clunie and Sheil-Small [3]). To this end, without loss of generality, we may write

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=2}^{\infty} b_n z^n. \tag{1}$$

Let $SH$ denote the family of functions $f = h + \overline{g}$ which are harmonic, univalent, and sense-preserving in $\mathbb{D}$ for which $f(0) = f_z(0) - 1 = 0$. Thus $SH$ contains contains the standard class $S$ of analytic univalent functions. Although the analytic part $h$ of a function $f \in SH$ is locally univalent, it will become apparent that it need not be univalent. One shows easily that the sense-preserving property implies that $|b_1| < 1$ .

The subclass $SH^0$ of $SH$ consists of all functions in $SH$ which have the additional property $f_{\bar{z}}(0) = b_1 = 0$.

In 1984 Clunie and Sheil-Small [3] investigated the class $SH$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on $SH$ and its subclasses. For example Avci and Zlotkiewicz [1], Flett [4], Jahangiri [5], Jahangiri et. al. [6], Silverman [8], Uralegaddi and Somanatha [9], Cho and Srivastava [2] etc.

Also note that $SH$ reduces to the class $S$ of normalized analytic univalent functions in $\mathbb{D}$, if the co-analytic part of $f$ is identically zero.

For $f \in S$, the differential operator $D^k$ $(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ of $f$ was introduced by Sălăgean [7]. For $f = h + \overline{g}$ given by (1), Jahangiri et al. [6] defined the modified Sălăgean operator of $f$ as

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)},$$

where

$$D^n h(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad D^n g(z) = \sum_{k=2}^{\infty} k^n b_k z^k.$$

Next for functions $f \in A$ , Cho and Srivastava [2] defined multiplier transformations.

For $f = h + \overline{g}$ given by (1), we introduced a differentian operator defined as follows

$$I_\beta^0 f(z) = D^0 f(z) = h(z) + \overline{g(z)},$$

$$I_\beta^n f(z) = z + \sum_{k=2}^{\infty} \left(\frac{\beta k + 1}{1 + \beta}\right)^n a_k z^k + (-1)^n \sum_{k=1}^{\infty} \left(\frac{\beta k - 1}{1 + \beta}\right)^n \overline{b_k z^k} \tag{2}$$

where $n \in \mathbb{N}_0$ $\beta \geq 1$. Denote by $SH(\beta, n, \mu)$ the subclass of $SH$ consisting of functions $f$ of the form (1) that satisfy the condition

Denote by $SH(\beta, n, \mu)$ the subclass of $SH$ consisting of functions $f$ of the form (1) that satisfy the condition

$$Re\left(\frac{I_\beta^{n+1} f(z)}{I_\beta^n f(z)}\right) \geq \mu, \ \ 0 \leq \mu < 1 \tag{3}$$

where $I_\beta^n f(z)$ is defined by (2).

We let the subclass $\overline{SH}(\beta, n, \mu)$ consisting of harmonic functions $f_n = h + \overline{g}_n$ in $SH$ so that $h$ and $g_n$ are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \ g_n(z) = (-1)^n \sum_{k=1}^{\infty} b_k z^k, \ \ \ a_k, \ b_k \geq 0. \tag{4}$$

In the first theorem, we introduce a sufficient coefficient condition for harmonic functions in $SH(\beta, n, \mu)$.

## 2 Main results

**Theorem 1.** *Let $f = h + \overline{g}$ be so that h and g are given by (1) with $b_1 = 0$ . Furthermore, let*

$$\sum_{k=2}^{\infty}\left(\frac{\beta k+1}{1+\beta}\right)^n\left(\frac{\beta k+1}{1+\beta}-\mu\right)|a_k| + \sum_{k=2}^{\infty}\left(\frac{\beta k-1}{1+\beta}\right)^n\left(\frac{\beta k-1}{1+\beta}+\mu\right)|b_k| \leq 1-\mu, \tag{5}$$

*where $2 \leq \beta, n \in \mathbb{N}_0, \frac{1}{1+\beta} \leq \mu \leq \frac{\beta}{1+\beta}$. Then $f$ is sense-preserving, harmonic univalent in $\mathbb{D}$ and $f \in SH^0(\beta, n, \mu)$.*

*Proof:* If $z_1 \neq z_2$,

$$\begin{aligned}
\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| \\
&= 1 - \left|\frac{\sum\limits_{k=2}^{\infty} b_k\left(z_1^k - z_2^k\right)}{(z_1-z_2)+\sum\limits_{k=2}^{\infty} a_k\left(z_1^k-z_2^k\right)}\right| \\
&> 1 - \frac{\sum\limits_{k=2}^{\infty} k|b_k|}{1-\sum\limits_{k=2}^{\infty} k|a_k|} \\
&\geq 1 - \frac{\sum\limits_{k=2}^{\infty}\frac{\left(\frac{\beta k-1}{1+\beta}\right)^n\left(\frac{\beta k-1}{1+\beta}+\mu\right)}{1-\mu}|b_k|}{1-\sum\limits_{k=2}^{\infty}\frac{\left(\frac{\beta k+1}{1+\beta}\right)^n\left(\frac{\beta k+1}{1+\beta}-\mu\right)}{1-\mu}|a_k|} \\
&\geq 0,
\end{aligned}$$

which proves univalence. Note that f is sense preserving in $\mathbb{D}$. This is because

$$\begin{aligned}
\left|h'(z)\right| &\geq 1 - \sum_{k=2}^{\infty} k|a_k||z|^{k-1} \\
&> 1 - \sum_{k=2}^{\infty}\frac{\left(\frac{\beta k+1}{1+\beta}\right)^n\left(\frac{\beta k+1}{1+\beta}-\mu\right)}{1-\mu}|a_k| \\
&\geq \sum_{k=2}^{\infty}\frac{\left(\frac{\beta k-1}{1+\beta}\right)^n\left(\frac{\beta k-1}{1+\beta}+\mu\right)}{1-\mu}|b_k| \\
&> \sum_{k=2}^{\infty} k|b_k||z|^{k-1} \\
&\geq \left|g'(z)\right|.
\end{aligned}$$

Using the fact that $Rew \geq \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$, it suffices to show that

$$\left|(1-\mu)I_\beta^n f(z)+I_\beta^{n+1}f(z)\right|-\left|(1+\mu)I_\beta^n f(z)-I_\beta^{n+1}f(z)\right|\geq 0. \tag{6}$$

Substituting for $I_\beta^n f(z)$ and $I_\beta^{n+1}f(z)$ in (6), we obtain

$$\begin{aligned}
& \left|(1-\mu)I_\beta^n f(z)+I_\beta^{n+1}f(z)\right|-\left|(1+\mu)I_\beta^n f(z)-I_\beta^{n+1}f(z)\right| \\
\geq\ & 2(1-\mu)\left|z\right| \\
-\ & \sum_{k=2}^{\infty}\left(\frac{\beta k+1}{1+\beta}\right)^n\left(\frac{\beta k+1}{1+\beta}+1-\mu\right)|a_k|\,|z|^k \\
& -\sum_{k=2}^{\infty}\left(\frac{\beta k-1}{1+\beta}\right)^n\left(\frac{\beta k-1}{1+\beta}-1+\mu\right)|b_k|\,|z|^k \\
& -\sum_{k=2}^{\infty}\left(\frac{\beta k+1}{1+\beta}\right)^n\left(\frac{\beta k+1}{1+\beta}-1-\mu\right)|a_k|\,|z|^k \\
& -\sum_{k=2}^{\infty}\left(\frac{\beta k-1}{1+\beta}\right)^n\left(\frac{\beta k-1}{1+\beta}+1+\mu\right)|b_k|\,|z|^k \\
>\ & 2(1-\mu)\left|z\right|\left\{1-\sum_{k=2}^{\infty}\left(\frac{\beta k+1}{1+\beta}\right)^n\left(\frac{\beta k+1}{1+\beta}-\mu\right)|a_k|\right. \\
& \left.\qquad -\sum_{k=2}^{\infty}\left(\frac{\beta k-1}{1+\beta}\right)^n\left(\frac{\beta k-1}{1+\beta}+\mu\right)|b_k|\right\}.
\end{aligned}$$

This last expression is non-negative by (5), and so the proof is complete. □

We will give the next theorems without proof.

**Theorem 2.** *Let $f_n=h+\overline{g}_n$ be given by (4) with $b_1=0$. Then $f_n\in\overline{SH}^0(\beta,n,\mu)$ if and only if*

$$\sum_{k=2}^{\infty}\left(\frac{\beta k+1}{1+\beta}\right)^n\left(\frac{\beta k+1}{1+\beta}-\mu\right)a_k+\sum_{k=2}^{\infty}\left(\frac{\beta k-1}{1+\beta}\right)^n\left(\frac{\beta k-1}{1+\beta}+\mu\right)b_k\leq 1-\mu, \tag{7}$$

*where $2\leq\beta, n\in\mathbb{N}_0, \frac{1}{1+\beta}\leq\mu\leq\frac{\beta}{1+\beta}$.*

**Theorem 3.** *Let $f_n\in\overline{SH}^0(\beta,n,\mu)$. Then for $|z|=r<1$ and $2\leq\beta, n\in\mathbb{N}_0, \frac{1}{1+\beta}\leq\mu\leq\frac{\beta}{1+\beta}$ we have*

$$|f_n(z)|\leq r+\frac{(1-\mu)}{\left(\frac{2\beta+1}{1+\beta}\right)^n\left(\frac{2\beta+1}{1+\beta}-\mu\right)}r^2,$$

*and*

$$|f_n(z)|\geq r-\frac{(1-\mu)}{\left(\frac{2\beta+1}{1+\beta}\right)^n\left(\frac{2\beta+1}{1+\beta}-\mu\right)}r^2.$$

**Theorem 4.** *Let $f_n$ be given by (4). Then $f_n\in\overline{SH}^0(\beta,n,\mu)$ if and only if*

$$f_n(z)=\sum_{k=1}^{\infty}\left(X_kh_k(z)+Y_kg_{n_k}(z)\right),$$

*where*

$$h_1(z)=z,\ \ h_k(z)=z-\frac{1-\mu}{\left(\frac{\beta k+1}{1+\beta}\right)^n\left(\frac{\beta k+1}{1+\beta}-\mu\right)}z^k\ (k=2,3,...),$$

*and*

$$g_{n_1}(z)=z,\ \ g_{n_k}(z)=z+(-1)^n\frac{1-\mu}{\left(\frac{\beta k-1}{1+\beta}\right)^n\left(\frac{\beta k-1}{1+\beta}+\mu\right)}\overline{z}^k\ (k=2,3,...),$$

$X_k\geq 0, Y_k\geq 0, \sum_{k=1}^{\infty}(X_k+Y_k)=1,\ 2\leq\beta, n\in\mathbb{N}_0, \frac{1}{1+\beta}\leq\mu\leq\frac{\beta}{1+\beta}.$

*In particular, the extreme points of* $\overline{SH}^0(\beta, n, \mu)$ *are* $\{h_k\}$ *and* $\{g_{n_k}\}$.

**Theorem 5.** *The class* $\overline{SH}^0(\beta, n, \mu)$ *is closed under convex combinations.*

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# Coefficient Estimates for Certain Subclasses of Analytic Functions Associated with a Differential Operator

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**Abstract:** In this paper, we investigate certain subclasses of analytic functions defined by generalized differential operators involving binomial series. Also, we obtain coefficient estimates involving of the nonhomogeneous Cauchy-Euler differential equation of order r.

**Keywords:** Analytic functions, Coefficient bounds, Differential operator, Subordination.

## 1 Introduction

Let $\mathcal{A}$ denote the class of all analytic functions in the open unit disc $\mathbb{D} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and having the form for $z \in \mathbb{D}$

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

A function $f \in \mathcal{A}$ is said to be belong to the class $S^*(\alpha)$ of starlike functions of order $\alpha$ if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \tag{2}$$

for $z \in \mathbb{D};\ 0 \leq \alpha < 1$ .

Also, a function $f \in \mathcal{A}$ is said to be belong to the class $K(\alpha)$ of convex functions of order $\alpha$ if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \tag{3}$$

for $z \in \mathbb{D};\ 0 \leq \alpha < 1$ .

The classes $S^*(\alpha)$ and $K(\alpha)$ considered by Silverman [9]. We consider that $S^*(0)$ and $K(0)$ are respectively, the classes of starlike functions and convex functions. For more details see [2], [3], [4], [10].

Let the functions $f,\ g \in \mathcal{A}$ be analytic in $\mathbb{D}$. Then $f$ is said to be subordinate $g$ if there exists a Schwarz function $w(z)$ on $\mathbb{D}$ with $w(0) = 0,\ |w(z)| < 1$, such that $f(z) = g(w(z))$ for $z \in \mathbb{D}$. We denote this subordination by $f(z) \prec g(z)\ (z \in \mathbb{D})$. In particular, if the function $g$ is univalent $\mathbb{D}$, then we get $f(z) \prec g(z) \Leftrightarrow f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

Making use of the binomial series for $k \in \mathbb{N} = \{1, 2, ...\},\ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$$(1-\lambda)^k = \sum_{m=0}^{k} \binom{k}{m} (-1)^m \lambda^m$$

and for $f \in \mathcal{A}$, $\lambda \in \mathbb{R},\ \mu \geq 0$ with $\lambda + \mu > 0$ and $\delta \in \mathbb{N}_0$, Wanas in [11] introduced the differential operator $W^{k,\delta}_{\lambda,\mu} f(z)$ which is defined as follows:

$$\begin{aligned}
W^{k,0}_{\lambda,\mu} f(z) &= f(z) \\
W^{k,1}_{\lambda,\mu} f(z) &= \frac{\left[1-(1-\lambda)^k\right] f(z) + \left[1-(1-\mu)^k\right] z f'(z)}{2-(1-\lambda)^k-(1-\mu)^k} \\
&\ \vdots \\
W^{k,\delta}_{\lambda,\mu} f(z) &= W^{k,1}_{\lambda,\mu}\left(W^{k,\delta-1}_{\lambda,\mu} f(z)\right).
\end{aligned} \tag{4}$$

If $f$ is given by (1), then from (4) we see that

$$W^{k,\delta}_{\lambda,\mu} f(z) = z + \sum_{n=2}^{\infty}\left[\sum_{m=1}^{k}\binom{k}{m}(-1)^m\left(\frac{\lambda^m + n\mu^m}{\lambda^m+\mu^m}\right)\right]^{\delta} a_n z^n. \tag{5}$$

### *1.1 Definition*

Let $p : \mathbb{D} \longrightarrow \mathbb{C}$ be a convex function such that $p(0) = 1$ and $Re\{p(z)\} > 0,\ z \in \mathbb{D}$. We denote by $S^{*,\delta}_g(\lambda, k; \alpha)$ the subclass of $\mathcal{A}$ given by

$$S^{*,\delta}_g(\lambda,k;\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } \frac{1}{1-\alpha}\left(\frac{z(W^{k,\delta}_{\lambda} f(z))'}{W^{k,\delta}_{\lambda} f(z)} - \alpha\right) \in g(\mathbb{D}),\ z \in \mathbb{D}\right\}, \tag{6}$$

where $\lambda \in \mathbb{R}$ with $\lambda > 0$ and $\delta \in \mathbb{N}_0,\ k \in \mathbb{N}$ and $\alpha \in [0, 1)$.

### *1.2 Definition*

A function $f \in \mathcal{A}$ in the class $C^{\delta}_g(\lambda, k, t; r)$ if it satisfies the following non-homogeneous Cauchy-Euler differential equation of order $t$;

$$z^t \frac{d^t w}{dz^t} + \binom{t}{1}(r+t-1) z^{t-1}\frac{d^{t-1}w}{dz^{t-1}} + ... + \binom{t}{t} w \prod_{i=0}^{t-1}(r+i) = g(z)\prod_{i=0}^{t-1}(r+i+1), \tag{7}$$

where $w = f(z),\ f \in \mathcal{A}$, $g(z) \in S^{*,\delta}_g(\lambda, k; \alpha),\ r \in \mathbb{R}\backslash(-\infty,\ -1]$ and $t \in \mathbb{N}_2 = \mathbb{N} - \{1\} = \{2, 3, ...\}$.

Clearly, by suitably specializing parameters for

$$g(z) = \frac{1+Az}{1+Bz}\ \ (-1 \leq B \leq A \leq 1,\ z \in \mathbb{D})$$

and

$$g(z) = \frac{1+(1-2\alpha)z}{1-z}\ \ (0 \leq \alpha < 1,\ z \in \mathbb{D}),$$

$S^{*,\delta}_g(\lambda, k; \alpha)$ reduces to the various subclasses of analytic functions (see [9]). Motivated from the recent work of Al-Hawary et al. [1] (see also, for example [5], [6], [7]) the main object of our investigation is to obtain some coefficient bounds for functions in the subclasses $S^{*,\delta}_g(\lambda, k; \alpha)$ and $C^{\delta}_g(\lambda, k, t; r)$ of analytic functions of order $\alpha$ by using the subordination principle between analytic functions.

To prove our main results, we recall lemma.

### *1.3 Lemma*

Let the function

$$g(z) = \sum_{n=1}^{\infty} b_n z^n\ \ (z \in \mathbb{D})$$

be convex in $\mathbb{E}$. Also let function $f(z)$ given by

$$f(z) = \sum_{n=1}^{\infty} a_n z^n\ \ (z \in \mathbb{D})$$

be analytic in $\mathbb{D}$. If $f(z) \prec g(z)\ (z \in \mathbb{D})$, then

$$|a_n|\ \leq |b_n|\ \ (n \in \mathbb{N}).$$

## 2 Coefficient bounds for the classes $S_g^{*,\delta}(\lambda, k; \alpha)$ and $C_g^{\delta}(\lambda, k, t; r)$.

We start by acquiring coefficient bounds for functions in the class $S_g^{*,\delta}(\lambda, k; \alpha)$.

**Theorem 1.** *Let the function $f \in \mathcal{A}$ be given by (1). If $f \in S_g^{*,\delta}(\lambda, k; \alpha)$, then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2} \left(j + (1-\alpha)|g'(0)|\right)}{(n-1)! \left| \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\lambda^m + n}{\lambda^m + 1} \right) \right|^{\delta}} \tag{8}$$

*for $n \in \mathbb{N}_2$.*

*Proof:* By the equation (5), the function $W_\lambda^{k,\delta} f(z)$ has the Taylor-Maclaurin series expansion

$$W_\lambda^{k,\delta} f(z) = z + \sum_{n=2}^{\infty} A_n z^n \quad (z \in \mathbb{D}),$$

where

$$A_n = \left[ \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\lambda^m + n}{\lambda^m + 1} \right) \right]^{\delta} a_n \tag{9}$$

for $n \in \mathbb{N}_2$. We observe that $W_\lambda^{k,\delta} f(z)$ is analytic in $\mathbb{D}$ with

$$W_\lambda^{k,\delta} f(0) = \left( W_\lambda^{k,\delta} f \right)'(0) - 1 = 0.$$

Now, from Definition 1.1 we have

$$\frac{1}{1-\alpha} \left\{ \frac{z \left( W_\lambda^{k,\delta} f(z) \right)'}{W_\lambda^{k,\delta} f(z)} - \alpha \right\} \in g(\mathbb{D}).$$

Let us define the function $p(z)$ by

$$p(z) = \frac{1}{1-\alpha} \left\{ \frac{z \left( W_\lambda^{k,\delta} f(z) \right)'}{W_\lambda^{k,\delta} f(z)} - \alpha \right\}, \tag{10}$$

we deduce that $p(0) = g(0) = 1$ and $p(z) \in g(\mathbb{D}),\ (z \in \mathbb{D})$. Therefore, we have $p(z) \prec g(z),\ (z \in \mathbb{D})$. Thus, according to Lemma 1.3, we obtain

$$\left| \frac{p^{(n)}(0)}{n!} \right| \leq |g'(0)| \quad (n \in \mathbb{N}), \tag{11}$$

where $p(z) = 1 + p_1 z + p_2 z + ...$ is analytic in $\mathbb{D}$. From (10), we easily get

$$z \left( W_\lambda^{k,\delta} f(z) \right)' - \alpha W_\lambda^{k,\delta} f(z) = (1-\alpha) p(z) W_\lambda^{k,\delta} f(z), \quad (z \in \mathbb{D}). \tag{12}$$

Since $A_1 = 1$, from (12), it follows that

$$(n-\alpha) A_n = (1-\alpha)(p_{n-1} + p_{n-2} A_2 + ... + p_1 A_{n-1}).$$

Especially, for $n = 2, 3, 4, ...$, we have

$$|A_2| \leq (1-\alpha)|g'(0)|,$$

$$|A_3| \leq \frac{(1-\alpha)|g'(0)| \left(1 + (1-\alpha)|g'(0)|\right)}{2!}$$

and

$$|A_4| \leq \frac{(1-\alpha)|g'(0)| \left(1 + (1-\alpha)|g'(0)|\right) \left(2 + (1-\alpha)|g'(0)|\right)}{3!},$$

respectively. Thus, by appealing to the principle of mathematical induction, we obtain

$$|A_n| \leq \frac{\prod_{j=0}^{n-2} \left(j + (1-\alpha)|g'(0)|\right)}{(n-1)!} \tag{13}$$

for $n \in \mathbb{N}_2$ . We now immediately find from (9) that

$$|a_n| \le \frac{\prod_{j=0}^{n-2} \big(j + (1-\alpha)|g'(0)|\big)}{(n-1)! \left| \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\lambda^m + n}{\lambda^m + 1} \right) \right|^{\delta}}.$$

This completes the proof. □

Next, we give coefficient bounds for functions in the $C_g^{\delta}(\lambda, k, t; r)$.

**Theorem 2.** *Let the function $f \in \mathcal{A}$ be given by* (1)*. If $f \in C_g^{\delta}(\lambda, k, t; r)$, then*

$$|a_n| \le \frac{\prod_{j=0}^{n-2} \big(j + (1-\alpha)|g'(0)|\big) \prod_{i=0}^{t-1} (r+i+1)}{(n-1)! \left| \sum_{m=1}^{k} \binom{k}{m} (-1)^{m+1} \left( \frac{\lambda^m + n}{\lambda^m + 1} \right) \right|^{\delta} \prod_{i=0}^{t-1} (r+i+n)} \tag{14}$$

*for $t, \ n \in \mathbb{N}_2$ where $r \in \mathbb{R} \backslash (-\infty, \ -1]$.*

*Proof:* Let the function $f \in \mathcal{A}$ be given by (1) and

$$g(z) = z + \sum_{n=1}^{\infty} b_n z^n \in S_g^{*,\delta}(\lambda, k; \alpha).$$

Then from (7), we get

$$a_n = \frac{\prod_{i=0}^{t-1} (r+i+1)}{\prod_{i=0}^{t-1} (r+i+n)} \, b_n$$

for $n \in \mathbb{N}_2, \ r \in \mathbb{R} \backslash (-\infty, \ -1]$ . Hence from Theorem 1, we obtain inequality (14). This completes the proof. □

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# A Generalization of New Periodicity Concept on Time Scales

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**Abstract:** In this study, the new periodicity concept on time scales introduced by Adıvar (see [1]) is revisited. We define two periodicity notions by utilizing shift operators as a generalization and relaxation of the new periodicity concept on time scales. In particular cases, our discussions cover periodic functions, anti-periodic functions, Bloch periodic functions, and unbounded functions.

**Keywords:** New periodicity concept, Shift operator, Time scale, $(T, \lambda)$-periodicity

## 1 Introduction

Qualitative analysis of dynamic equations on continuous and discrete time domains is still a voguish research area in mathematical sciences due to its potential for application in a wide range of disciplines. Motivated by the popularity of qualitative theory for differential and difference equations, scholars have started generalizing and unifying the already established literature on arbitrary, nonempty, and closed subset of real numbers so-called time scales in the last three decades. The theory of dynamic equations has become a very fruitful research subject for mathematicians since it provides a joint study of continuous and discrete time models. By a quick literature review, one may easily see remarkable applications of time scales in mathematical analysis [2]-[4], fractals and fractional calculus [5]-[8], biology [9]-[11], economics [12]-[13], and optimization [14].

Study of periodic structures and the existence of periodic solutions of dynamic equations is one of the hot topics in applied mathematics. Undoubtfully, periodic solutions of differential and difference equations are extensively studied in the existing literature, and these works have been unified and generalized on time scales. In this work, we give a special emphasis on periodicity notion for classes of functions defined on hybrid time domains. The motivation of the study is highlighted below:

• By conventional periodicity notion for a function, we indicate the property $f(t+T) = f(t)$ which holds for a fixed $T > 0$ and for all $t$. Notice that, in order to define a periodic function on a time scale $\mathbb{T}$ one has to ensure that $\mathbb{T}$ is translation invariant (additively periodic), that is there exists a $P > 0$ such that $t \pm P \in \mathbb{T}$ for all $t \in \mathbb{T}$ (see [15]).
• Addition is not the only way to step forward and backward on a time scale. It should be emphasized that additive periodicity condition is a very restrictive condition for time scales, and it rules out several time scales involving $q^{\mathbb{N}_0}$ on which $q$-difference equations are constructed. In [16], shift operators $\delta_\pm$ are used the define forward and backward motion on time scales, and a new periodicity concept for time scales is introduced with respect to shift operators in [1]. There are numerous papers handling existence of periodic solutions in shifts $\delta_\pm$ for dynamic equations on time scales. We refer to [17]-[22] as related studies.
• Periodicity is a relaxable and generalizable property for function classes. As a relaxation of periodicity notion, we can address almost periodicity and almost automorphy concepts which have been introduced in $20^{th}$ century. Besides, a generalization of periodicity is introduced and studied as $(\omega, c)$-periodicity in recent papers [23]-[25]. Also, the discrete counter part of $(\omega, c)$-periodicity is defined in [26]. Precisely, a function $f$ is said to be $(\omega, c)$-periodic if

$$f(t+\omega) = cf(t)$$

for $c \in \mathbb{C}\setminus\{0\}$ and $\omega > 0$. Observe that
◦ If $c = 1$, then $(\omega, c)$-periodicity coincides with conventional periodicity.
◦ If $c = -1$, then $(\omega, c)$-periodicity coincides with anti-periodicity.
◦ If $c = e^{ikN}$, then $(\omega, c)$-periodicity coincides with Bloch periodicity.

It should be noted that one may see a similar approach in literature for generalization of periodicity notion regarding vector/matrix valued functions so-called affine periodicity (see [27]).

In this work, we aim to propose a generalized periodicity, $(T, \lambda)$-periodicity, by employing shift operators $\delta_\pm$. We generalize the new periodicity concept for time scales given in [1], and use shift operators avoids separate definitions of periodicity for continuous, discrete and hybrid calculus.

The organization of the manuscript is as follows: The next section is devoted to time scale essentials. In Section 3, we give basics of shift operators, and recall the new periodicity concept on time scales. Section 4 contains the outcomes of the study. In the last section, we provide concluding comments which involve our future directions.

## 2 Preliminaries on time scale calculus

Throughout the manuscript, we assume a familiarity with the theory of time scales. In this section, we just give a very short summary about the fundamentals of time scale calculus. Given definitions, results and examples can be found in excellent books [28] and [29].

A time scale, denoted by $\mathbb{T}$, is an arbitrary, nonempty and closed subset of real numbers. The operator $\sigma : \mathbb{T} \to \mathbb{T}$ called forward jump operator is defined by $\sigma(t) := \inf\{s \in \mathbb{T}, s > t\}$. The step size function $\mu : \mathbb{T} \to \mathbb{R}$ is given by $\mu(t) := \sigma(t) - t$. We say a point $t \in \mathbb{T}$ is right dense if $\mu(t) = 0$, and right scattered if $\mu(t) > 0$. Furthermore, a point $t \in \mathbb{T}$ is said to be left dense if $\rho(t) := \sup\{s \in \mathbb{T}, s < t\} = t$ and left scattered if $\rho(t) < t$. The notation $[s, t)_{\mathbb{T}}$ indicates the intersection $[s, t) \cap \mathbb{T}$ and the intervals $[s, t]_{\mathbb{T}}, (s, t)_{\mathbb{T}}$, and $(s, t]_{\mathbb{T}}$ can be defined similarly.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be $rd$-continuous if it is continuous at right dense points and its left sided limits exists at left dense points. Moreover, we use the notation $C_{rd}$ in order to represent all $rd$-continuous functions on $\mathbb{T}$. The set $\mathbb{T}^k$ is defined in the following way: If $\mathbb{T}$ has a left-scattered maximum $m$, then $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. Moreover, the delta derivative of a function $f : \mathbb{T} \to \mathbb{R}$ at a point $t \in \mathbb{T}^k$ is defined by

$$f^{\Delta}(t) := \lim_{\substack{s \to t \\ s \neq \sigma(t)}} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}.$$

The following table shows three important examples of time scales:

| $\mathbb{T}$ | $\mathbb{R}$ | $\mathbb{Z}$ | $q^{\mathbb{Z}} \cup \{0\}, q < 1$ |
|---|---|---|---|
| $\rho(t)$ | $t$ | $t-1$ | $qt$ |
| $\sigma(t)$ | $t$ | $t+1$ | $\frac{t}{q}$ |
| $\mu(t)$ | $0$ | $1$ | $(1-q)t$ |
| $f^{\Delta}(t)$ | $f'(t)$ | $\Delta f(t)$ | $D_q f(t) = \frac{f(t)-f(qt)}{(1-q)t}$ |
| $\int_0^b f(t)\Delta t$ | $\int_0^b f(t)dt$ | $\sum_{t=0}^{b-1} f(t),\ (0 < b)$ | $\int_0^b f(t)d_q t = (1-q)\, b \sum_{t=0}^{\infty} q^j f(q^j b)$ |

Table 1

**Theorem 1** (Substitution, [28, Theorem 1.98]). *Assume $\upsilon : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \upsilon(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an $rd$-continuous function and $\upsilon$ is differentiable with $rd$-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b g(s)\, \upsilon^{\Delta}(s)\, \Delta s = \int_{\upsilon(a)}^{\upsilon(b)} g\left(\upsilon^{-1}(s)\right) \widetilde{\Delta} s.$$

**Definition 1.** *A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive if $1 + \mu(t)\, p(t) \neq 0$ for all $t \in \mathbb{T}^k$. We denote the set of all regressive functions by $\mathcal{R}$. Also, $\mathcal{R}^+$ stands for the set of all positively regressive elements of $\mathcal{R}$ defined by*

$$\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)\, p(t) > 0 \text{ for all } t \in \mathbb{T}\}.$$

**Definition 2** (Exponential function). *For $h > 0$, set $\mathbb{C}_h := \{z \in \mathbb{C} : \ z \neq -1/h\}$, $\mathbb{J}_h := \{z \in \mathbb{C} : \ -\pi/h < Im(z) \leq \pi/h\}$ and $\mathbb{C}_0 := \mathbb{J}_0 := \mathbb{C}$. For $h \geq 0$ and $z \in \mathbb{C}_h$, the* cylinder transformation *$\xi_h : \mathbb{C}_h \to \mathbb{J}_h$ is given by*

$$\xi_h(z) := \begin{cases} z, & h = 0 \\ \dfrac{1}{h} \operatorname{Log}(1 + zh), & h > 0. \end{cases}$$

*Then, the exponential function on $\mathbb{T}$ is presented in the form*

$$e_p(t, s) := \exp\left\{ \int_s^t \xi_{\mu(\tau)}\big(p(\tau)\big) \Delta\tau \right\} \quad \text{for } s, t \in \mathbb{T}.$$

**Lemma 1.** *Let $p, q \in \mathcal{R}$. Then*

*i.* $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;
*ii.* $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
*iii.* $\frac{1}{e_p(t,s)} = e_{\ominus p}(t, s)$, *where* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$;
*iv.* $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
*v.* $e_p(t, s)e_p(s, r) = e_p(t, r)$;
*vi.* $\left(\frac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(\cdot,s)}$;
*vii.* $e_p(t, s)e_q(s, r) = e_{p \oplus q}(t, r)$, *where* $p \oplus q = p(t) + q(t) + p(t)\, q(t)\, \mu(t)$.

The following table demonstrates some exponential functions over particular time scales

| $\mathbb{T}$ | $e_\alpha(t,t_0)$ |
|---|---|
| $\mathbb{R}$ | $e^{\alpha(t-t_0)}$ |
| $\mathbb{Z}$ | $(1+\alpha)^{t-t_0}$ |
| $h\mathbb{Z}$ | $(1+h\alpha)^{(t-t_0)/h}$ |
| $q^{\mathbb{N}_0}$ | $\prod_{s\in[t_0,t)} [1+(q-1)\alpha s]\,, t>t_0$ |
| $\frac{1}{n}\mathbb{Z}$ | $\left(1+\frac{\alpha}{n}\right)^{n(t-t_0)}$ |

**Theorem 2** (Variation of constants)**.** *Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. The unique solution of the regressive initial value problem*

$$\begin{cases} y^{\Delta}(t) = p(t)\,y(t) + f(t) \\ y(t_0) = y_0 \end{cases}$$

*is given by*

$$y(t) = e_p(t,t_0)\,y_0 + \int_{t_0}^{t} e_p(t,\sigma(s))\,f(s)\,\Delta s.$$

We direct readers to the pioneering book [28] for further reading on time scale calculus.

## 3 Shift operators and periodic time scales in shifts

In this section, we introduce basics of shift operators and the new periodicity concept on time scales under the guidance of [1] and [16]. The definitions, results and examples given in this part of the paper can be found in [1].

**Definition 3.** *Let $\mathbb{T}^*$ be a nonempty subset of the time scale $\mathbb{T}$ including a fixed number $t_0 \in \mathbb{T}^*$ such that there exists operators $\delta_\pm : [t_0,\infty)_\mathbb{T} \times \mathbb{T}^* \to \mathbb{T}^*$ satisfying the following properties:*

1. The function $\delta_\pm$ are strictly increasing with respect to their second arguments, if

$$(T_0,t), (T_0,u) \in \mathcal{D}_\pm := \left\{(s,t) \in [t_0,\infty)_\mathbb{T} \times \mathbb{T}^* : \delta_\pm(s,t) \in \mathbb{T}^*\right\},$$

then

$$T_0 \le t \le u \text{ implies } \delta_\pm(T_0,t) \le \delta_\pm(T_0,u)\,;$$

2. If $(T_1,u), (T_2,u) \in \mathcal{D}_-$ with $T_1 < T_2$, then $\delta_-(T_1,u) > \delta_-(T_2,u)$ and if $(T_1,u), (T_2,u) \in \mathcal{D}_+$ with $T_1 < T_2$, then $\delta_+(T_1,u) < \delta_+(T_2,u)$;
3. If $t \in [t_0,\infty)_\mathbb{T}$, then $(t,t_0) \in \mathcal{D}_+$ and $\delta_+(t,t_0) = t$. Moreover, if $t \in \mathbb{T}^*$, then $(t_0,t) \in \mathcal{D}_+$ and $\delta_+(t_0,t) = t$;
4. If $(s,t) \in \mathcal{D}_\pm$, then $(s,\delta_\pm(s,t)) \in \mathcal{D}_\mp$ and $\delta_\mp(s,\delta_\pm(s,t)) = t$;
5. If $(s,t) \in \mathcal{D}_\pm$ and $(u,\delta_\pm(s,t)) \in \mathcal{D}_\mp$, then $(s,\delta_\mp(u,t)) \in \mathcal{D}_\pm$ and $\delta_\mp(u,\delta_\pm(s,t)) = \delta_\pm(s,\delta_\mp(u,t))$.

Then the operators $\delta_+$ and $\delta_-$ are called forward and backward shift operators associated with the initial point $t_0$ on $\mathbb{T}^*$ and the sets $\mathcal{D}_+$ and $\mathcal{D}_-$ are domain of the operators, respectively.

**Example 1.** *The following table shows shift operators $\delta_\pm(s,t)$ on some time scales:*

| $\mathbb{T}$ | $t_0$ | $\mathbb{T}^*$ | $\delta_-(s,t)$ | $\delta+(s,t)$ |
|---|---|---|---|---|
| $\mathbb{R}$ | 0 | $\mathbb{R}$ | $t-s$ | $t+s$ |
| $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | $t-s$ | $t+s$ |
| $q^\mathbb{Z}\cup\{0\}$ | 1 | $q^\mathbb{Z}$ | $\frac{t}{s}$ | $st$ |
| $\mathbb{N}^{1/2}$ | 0 | $\mathbb{N}^{1/2}$ | $\left(t^2-s^2\right)^{1/2}$ | $\left(t^2+s^2\right)^{1/2}$ |

**Lemma 2.** *Let $\delta_\pm$ be the shift operators associated with the initial point $t_0$. Then we have the following:*

1. $\delta_-(t,t) = t_0$ for all $t \in [t_0,\infty)_\mathbb{T}$;
2. $\delta_-(t_0,t) = t$ for all $t \in \mathbb{T}^*$;
3. If $(s,t) \in \mathcal{D}_+$, then $\delta_+(s,t) = u$ implies $\delta_-(s,u) = t$ and if $(s,u) \in \mathcal{D}_-$, then $\delta_-(s,u) = t$ implies $\delta_+(s,t) = u$;
4. $\delta_+(t,\delta_-(s,t_0)) = \delta_-(s,t)$ for all $(s,t) \in \mathcal{D}_+$ with $t \ge t_0$;
5. $\delta_+(u,t) = \delta_+(t,u)$ for all $(u,t) \in \left([t_0,\infty)_\mathbb{T} \times [t_0,\infty)_\mathbb{T}\right) \cap \mathcal{D}_+$;
6. $\delta_+(s,t) \in [t_0,\infty)_\mathbb{T}$ for all $(s,t) \in \mathcal{D}_+$ with $t \ge t_0$;
7. $\delta_-(s,t) \in [t_0,\infty)_\mathbb{T}$ for all $(s,t) \in \left([t_0,\infty)_\mathbb{T} \times [s,\infty)_\mathbb{T}\right) \cap \mathcal{D}_-$;
8. If $\delta_+(s,.)$ is $\Delta$-differentiable in its second variable, then $\delta_+^{\Delta_t}(s,.) > 0$;
9. $\delta_+(\delta_-(u,s), \delta_-(s,v)) = \delta_-(u,v)$ for all $(s,v) \in \left([t_0,\infty)_\mathbb{T} \times [s,\infty)_\mathbb{T}\right) \cap \mathcal{D}_-$ and $(u,s) \in \left([t_0,\infty)_\mathbb{T} \times [u,\infty)_\mathbb{T}\right) \cap \mathcal{D}_-$;

 

10. If $(s,t) \in \mathcal{D}_-$ and $\delta_-(s,t) = t_0$, then $s = t$.

**Definition 4** (Periodicity in shifts)**.** *Let $\mathbb{T}$ be a time scale with the shift operators $\delta_\pm$ associated with the initial point $t_0 \in \mathbb{T}^*$, then $\mathbb{T}$ is said to be periodic in shifts $\delta_\pm$, if there exists a $p \in (t_0,\infty)_{\mathbb{T}^*}$ such that $(p,t) \in \mathcal{D}_\mp$ for all $t \in \mathbb{T}^*$. $P$ is called the period of $\mathbb{T}$ if*

$$P = \inf\left\{p \in (t_0,\infty)_{\mathbb{T}^*} : (p,t) \in \mathcal{D}_\mp \text{ for all } t \in \mathbb{T}^*\right\} > t_0.$$

Observe that an additive periodic time scale must be unbounded. However, unlike additive periodic time scales a time scale, periodic in shifts, may be bounded.

**Example 2.** *The following time scales are not additive periodic but periodic in shifts $\delta_\pm$.*

1. $\mathbb{T}_1 = \left\{\pm n^2 : n \in \mathbb{Z}\right\}$, $\delta_\pm(P,t) = \begin{cases} \left(\sqrt{t} \pm \sqrt{P}\right)^2 & \text{if } t > 0 \\ \pm P & \text{if } t = 0 \\ -\left(\sqrt{-t} \pm \sqrt{P}\right)^2 & \text{if } t < 0 \end{cases}$, $P = 1$, $t_0 = 0$,
2. $\mathbb{T}_2 = \overline{q^{\mathbb{Z}}}$, $\delta_\pm(P,t) = P^{\pm 1}t$, $P = q$, $t_0 = 1$,
3. $\mathbb{T}_3 = \cup_{n\in\mathbb{Z}}\left[2^{2n}, 2^{2n+1}\right]$, $\delta_\pm(P,t) = P^{\pm 1}t$, $P = 4$, $t_0 = 1$,
4. $\mathbb{T}_4 = \left\{\frac{q^n}{1+q^n} : q > 1 \text{ is constant and } n \in \mathbb{Z}\right\} \cup \{0,1\}$,

$$\delta_\pm(P,t) = \frac{q^{\left(\frac{\ln\left(\frac{t}{1-t}\right) \pm \ln\left(\frac{P}{1-P}\right)}{\ln q}\right)}}{1 + q^{\left(\frac{\ln\left(\frac{t}{1-t}\right) \pm \ln\left(\frac{P}{1-P}\right)}{\ln q}\right)}}, \quad P = \frac{q}{1+q}.$$

Notice that the time scale $\mathbb{T}_4$ in Example 2 is bounded above and below and

$$\mathbb{T}_4^* = \left\{\frac{q^n}{1+q^n} : q > 1 \text{ is constant and } n \in \mathbb{Z}\right\}.$$

**Corollary 1.** *Let $\mathbb{T}$ be a time scale that is periodic in shifts $\delta_\pm$ with the period $P$. Then we have*

$$\delta_\pm(P,\sigma(t)) = \sigma(\delta_\pm(P,t)) \text{ for all } t \in \mathbb{T}^*. \tag{1}$$

**Definition 5** (Periodic function in shifts $\delta_\pm$)**.** *Let $\mathbb{T}$ be a time scale $P$-periodic in shifts. We say that a real valued function $f$ defined on $\mathbb{T}^*$ is periodic in shifts $\delta_\pm$ if there exists a $T \in [P,\infty)_{\mathbb{T}^*}$ such that*

$$(T,t) \in \mathcal{D}_\pm \text{ and } f\left(\delta_\pm^T(t)\right) = f(t) \text{ for all } t \in \mathbb{T}^*, \tag{2}$$

*where $\delta_\pm^T(t) = \delta_\pm(T,t)$. $T$ is called period of $f$, if it is the smallest number satisfying (2).*

**Definition 6** ($\Delta$-periodic function in shifts $\delta_\pm$)**.** *Let $\mathbb{T}$ be a time scale $P$-periodic in shifts. A real valued function $f$ defined on $\mathbb{T}^*$ is $\Delta$-periodic function in shifts if there exists a $T \in [P,\infty)_{\mathbb{T}^*}$ such that*

$$(T,t) \in \mathcal{D}_\pm \text{ for all } t \in \mathbb{T}^* \tag{3}$$

$$\text{the shifts } \delta_\pm^T \text{ are } \Delta\text{-differentiable with rd-continuous derivatives} \tag{4}$$

*and*

$$f\left(\delta_\pm^T(t)\right)\delta_\pm^{\Delta T}(t) = f(t) \tag{5}$$

*for all $t \in \mathbb{T}^*$, where $\delta_\pm^T(t) = \delta_\pm(T,t)$. The smallest number $T$ satisfying (3-5) is called period of $f$.*

**Lemma 3** ([21, Lemma 3.1])**.** *Let $\mathbb{T}$ be a time scale that is periodic in shift operators $\delta_\pm$ with period $P$. Suppose that the shift operators $\delta_\pm^T$ are $\Delta$-differentiable on $t \in \mathbb{T}^*$ where $T \in [P,\infty)_{\mathbb{T}^*}$. Then the graininess function $\mu : \mathbb{T} \to [0,\infty)$ satisfies*

$$\mu\left(\delta_\pm^T(t)\right) = \delta_\pm^{\Delta T}(t)\,\mu(t).$$

## 4 Design of a generalized periodicity with respect to shift operators

Henceforth, we suppose that $\mathbb{T}$ is a $P$-periodic time scale in shifts $\delta_\pm$, and the shift operators $\delta_\pm$ are $\Delta$-differentiable with $rd$-continuous derivatives. We use the phrase "periodic in shifts" to indicate periodicity in shifts $\delta_\pm$. Moreover, by $\delta_\pm^{(k)}(T,t)$, $k \in \mathbb{N}$ we denote $k$-times composition of shifts of $\delta_\pm^T$ with itself, namely,

$$\delta_\pm^{(k)}(T,t) := \underbrace{\delta_\pm^T \circ \delta_\pm^T \circ \ldots \circ \delta_\pm^T}_{k-times}(t).$$

**Definition 7.** *A function $f$ defined on $\mathbb{T}^*$ is said to be $(T, \lambda)$-periodic in shifts if for a fixed $\lambda \in \mathbb{C}\backslash\{0\}$ there exists a $T \in [P, \infty)_{\mathbb{T}^*}$ such that*

$$(T, t) \in \mathcal{D}_+ \text{ and } f\left(\delta_+^T(t)\right) = \lambda f(t) \text{ for all } t \in \mathbb{T}^*.$$

In preparation for auxiliary results, we define

$$P(t_0) := \left\{\delta_+^{(k)}(T, t_0),\ k = 0, 1, 2, \ldots\right\}, \tag{6}$$

$$m(t) := \max\left\{k \in \mathbb{N} : \delta_+^{(k)}(T, t_0) \leq t\right\}, \tag{7}$$

and accordingly any point $t \geq t_0$ of $\mathbb{T}^*$ can be decomposed as

$$t = \delta_+^{(m(t))}(T, t_0) + t_r, \tag{8}$$

where

$$t_r := \begin{cases} 0 & \text{if } t \in P(t_0) \\ \delta_-\left(\delta_+^{(m(t))}(T, t_0), t\right) & \text{if } t \notin P(t_0) \end{cases}. \tag{9}$$

**Lemma 4.** *Let $f$ be a $(T, \lambda)$-periodic function in shifts. Then, $f$ can be represented as*

$$f(t) = \begin{cases} \lambda^{m(t)} f(t_0) & \text{if } t \in P(t_0) \\ \lambda^{m(t)} f(t_r) & \text{if } t \notin P(t_0) \end{cases}.$$

*The proof of the above result is omitted since it is a direct consequence of Definition 7 and (6-9).*

**Lemma 5.** *A function $f$ is $(T, \lambda)$-periodic in shifts if and only if there exists a function $g$ which is $T$-periodic in shifts so that*

$$f(t) = \lambda^{m(t)} g(t).$$

*Proof:* Assume that $f$ is $(T, \lambda)$-periodic function in shifts, and define

$$g(t) = \lambda^{-m(t)} f(t).$$

Then,

$$g\left(\delta_+^T(t)\right) = \lambda^{-m\left(\delta_+^T(t)\right)} f\left(\delta_+^T(t)\right) = \lambda^{-m(t)-1} \lambda f(t) = g(t)$$

which shows $g$ is $T$-periodic in shifts.

On the other hand, we suppose that $f(t) = \lambda^{m(t)} g(t)$. Then, one may easily obtain $(T, \lambda)$-periodicity of $f$ as

$$f\left(\delta_+^T(t)\right) = \lambda^{m\left(\delta_+^T(t)\right)} g\left(\delta_+^T(t)\right) = \lambda^{m(t)+1} g(t) = \lambda f(t).$$

The proof is complete. □

**Example 3.** *Let $\mathbb{T} = q^{\mathbb{Z}} \cup \{0\}$, $q > 1$ which is a q-periodic time scale with shift operators $\delta_\pm(P, t) = P^{\pm 1} t$. Then the function*

$$f(t) = (-2)^{\log_q t}$$

*is $\left(q^2, 4\right)$-periodic in shifts. To see this, we write*

$$f\left(\delta_+\left(q^2, t\right)\right) = (-2)^{\log_q q^2 t} = (-1)^{2+\log_q t} (2)^{2+\log_q t} = 4(-2)^{\log_q t} = 4f(t).$$

**Remark 1.** *In [1], it is highlighted that set of real numbers $\mathbb{R}$ is not only an additively periodic time scale but also a time scale periodic in shifts with $t_0 = 1$, where*

$$\delta_-(s, t) = \begin{cases} t/s & \text{if } t \geq 0 \\ st & \text{if } t < 0 \end{cases}, \text{ for } s \in [1, \infty), \tag{10}$$

*and*

$$\delta_+(s, t) = \begin{cases} st & \text{if } t \geq 0 \\ t/s & \text{if } t < 0 \end{cases}, \text{ for } s \in [1, \infty). \tag{11}$$

*In the next example, we slightly modify [1, Example 5] in order to give an example of $(T, \lambda)$-periodic function on $\mathbb{R}$ corresponding to shift operator given in (11).*

 

**Example 4.** *Let $\mathbb{T}=\mathbb{R}$ with shift operators given in (*10-11*). We define the function*

$$f(t)=\sin\left(-\frac{\ln t}{\ln 3}\pi\right)3^{-\log_3 t},\ t>0$$

*as a $\left(9,\frac{1}{9}\right)$-periodic function on the half line since*

$$\begin{aligned}f(\delta_+(9,t))&=\sin\left(-\frac{\ln 9t}{\ln 3}\pi\right)3^{-\log_3 9t}\\&=\sin\left(-\frac{\ln 9+\ln t}{\ln 3}\right)3^{-2-\log_3 t}\\&=\frac{1}{9}\sin\left(-\frac{\ln t}{\ln 3}\pi\right)3^{-\log_3 t}\\&=\frac{1}{9}f(t).\end{aligned}$$

*Next, we introduce $(T,\lambda)$-$\Delta$-periodic function in shifts in a similar fashion with [1, Definition 6] (see Definition* 6*).*

**Definition 8.** *A function $f$ defined on $\mathbb{T}^*$ is said to be $(T,\lambda)$-$\Delta$-periodic function in shifts if there exists a $T\in[P,\infty)_{\mathbb{T}^*}$ such that*

$$(T,t)\in\mathcal{D}_+ \text{ and } f\left(\delta_+^T(t)\right)\delta_+^{\Delta T}(t)=\lambda f(t) \text{ for all } t\in\mathbb{T}^*.$$

We present the next result whose proof is omitted since it is similar to the proof of Lemma 5.

**Lemma 6.** *A function $f$ is $(T,\lambda)$-$\Delta$-periodic in shifts if and only if there exists a function $h$ which is $\Delta$-periodic function in shifts with period $T$ such that*

$$f(t)=\lambda^{m(t)}h(t).$$

**Example 5.** *Let $\mathbb{T}=q^{\mathbb{Z}}\cup\{0\}$, $q>1$ with shift operators $\delta_\pm(P,t)=P^{\pm1}t$. Then, the function*

$$f(t)=\frac{2^{-\log_q t}}{t}$$

*is $\left(q,\frac{1}{2}\right)$-$\Delta$-periodic in shifts, that is*

$$f\left(\delta_+^q(t)\right)\delta_+^{\Delta q}(t)=\frac{2^{-\log_q qt}}{qt}q=\frac{1}{2}\frac{2^{-\log_q t}}{t}=\frac{1}{2}f(t).$$

*The following result is straightforward due to [1, Theorem 2].*

**Theorem 3.** *Let $f$ be a $(T,\lambda)$-$\Delta$-periodic function in shifts $\delta_\pm$. If $f\in C_{rd}(\mathbb{T})$, then*

$$\lambda\int_{t_0}^{t}f(s)\Delta s=\int_{\delta_+^T(t_0)}^{\delta_+^T(t)}f(s)\Delta s.$$

Next, we give a remarkable property regarding time scale exponential function:

**Lemma 7.** *Let $p\in\mathcal{R}$ be a $(T,\lambda)$-$\Delta$-periodic function in shifts on $\mathbb{T}$ and suppose that also $\lambda p\in\mathcal{R}$. Then*

$$e_p\left(\delta_+^T(t),\delta_+^T(t_0)\right)=e_{\lambda p}(t,t_0) \text{ for } t,t_0\in\mathbb{T}^*.$$

*Proof:* We assume $p$ is a $(T,\lambda)$-$\Delta$-periodic function and $p,\lambda p\in\mathcal{R}$. Then, we present the time scale exponential function as

$$e_p\left(\delta_+^T(t),\delta_+^T(t_0)\right)=\begin{cases}\exp\left(\displaystyle\int_{\delta_+^T(t_0)}^{\delta_+^T(t)}\frac{1}{\mu(z)}\mathrm{Log}(1+\mu(z)p(z))\,\Delta z\right) & \text{if } \mu(z)\neq 0\\ \exp\left(\displaystyle\int_{\delta_+^T(t_0)}^{\delta_+^T(t)}p(z)\,\Delta z\right) & \text{if } \mu(z)=0\end{cases}.$$

By using Theorem 1, Lemma 3, and $(T,\lambda)$-$\Delta$-periodicity of $p$ in shifts, we obtain

$$\begin{aligned}
e_p\left(\delta_+^T(t),\delta_+^T(t_0)\right) &= \begin{cases} \exp\left(\int\limits_{t_0}^{t}\frac{\delta_+^{\Delta T}(z)}{\mu(\delta_+^T(z))}\text{Log}(1+\mu(\delta_+^T(z))p(\delta_+^T(z)))\,\Delta z\right) & \text{if } \mu(z)\neq 0 \\ \exp\left(\int\limits_{t_0}^{t}p\left(\delta_+^T(z)\right)\,\delta_+^{\Delta T}(z)\,\Delta z\right) & \text{if } \mu(z)=0 \end{cases} \\
&= \begin{cases} \exp\left(\int\limits_{t_0}^{t}\frac{\delta_+^{\Delta T}(z)}{\mu(\delta_+^T(z))}\text{Log}(1+\frac{\delta_+^{\Delta T}(z)}{\delta_+^{\Delta T}(z)}\mu(\delta_+^T(z))p(\delta_+^T(z)))\,\Delta z\right) & \text{if } \mu(z)\neq 0 \\ \exp\left(\int\limits_{t_0}^{t}\lambda p(z)\,\Delta z\right) & \text{if } \mu(z)=0 \end{cases} \\
&= \begin{cases} \exp\left(\int\limits_{t_0}^{t}\frac{1}{\mu(z)}\text{Log}(1+\lambda\mu(z)p(z))\,\Delta z\right) & \text{if } \mu(z)\neq 0 \\ \exp\left(\int\limits_{t_0}^{t}\lambda p(z)\,\Delta z\right) & \text{if } \mu(z)=0 \end{cases} \\
&= e_{\lambda p}(t,t_0),
\end{aligned}$$

which proves our assertion. □

## 5 Concluding comments

Consider the following dynamic equation

$$\begin{cases} x^{\Delta}(t)=a(t)\,x(t)+f(t) \\ x(t_0)=x_0 \end{cases},\ t\in\mathbb{T}, \tag{12}$$

where $a$ and $f$ are $(T,\lambda)$-$\Delta$-periodic functions in shifts, $a\in\mathcal{R}$, and $f\in C_{rd}$.

One may focus on the relationship between the existence of a bounded solution and a $(T,\lambda)$-periodic solution of (12). Inspired by [30, Definition 1], we introduce an alternative boundedness concept called $\lambda$-boundedness for a function defined on a time scale.

**Definition 9.** *Let $T\in[P,\infty)_{\mathbb{T}^*}$ be fixed constant, where $P$ is the period of the time scale. A function $x:\mathbb{T}^*\to\mathbb{R}$ is said to be $\lambda$-bounded if*

$$\left|\lambda^{-m(t)}x(t)\right|\leq M \text{ for all } t\in\mathbb{T}^*,$$

*where $\lambda$ is a fixed nonzero constant and $m(t)$ is as in (7).*

Then, we present the following result:

**Theorem 4.** *If the dynamical equation (12) has a $(T,\lambda)$-periodic solution in shifts, then it has a $\lambda$-bounded solution.*

*Proof:* Suppose that the dynamical equation in (12) has a $(T,\lambda)$-periodic solution in shifts, and fix

$$M=\sup_{k\in[t_0,T)}|x(k)|.$$

Then, we write

$$\begin{aligned}
\left|\lambda^{-m(t)}x(t)\right| &= \begin{cases} \left|\lambda^{-m(t)}x\left(\delta_+^{(m(t))}(T,t_0)\right)\right| & \text{if } t\in P(t_0) \\ \left|\lambda^{-m(t)}x\left(\delta_+^{(m(t))}(T,t_r)\right)\right| & \text{if } t\notin P(t_0) \end{cases} \\
&= \begin{cases} \left|\lambda^{-m(t)}\lambda^{m(t)}x(t_0)\right| & \text{if } t\in P(t_0) \\ \left|\lambda^{-m(t)}\lambda^{m(t)}x(t_r)\right| & \text{if } t\notin P(t_0) \end{cases} \\
&\leq M,
\end{aligned}$$

where we employ the representation given in (9) and use the fact $t_0<t_r<T$ for the case $t\notin P(t_0)$. Thus, the proof is complete. □

On the other hand, by assuming $x$ be a $\lambda$-bounded solution of the dynamical equation given in (12), we aim to prove a Massera type theorem by employing Brouwer's fixed point theorem as the continuation of this work. By obtaining some more detailed results regarding the time scale exponential function based on the new defined periodicity notion, we believe $(T,\lambda)$-periodic solutions of several types of dynamic equations can be studied.

 

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# Modeling of Variables Affecting Success with Deep Learning Methods

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**Abstract:** Deep learning methods are one of the machine learning models that have spread rapidly in the field of education as well as in many other fields in the last decade. This method is a fairly new method in educational literature. The aim of this study is to model and predict the mathematics achievement of the successful and the unsuccessful students using the deep learning method. For this purpose, Turkey's International Student Assessment Program (PISA) survey data was used. As a result of the analysis, Jordan method was found the most successful method among Elman, Logistics and MLP methods.

**Keywords:** Deep Learning, Elman Method, Jordan Method, Mathematics Achievement, PISA.

## 1 Introduction

Deep learning is one of the key artificial intelligence methods for prediction, classification or clustering. Deep learning is widely used in areas such as engineering, medicine and finance [1]. The success of the method is due to the functions that allow it to easily model complex relationships. Also, deep learning approach obtains higher accuracy than traditional artificial neural networks [2]. These features encourage the researchers to use Deep learning in the field of education for educational data mining purposes [3]. Studies show that the deep learning approach produces successful results in educational researches [1].

The complex relationship between variables that affect mathematics achievement encourages the use of deep learning in educational era [4]. For this purpose we used Turkish students' PISA mathematics score. PISA has been conducted by the OECD every 3 years since 2000 to 15-year-old students [5]. PISA consists of student, school and teacher questionnaires. Main purpose of this program is providing educational equality [6]. It also provides a comparison of the education systems of the countries.

Turkey got an average of 454 points in mathematics. The OECD average was 459 points. Turkey became 42th in mathematics with this score. Additionally, Turkey ranked 48th among 72 countries in mathematics achievement in the PISA 2015 [7]. Turkey's average mathematics score reached its highest level since 2003 [8]. Although this improvement is important, it is not sufficient. As it is seen, we need to detect and provide effective variables for continuous success. The main object of this study is detecting students' key predictors for mathematical success with deep learning approach. Also we aim to compare results with MLP and logistic regression.

## 2 Method and materials

### 2.1 Variables

Many variables are used to evaluate mathematics achievement. While some of these variables consist of indices, some of them are the questions in the PISA questionnaire. In this study, we preferred student gender, home possessions, Subjective well-being: sense of belenging to school, economic, social and cultural status, Parents' emotional support perceived by student and digital device usesage for learning or teaching during mathematics lessons in the last month as independent variables. We used the most successful $30\%$ and the least successful $30\%$ of the students in mathematics achievement as dependent variable.

### 2.2 Methodology

In this study, four different methods were used, namely Multilayer Perceptron, Elman neural network, Jordan neural network and Logistic regression.

Multi Layer Perceptrion (MLP): Multilayer Perceptron is one of the most important artificial neural network algorithms used for educational data mining. In many studies, artificial neural networks are used to predict achievement [9, 10]. However, deep learning methods are new in educational researches. A Deep Neural Network (DNN) is form from an input and output layers with a number of hidden layers between them. DNN is similar to MLP but it has many hidden layers than MLP. The multiple hidden layers are advantageous to solve complex decision problems.

A Deep Neural Network (DNN) is form from an input and output layers with a number of hidden layers between them. DNN is similar to MLP but it has many hidden layers than MLP. The multiple hidden layers are advantageous to solve complex decision problems.

Hidden layers produce new weighted values to transmit input values to the output layers. In learning DNN, network weights are updated to reduce error with output and the target values. The last DNN solution reveals the most appropriate symphysis of weights, and so the network characteristic approaches a given decision function. So, the DNN learns the decision function via implicit examples [11].

Elman Neural Network: This network is a kind of recurrent neural network (RNN). In a recurrent neural network, as neurons reconnect to other neurons, activation of neurons flows in a loop, making the flow of information multi-directional.

Elman neural networks are similar to MLP powered by context layers. Context layer neurons are equal number to the hidden layer neurons. And neurons in the context layer and neurons in the hidden layer are completely interconnected [12].

Jordan neural networks: Although Jordan and Elman neural networks are similar, the only difference is that in Jordan neural network the context neurons are fed with the output layer, not the hidden layer [13].

Logistic Regression: Logistic regression is a special case of regression analysis. Logistic regression method is widely used in the field of education to determine the factors affecting achievement [14]. Assumptions in logistic regression are same with classical regression. But, in the logistic regression models dependent variable is categorical [15].

## 3 Results and conclusions

A confusion matrix is a precise table format that approves evaluation of the algorithm performances. Each row of the matrix represents the cases in a true category while each column represents the cases in a estimated class [16]. The Confusion matrixes of the models are given in Table 1.

| | | MLP Method | | Elman Method | | Jordan Method | | Logistic Method | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | Unsuccessful | Successful | Unsuccessful | Successful | Unsuccessful | Successful | Unsuccessful | Successful | Total |
| | Unsuccessful | 1117 | 503 | 1190 | 430 | 1215 | 405 | 1100 | 520 | **1620** |
| | Successful | 442 | 1071 | 499 | 1014 | 447 | 1066 | 462 | 1051 | **1513** |
| Actual | Total | **1559** | **1574** | **1689** | **1444** | **1662** | **1471** | **1562** | **1571** | |

**Table 1** Confusion matrixes of the Models

Comparison criteria are used in order to understand the results obtained in this Confusion matrix correctly. The results of some comparison criteria are given in Table 2 below. Accuracy rate refers to the total rate of predictions with correct predictions.

Precision is the ratio of successfully estimated positive observations to the complete estimated positive observations. Specificity measures how exact the assignment to the positive class. The recall rate is the ratio of positive correct prediction all observations in the real class. The F1 score is an equirilium measure between precision and recall [15, 17].

In Table 2, we obtained the highest comparison values with Jordan method. As a result of the analysis, Jordan method was found as the most successful method among Elman, Logistics and MLP methods.

| | MLP | Elman | Jordan | Logistic |
|---|---|---|---|---|
| Accuracy | 0.698 | 0.703 | **0.728** | 0.687 |
| Precision | 0.716 | 0.705 | **0.731** | 0.704 |
| Recall | 0.690 | 0.735 | **0.750** | 0.679 |
| Specificity | 0.708 | 0.670 | **0.705** | 0.695 |
| F1 | 0.703 | 0.719 | **0.740** | 0.691 |

**Table 2** Comparison of the Model Performance

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# On the Solution of Mathematical Problem Including Sequential Time Fractional Wave Equation

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**Abstract:** The purpose of this study is to establish the analytic solution of sequential time fractional wave equation subject to Dirichlet boundary and initial conditions, by separation of variables method. The fractional derivative is taken in Caputo sense. The analytic solution is constructed in series form in terms of fractional trigonometric functions.

**Keywords:** Caputo fractional derivative, Fractional trigonometric function, Mittag-Leffler function, Time fractional wave equation.

## 1 Introduction

Analytical solutions of fractional order differential equations has been a very intriguing topic for many researchers since various fields such as mathematics, physics, chemistry, biology, and engineering focus on this subject [1]-[14]. Due to these interests, various fractional derivative methods have emerged. The time and space-fractional derivatives are taken in Caputo sense which is more common and more suitable than other fractional derivatives. In this study, we investigate the analytical solution of the wave equation subject to the Dirichlet boundary and initial conditions. In this study, we deal with the following initial-boundary value problem involving time fractional wave equation by separation of variables

$$\begin{cases} D_t^{2\alpha}u(x,t) = c^2u_{xx}(x,t) \,, 0 < x < 1, 0 < \alpha < 1, \\ u(x,0) = \Phi(x), D_t^{\alpha}u(x,0) = \Psi(x), \\ u(0,t) = u(1,t) = 0. \end{cases} \tag{1}$$

For $\alpha = 1$, we have the following problem:

$$\begin{cases} u_{tt}\left(x,t\right) - c^2u_{xx}\left(x,t\right) = 0, 0 < x < 1, \\ u\left(x,0\right) = \Phi\left(x\right), u_t\left(x,t\right) = \Psi\left(x\right), \\ u\left(0,t\right) = u\left(1,t\right) = 0. \end{cases}$$

The separation of variables method is applied to reduce the problem to two separate fractional ODEs. The analytic solution is acquired in the form of a Fourier series with respect to the eigenfunctions of a certain eigenvalue problem.

## 2 Preliminary results

In this section, the fundamental definitions and properties of fractional calculus are presented.

**Definition 1.** *Riemann-Liouville time-fractional integral of a real valued function $u\left(x,t\right)$ is defined as*

$$I_t^{\alpha}u\left(x,t\right) = \frac{1}{\Gamma\left(\alpha\right)}\int_0^t\left(t-s\right)^{\alpha-1}u\left(x,s\right)ds, \tag{2}$$

*where $\alpha > 0$ denotes the order of the integral.*

**Definition 2.** *The $q^{th}$ order Caputo fractional derivative of $u(t)$ is defined as follows:*

$$D_t^q u\left(t\right) = \frac{1}{\Gamma\left(m-q\right)}\int_0^t\left(t-s\right)^{m-q-1}\frac{d^m}{ds^m}u\left(s\right)ds, m-1 < q < m(m \in \mathbb{Z}). \tag{3}$$

**Definition 3.** *The two parameter Mittag-Leffler function is defined as:*

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},Re(\alpha)>0,\ z,\beta\in\mathbb{C}. \tag{4}$$

Some properties of Mittag-Leffler function are given as follows:

- If $\alpha=1$ and $\beta=1$, $E_{1,1}(t)=e^t$. Thus, it can be said that Mittag-Leffler function is generalization of usual exponential function.
- $E_{q,1}(t)>0 if 0<q<1$.
- $E_{q,1}(t)$ is monotone increasing if $0<q<1$ and $t>0$.
- $D^q(E_{q,1}(t^q))=E_{q,1}(t^q)$.
- $D^{nq}(E_{q,1}(\mathrm{rt}^q))=r^nE_{q,1}(\mathrm{rt}^q)$ where $0<q<1$, $r$ is a constant and $n\in\mathbb{N}$.

**Definition 4.** *The functions*

$$\sin_q(\mu t)=\frac{E_{q,1}(i\mu t^q)-E_{q,1}(-i\mu t^q)}{2i}=\sum_{k=0}^{\infty}\frac{(-1)^k(\mu t^q)^{2k+1}}{\Gamma((2k+1)q+1)}$$

*and*

$$\cos_q(\mu t)=\frac{E_{q,1}(i\mu t^q)+\ E_{q,1}(-i\mu t^q)}{2}=\sum_{k=0}^{\infty}\frac{(-1)^k(\mu t^q)^{2k}}{\Gamma(2kq+1)}$$

*are called fractional trigonometric functions. Notice that these functions are usual trigonometric functions $sin(\mu t)$ and $cos(\mu t)$ when* $q=1$.

## 3 Main results

Since we assume $D_t^{2\alpha}u(x,t)$ to be sequential, it obeys the following composite rule: $\mathrm{D}^{2\alpha}u=\mathrm{D}^{\alpha}(\mathrm{D}^{\alpha}u)$.

A particular solution of the problem (1), we seek is in the following form:

$$u(x,t;\alpha)=X(x)T(t;\alpha) \tag{5}$$

where $0\le x\le l$, $0\le t\le T$.

Let us try to find all the separated solutions of the wave equation. Substituting (5) into the equation leads to the following:

$$X(x)\mathrm{D_t^{2\alpha}T(t;\alpha)}=\ \mathrm{c^2X}''(x)T(t;\alpha).$$

Dividing both sides of this equation by $-\mathrm{c}^2X(x)T(t;\alpha)$ results in

$$-\frac{\mathrm{X}''(\mathrm{x})}{\mathrm{X}(\mathrm{x})}=-\frac{D_t^{2\alpha}T(t;\alpha)}{T(t;\alpha)}=\lambda. \tag{6}$$

Clearly $\lambda$ is a constant, since it is independent of space variable $x$ since $\lambda=-\frac{\mathrm{D_t^{2\alpha}T(t;\alpha)}}{\mathrm{c^2T(t;\alpha)}}$ and is independent of $t$ since $\lambda=-\frac{\mathrm{X}''(\mathrm{x})}{\mathrm{X(x)}}$. From the boundary conditions in (1), we have

$$X(0)T(t)=X(l)T(t)=0,\forall t\to X(0)=X(l)=0. \tag{7}$$

The equations in (6) reduce the problem to two separate ODEs including fractional derivatives with respect to time. The equation on the left with boundary conditions (7) gives the following fractional differential equation

$$X''(x)+\lambda X(x)=0, \tag{8}$$

$$X(0)=0,\ X(l)=0$$

which has the solution in the following form

$$X(x)=Ccos(\beta x)+Dsin(\beta x) \tag{9}$$

Boundary condition for $X(x)$ leads to $X(0)=C=0$ and $X(l)=Dsin(\beta l)=0$. The solution with $D=0$ will lead to the trivial zero solution, so we consider the case when $sin(\beta l)=0$. As a result we get $\beta l=n\pi$ for $n=1,2,3,\ldots$ and $\lambda_{\mathrm{n}}=(\frac{\mathrm{n\pi}}{\mathrm{l}})^2$, $\mathrm{X_n}(x)=sin(\frac{\mathrm{n\pi x}}{\mathrm{l}})$ for $n=1,2,3,\ldots$

The equation on the right of (7) for each eigenvalue $\lambda_n$ gives the following fractional differential equation

$$D^{2\alpha}T_n(t;\alpha)+c^2\left(\frac{n\pi}{l}\right)^2T_n(t;\alpha)=0$$

which has the following solution for $n=1,2,3,\ldots$

$$T_n(t;\alpha) = A_n \cos_\alpha\left(\frac{cn\pi}{l}t^\alpha\right) + B_n \sin_\alpha\left(\frac{cn\pi}{l}t^\alpha\right).$$

For each eigenvalue $\lambda_n = (\frac{n\pi}{l})^2$, we construct the following functions for $n = 1, 2, 3, ...$

$$u_n(x,t;\alpha) = X_n(x)\,T_n(t;\alpha) = \left[A_n \cos_\alpha\left(\frac{cn\pi}{l}t^\alpha\right) + \ B_n \sin_\alpha\left(\frac{cn\pi}{l}t^\alpha\right)\right]\sin\left(\frac{n\pi x}{l}\right) \tag{10}$$

where $A_n$, $B_n$ are arbitrary constants. Since a linear combination of solutions of the wave equation is also a solution, any finite sum $u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos_\alpha\left(\frac{cn\pi}{l}t^\alpha\right) + B_n \sin_\alpha\left(\frac{cn\pi}{l}t^\alpha\right)\right] sin\left(\frac{n\pi x}{l}\right)$ is also a solution which satisfies both the fractional equations and boundary conditions in (1). Returning to our boundary value problem (1), we would like to find the solution as a linear combination of separated solutions. However, finite sums in the form (10) are very special, since not every function is as a sum of sines and cosines. Writing the initial conditions, we have

$$\begin{aligned}
\Phi(x) &= \sum_{n=1}^{\infty} A_n sin\left(\frac{n\pi x}{l}\right),\\
D^\alpha u(x,t) &= \sum_{n=1}^{\infty}\left[A_n \frac{cn\pi}{l}\sin_\alpha\left(\frac{cn\pi}{l}t^\alpha\right) + B_n \frac{cn\pi}{l} cos_\alpha\left(\frac{cn\pi}{l}t^\alpha\right)\right] sin\left(\frac{n\pi x}{l}\right),\\
\Psi(x) &= \sum_{n=1}^{\infty} B_n \frac{cn\pi}{l} sin\left(\frac{n\pi x}{l}\right)
\end{aligned}$$

In order to determine the coefficients $A_n, B_n$, taking the initial conditions $u(x,0) = \Phi(x), D^\alpha u(x,0) = \Psi(x)$ into account produce the following:

$$\begin{aligned}
A_n &= \frac{2}{l}\int_0^l sin\left(\frac{n\pi x}{l}\right)\Phi(x)dx,\\
\frac{n\pi c}{l}B_n &= \frac{2}{l}\int_0^l sin\left(\frac{n\pi x}{l}\right)\Psi(x)dx.
\end{aligned}$$

Finally the solution becomes

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos_\alpha\left(\frac{cn\pi}{l}t^\alpha\right) + B_n \sin_\alpha\left(\frac{cn\pi}{l}t^\alpha\right)\right] sin\left(\frac{n\pi x}{l}\right).$$

## 4 Illustrative example

In this section some illustrative examples are presented to prove the effectiveness and accuracy of the method , used in this study.

**Example 1.** *Consider following initial-boundary value problem involving time fractional wave equation:*

$$\begin{cases}
D_t^{2\alpha}u(x,t) = c^2 u_{xx}(x,t)\,, 0 < x < 1,\ t > 0, 0 < \alpha < 1,\\
u(x,0) = \sin(5\pi x) + 2\sin(7\pi x)\,, D_t^\alpha u(x,0) = 0,\\
u(0,t) = u(1,t) = 0.
\end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos_\alpha\left(cn\pi t^\alpha\right) + B_n \sin_\alpha\left(cn\pi t^\alpha\right)\right]\sin(n\pi x).$$

*This time it is not necessary to use the integral formula to evaluate* $A_n$ *and* $B_n$. *It is easier to observe directly, just by matching the coefficients, that*

$$\sin(5\pi x) + 2\sin(7\pi x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \Rightarrow A_n = \begin{cases} 1\ if\ n = 5,\\ 2\ if\ n = 7,\\ 0\ if\ n \neq 5, 7.\end{cases}$$

$$0 = D_t^\alpha u(x,0) = \sum_{n=1}^{\infty} \mathrm{cn}\pi B_n \mathrm{sin(n\pi x)} \ \Rightarrow\ B_n = 0\ for\ n \in \mathbb{N}$$

*As a result we have the following solution*

$$u(x,t) = \sin(5\pi x)\,\frac{E_{q,1}(i\mu t^q) +\ E_{q,1}(-i\mu t^q)}{2} + 2\sin(7\pi x)\,\frac{E_{q,1}(i\mu t^q) - E_{q,1}(-i\mu t^q)}{2i}$$

*where* $\mu = 5c\pi$.

**Example 2.** *Consider following initial-boundary value problem involving time fractional wave equation:*

$$\begin{cases} D_t^{2\alpha} u(x,t) = c^2 u_{xx}(x,t)\ , 0 < x < 1,\ t > 0, 0 < \alpha < 1, \\ u(x,0) = x(1-x), D_t^{\alpha} u(x,0) = 0, \\ u(0,t) = u(1,t) = 0. \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} \left[ A_n \cos_\alpha \left( cn\pi t^\alpha \right) + B_n \sin_\alpha \left( cn\pi t^\alpha \right) \right] sin\left( n\pi x \right).$$

*where the coefficients* $\mathrm{A_n}$ *and* $\mathrm{B_n}$ *are computed as*

$$\begin{aligned} A_n &= 2\int_0^1 x(1-x)\sin\left(n\pi x\right)dx, \\ B_n &= 2\int_0^1 0\sin\left(n\pi x\right)dx = 0. \end{aligned}$$

*We have*

$$\mathrm{A_n} = \begin{cases} \frac{8}{n^3\pi^3}\ for\ n\ odd, \\ 0\ for\ n\ even. \end{cases}$$

*and*

$$u\left(x,t\right) = \sum_{n=1,\ \ n\ odd}^{\infty} \frac{8}{n^3\pi^3} \sin\left(n\pi x\right) \cos_\alpha \left( cn\pi t^\alpha \right)$$

## 5 Conclusion

In this research, analytical solutions of the boundary time fractional wave equation using the Dirichlet boundary conditions are established. By means of Mittag-Leffler, the solution is constructed in terms Mittag-Leffler function and fractional trigonometric functions in the series form.

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# On Rough $\mathcal{I}$-Convergence and Rough $\mathcal{I}$-Cauchy Sequence

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**Abstract:** In this study, we first defined the concept of rough $\mathcal{I}^*$-convergence and investigated the relations between rough $\mathcal{I}$-convergence and rough $\mathcal{I}^*$-convergence. Then, we introduced the notion of rough $\mathcal{I}$-Cauchy sequence and examined the relations between rough $\mathcal{I}$-convergence and rough $\mathcal{I}$-Cauchy sequence. Finally, we introduced the notion of rough $\mathcal{I}^*$-Cauchy sequence and investigated the relations between rough $\mathcal{I}$-Cauchy sequence and rough $\mathcal{I}^*$-Cauchy sequence.

**Keywords:** Ideal, Rough convergence, $\mathcal{I}$-Convergence, Rough $\mathcal{I}$-Convergence.

## 1 Introduction and Background

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [11] and Schoenberg [19]. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [14] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers. Kostyrko et al. [13] studied the idea of $\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points. Nabiev, Pehlivan and Gürdal [15] introduced the notions of $\mathcal{I}$-Cauchy sequence and $\mathcal{I}^*$-Cauchy sequence and then studied their certain properties.

The idea of rough convergence was first introduced by Phu [16] in finite-dimensional normed spaces. In [16], he showed that the set $\text{LIM}^r x$ is bounded, closed, and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of $\text{LIM}^r x$ on the roughness degree $r$. In another paper [17] related to this subject, he defined the rough continuity of linear operators and showed that every linear operator $f : X \to Y$ is $r$ -continuous at every point $x \in X$ under the assumption $dimY < \infty$ and $r > 0$ where $X$ and $Y$ are normed spaces. In [18], he extended the results given in [16] to infinite-dimensional normed spaces. In [5] Aytar studied rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, in [6] Aytar studied that the $r$-limit set of the sequence is equal to the intersection of these sets and that $r$-core of the sequence is equal to the union of these sets. Recently, Dündar and Çakan [8, 9] and Dündar [7] introduced the notion of rough $\mathcal{I}$-convergence and the set of rough $\mathcal{I}$-limit points of a sequence and studied the notions of rough convergence, $\mathcal{I}_2$-convergence and the sets of rough limit points and rough $\mathcal{I}_2$-limit points of a double sequence. Also, Arslan and Dündar [3, 4] introduced rough convergence and investigated some properties in 2-normed spaces.

In this paper, we first defined the concept of rough $\mathcal{I}^*$-convergence and investigated the relations between rough $\mathcal{I}$-convergence and rough $\mathcal{I}^*$-convergence. Then, we introduced the notion of rough $\mathcal{I}$-Cauchy sequence and examined the relations between rough $\mathcal{I}$-convergence and rough $\mathcal{I}$-Cauchy sequence. Finally, we introduced the notion of rough $\mathcal{I}^*$-Cauchy sequence and investigated the relations between rough $\mathcal{I}$-Cauchy sequence and rough $\mathcal{I}^*$-Cauchy sequence. We note that our results and proof techniques presented in this paper are $\mathcal{I}$ analogues of those in Phu's [16] paper, Aytar' s [5] paper and Dündar and Çakan's [8] paper. Their papers include the actual origin of most of these results and proof techniques. Our following theorems and results are the $\mathcal{I}$-extension of theorems and results in [5, 16].

Now, we recall certain fundamental definitions and notations (See [1]-[4], [7]-[10], [12], [14]-[18], [20]-[21]).

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of $X$ is said to be an ideal in $X$ provided:
i) $\emptyset \in \mathcal{I}$,
ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.
$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of $X$ is said to be a filter in $X$ provided:
i) $\emptyset \notin \mathcal{F}$,
ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,
iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

 

**Lemma 1.** *[14] If $\mathcal{I}$ is a nontrivial ideal in $X$, $X \neq \emptyset$, then the class*

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$$

*is a filter on $X$, called the filter associated with $\mathcal{I}$.*

A nontrivial ideal $\mathcal{I}$ in $X$ is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Throughout the paper, let $\mathbb{R}^n$ be a real $n$-dimensional space with the norm $\|.\|$, $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal and $r$ be a nonnegative real number. Consider a sequence $x = (x_k)$ such that $x_k \in \mathbb{R}^n$.

A sequence $x = (x_k)$ is said to be $r$-convergent to $L$, denoted by $x_k \xrightarrow{r} L$ provided that

$$\forall \varepsilon > 0 \ \exists k_\varepsilon \in \mathbb{N} : \ k \geq k_\varepsilon \Rightarrow \|x_k - L\| < r + \varepsilon.$$

The set

$$\text{LIM}^r x := \{L \in \mathbb{R}^n : x_k \to^r L\}$$

is called the $r$-limit set of the sequence $x = (x_k)$. A sequence $x = (x_k)$ is said to be $r$-convergent if

$$\text{LIM}^r x \neq \emptyset.$$

In this case, $r$ is called the convergence degree of the sequence $x = (x_k)$.

For $r = 0$, we get the ordinary convergence. There are several reasons for this interest (see [16]).

The sequence $(x_k)$ is said to be a rough Cauchy sequence satisfying

$$\forall \varepsilon > 0, \exists n_\varepsilon : k, n \geq n_\varepsilon \Rightarrow \|x_k - x_n\| < \rho + \varepsilon$$

for $\rho > 0$. $x = (x_k)$ is said to be a rough Cauchy sequence with roughness degree $\rho$, or $\rho$-Cauchy sequence for short. $\rho$ is also called a Cauchy degree of $(x_k)$.

A sequence $x = (x_k)$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}^n$, written as $\mathcal{I}$-$\lim x = L$, provided that the set

$$\{k \in \mathbb{N} : \|x_k - L\| \geq \varepsilon\}$$

belongs to $\mathcal{I}$ for every $\varepsilon > 0$. In this case, $L$ is called the $\mathcal{I}$-limit of the sequence $x$.

Note that if $\mathcal{I}$ is an admissible ideal, then usual convergence implies $\mathcal{I}$-convergence.

A sequence $x = (x_k)$ is said to be rough $\mathcal{I}$-convergent to $L$, denoted by $x_k \xrightarrow{r-\mathcal{I}} L$ provided that

$$\{k \in \mathbb{N} : \|x_k - L\| \geq r + \varepsilon\}$$

belongs to $\mathcal{I}$ for every $\varepsilon > 0$; or equivalently, if the condition

$$\mathcal{I} - \limsup \|x_k - L\| \leq r \tag{1}$$

is satisfied. In addition, we can write $x_k \xrightarrow{r-\mathcal{I}} L$ iff the inequality

$$\|x_k - L\| < r + \varepsilon$$

holds for every $\varepsilon > 0$ and almost all $k$.

Here $r$ is called the roughness degree. If we take $r = 0$, then we obtain the ordinary ideal convergence. In a similar fashion to the idea of classic rough convergence, the idea of rough $\mathcal{I}$-convergence of a sequence can be interpreted as follows.

In general, the rough $\mathcal{I}$-limit of a sequence may not be unique for the roughness degree $r > 0$. So we have to consider the so-called rough $\mathcal{I}$-limit set of a sequence $x = (x_k)$, which is defined by

$$\mathcal{I} - \text{LIM}^r x = \{L \in \mathbb{R}^n : x_k \xrightarrow{r-\mathcal{I}} L\}.$$

A sequence $x = (x_k)$ is said to be rough $\mathcal{I}$-convergent if

$$\mathcal{I} - \text{LIM}^r x \neq \emptyset.$$

It is clear that if

$$\mathcal{I} - \text{LIM}^r x \neq \emptyset$$

for a sequence $x = (x_k)$ of real numbers, then we have

$$\mathcal{I} - \text{LIM}^r x = [\mathcal{I} - \limsup x - r, \mathcal{I} - \liminf x + r]. \tag{2}$$

We know that $\text{LIM}^r x = \emptyset$ for an unbounded sequence $x = (x_k)$. But such a sequence might be rough $\mathcal{I}$-convergent.

### *1.1 Main Results*

In this section, we first defined the concept of rough $\mathcal{I}^*$-convergence and investigated the relations between rough $\mathcal{I}$-convergence and rough $\mathcal{I}^*$-convergence.

**Definition 1.** *A sequence $(x_k)$ is said to be rough $\mathcal{I}^*$-convergent to $L \in \mathbb{R}^n$ if and only if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \dots\}$, $M \in \mathcal{F}(\mathcal{I})$ (that is $\mathbb{N}\backslash M \in \mathcal{I}$) such that*

$$r - \lim_{k\to\infty} \|x_{m_k} - L\| = 0,$$

*that is,*

$$\forall\varepsilon > 0\ \exists k_0 = k_0(\varepsilon) \in \mathbb{N} :\ k \geq k_0 \Rightarrow \|x_{m_k} - L\| < r + \varepsilon.$$

*In this case, we write*

$$x_k \overset{r-\mathcal{I}^*}{\longrightarrow} L.$$

**Theorem 1.** *For a sequence $(x_k)$, if $x_k \overset{r-\mathcal{I}^*}{\longrightarrow} L$, then $x_k \overset{r-\mathcal{I}}{\longrightarrow} L$.*

*Proof:* Let $(x_k)$ be a rough $\mathcal{I}^*$-convergent to $L \in \mathbb{R}^n$. Then, there exists a set $H \in \mathcal{I}$ such that for

$$M = \mathbb{N}\backslash H = \{m_1 < m_2 < \cdots < m_k < \dots\}$$

we have

$$r - \lim_{k\to\infty} \|x_{m_k} - L\| = 0. \tag{3}$$

Let $\varepsilon > 0$. By virtue of 3, there exists $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that

$$\|x_{m_k} - L\| < r + \varepsilon$$

for all $k > k_0$. Then, it is clearly that

$$A(\varepsilon) : \{n \in \mathbb{N} : \|x_k - L\| \geq r + \varepsilon\} \subset H \cup \{m_1 < m_2 < \cdots < m_{k_0}\}. \tag{4}$$

Since $\mathcal{I}$ is admissible, by 4

$$H \cup \{m_1 < m_2 < \cdots < m_{k_0}\} \in \mathcal{I}$$

and so by the definition of ideal

$$A(\varepsilon) \in \mathcal{I}.$$

Hence, we have

$$x_k \overset{r-\mathcal{I}}{\longrightarrow} L.$$

□

Now, we give definition of rough $\mathcal{I}$-Cauchy sequence and examined the relations between rough $\mathcal{I}$-convergence and rough $\mathcal{I}$-Cauchy sequence.

**Definition 2.** *A sequence $(x_k)$ is called rough $\mathcal{I}$-Cauchy sequence in $\mathbb{R}^n$, if for $\varepsilon > 0$ there exists $n = n(\varepsilon) \in \mathbb{N}$ such that*

$$A(\varepsilon) = \{k \in \mathbb{N} : \|x_k - x_n\| \geq r + \varepsilon\} \in \mathcal{I}$$

*for $r > 0$. $x = (x_k)$ is said to be a rough $\mathcal{I}$-Cauchy sequence with roughness degree $r$, or $r$-$\mathcal{I}$-Cauchy sequence for short. $r$ is also called an $\mathcal{I}$-Cauchy degree of $(x_k)$.*

**Theorem 2.** *If a sequence $(x_k)$ is rough $\mathcal{I}$-convergent, then it is rough $\mathcal{I}$-Cauchy sequence.*

*Proof:* Suppose that $(x_k)$ is rough $\mathcal{I}$-convergent to $L \in \mathbb{R}^n$. Then for $\varepsilon > 0$

$$A\left(\frac{\varepsilon}{2}\right) = \left\{k \in \mathbb{N} : \|x_k - L\| \geq r + \frac{\varepsilon}{2}\right\} \in \mathcal{I}.$$

This implies that

$$A^c\left(\frac{\varepsilon}{2}\right) = \left\{k \in \mathbb{N} : \|x_k - L\| < r + \frac{\varepsilon}{2}\right\} \in \mathcal{F}(\mathcal{I})$$

and therefore $A^c(\frac{\varepsilon}{2})$ is non-empty. So, we can choose a positive integer $n$ such that $n \notin A(\frac{\varepsilon}{2})$, we have

$$\|x_n - L\| < r + \frac{\varepsilon}{2}.$$

Now, we define the set

$$B(\varepsilon) = \{k \in \mathbb{N} : \|x_k - x_n\| \geq 2r + \varepsilon\} \in \mathcal{I}$$

such that we show that

$$B(\varepsilon) \subset A(\frac{\varepsilon}{2}).$$

Let $k \in B(\varepsilon)$, then we have

$$\begin{aligned} 2r + \varepsilon \leq \|x_k - x_n\| &\leq \|x_k - L\| + \|x_n - L\| \\ &< \|x_k - L\| + r + \frac{\varepsilon}{2}. \end{aligned}$$

This implies

$$r + \frac{\varepsilon}{2} < \|x_k - L\|$$

and so $k \in A(\frac{\varepsilon}{2})$. Hence, we have

$$B(\varepsilon) \subset A(\frac{\varepsilon}{2})$$

and $(x_k)$ is a rough $\mathcal{I}$-Cauchy sequence. □

Finally, we introduced the notion of rough $\mathcal{I}^*$-Cauchy sequence and investigated the relations between rough $\mathcal{I}$-Cauchy sequence and rough $\mathcal{I}^*$-Cauchy sequence.

**Definition 3.** *A sequence $(x_k)$ is called rough $\mathcal{I}^*$-Cauchy sequence in $\mathbb{R}^n$, if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \dots\} \subset \mathbb{N}$, $M \in \mathcal{F}(\mathcal{I})$ (that is, $\mathbb{N} \backslash M \in \mathcal{I}$), such that the subsequence $x_K = (x_{m_k})$ is an ordinary Cauchy sequence in $\mathbb{R}^n$, that is,*

$$r - \lim_{k,j \to \infty} \|x_{m_k} - x_{m_j}\| = 0.$$

**Theorem 3.** *If a sequence $(x_k)$ is a rough $\mathcal{I}^*$-Cauchy sequence then $(x_k)$ is rough $\mathcal{I}$-Cauchy sequence in $\mathbb{R}^n$.*

*Proof:* Let $(x_k)$ be a rough $\mathcal{I}^*$-Cauchy sequence in $\mathbb{R}^n$. Then, by definition, for every $\varepsilon > 0$ there exist $k_0 = k_0(\varepsilon) \in \mathbb{N}$ and a set $M = \{m_1 < m_2 < \cdots < m_k < \dots\} \subset \mathbb{N}$, $M \in \mathcal{F}(\mathcal{I})$ such that for all $k, j > k_0$

$$\|x_{m_k} - x_{m_j}\| < r + \varepsilon.$$

Let $K = K(\varepsilon) = m_{k_0+1}$. Then, for every $\varepsilon > 0$ and all $k > k_0$, we have

$$\|x_{m_k} - x_K\| < r + \varepsilon.$$

Now, let $H = \mathbb{N} \backslash M$. Then obviously, $H \in \mathcal{I}$ and

$$A(\varepsilon) = \{k \in \mathbb{N} : \|x_{m_k} - x_K\| > r + \varepsilon\} \subset H \cup \{m_1 < m_2 < \cdots < m_{k_0}\}. \tag{5}$$

Since $\mathcal{I}$ is admissible, by 5, we have

$$H \cup \{m_1 < m_2 < \cdots < m_{k_0}\} \in \mathcal{I}$$

and so by the definition of ideal

$$A(\varepsilon) \in \mathcal{I}.$$

Hence, $(x_k)$ is rough $\mathcal{I}$-Cauchy sequence in $\mathbb{R}^n$. □

## Conclusion

We gave definitions of rough $\mathcal{I}^*$-convergence, rough $\mathcal{I}$-Cauchy sequence and rough $\mathcal{I}^*$-Cauchy sequence. Our results include that for a sequence $x = (x_n)$, relations between these new concepts.

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# A Novel Decision-Making Method in Pythagorean Fuzzy Soft Sets with Heart Disease Application

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**Abstract:** Pythagorean fuzzy set characterized by a membership degree and a non-membership degree, which satisfies the condition that the square sum of its membership degree and non-membership degree is less than or equal to 1. As a generalized set, Pythagorean fuzzy sets have a close relationship with intuitionistic fuzzy sets. The intuitionistic fuzzy set played an important role in decision-making problems in a very short period of time and was successfully used in many decision-making problems. However, in some real-life problems, the sum of membership degree and nonmembership degree may be greater than 1. The sum of the squares of these degrees is less than 1. In this case, the Pythagorean fuzzy set is a very useful tool and enables more effective results in multiple attribute decision-making problems. In the present study, for the medical decision-making problem, the new method is proposed related to the Pythagorean fuzzy soft set. The real dataset which is called the Cleveland heart disease dataset is applied to this problem.

**Keywords:** Cleveland heart dataset, Comparison table, Decision-making, Pythagorean fuzzy soft set.

## 1 Introduction

Uncertainty is a crucial concept for decision-making problems. It is not easy to make precise decisions in life since each information contains vagueness, uncertainty, imprecision. Therefore, for such situations, different approaches were improved like fuzzy sets [1], intuitionistic fuzzy sets(IFS) [2], soft sets(SS) [3], Pythagorean fuzzy sets(PFS) [4].

Yager [5] offered a new FS called Pythagorean fuzzy set(PFS). PFS attracted the attention of many researchers in a short time. The formulation of the negation for IFSs and PFSs is examined by Yager [4]. In [6],PF subsets and its relationship with IF subsets were debated and some set operations on PF subsets were defined. In [7], the properties such as boundedness, idempotency, and monotonicity related to the Pythagorean fuzzy aggregation operators are investigated. Further, to solve uncertainty, multiple attribute group, DM problems Pythagorean fuzzy superiority and inferiority ranking method was developed in [7]. Pythagorean fuzzy soft set(PFSS) is a combination of PFS and SS and was defined by Peng et.al [8]. A PFSS is a parametrized family of PFSs and a generalization of IFSSs. Kirisci extended PFSS to Pythagorean fuzzy parametrized pythagorean fuzzy soft set [9].

In this presentation, we propose a new decision-making method for Pythagorean fuzzy soft sets(PFSSs). The characteristic of this method is that it has object recognition from a number of multi-observer data, which is imprecise with PFSS theoretic approach. The real dataset which is called Cleveland heart disease dataset applied in this method.

## 2 Preliminaries

Some knowledge that will be used throughout the article will be given. Let $E, N$ be an initial universe and parameter sets, respectively.

For $a \in E$, the set $B = \{(a, d_B(a), y_B(a)) : a \in E\}$ in $E$ is called *Pythagorean fuzzy set*(PFS), where $d_B : E \to [0,1]$ and $y_B : I \to [0,1]$ together the situation that $0 \leq [d_B(a)]^2 + [y_B(a)]^2 \leq 1$ in [4, 5, 10]. The degree of indeterminacy $b_B = \sqrt{1 - [d_B(a)]^2 - [y_B(a)]^2}$.

For $B \subseteq N$, choose $\widetilde{F} : B \to \upsilon(E)$, where the set of all PFSs over $E$ is indicated by $\upsilon(E)$. Then, a pair $\widetilde{T}_B = (\widetilde{I}, B)$ is called *Pythagorean Fuzzy Soft Set*(PFSS) on $E$ [8].

 

Take $k = \{(r,s) : r^2 + s^2 < 1, r, s \in [0,1], \}$. Let $(K, \leq_K)$ be a complete lattice. The corresponding partial order $\leq_K$ is defined by

$$(r,s) \leq_K (t,u) \quad \Leftrightarrow \quad r \leq t \quad and \quad s \geq u$$

for all $(r,s),(t,u) \in K$. The Pythagorean fuzzy value(PFV) is denoted by an ordered pair $(r,s) \in K$ [11].

Choose two PFVS $P = (d_P, y_P), R = (d_R, y_R) \in K$.

i. $P \oplus R = \left(\sqrt{d_P^2 + d_R^2 - d_P^2 d_R^2}, y_P^2 y_R^2\right)$,
ii. $P \otimes R = \left(d_P^2 d_R^2, \sqrt{y_P^2 + y_R^2 - y_P^2 y_R^2}\right)$,
ii. $\alpha P = \left(\sqrt{1-(1-d_P^2)^\alpha}, (y_P)^\alpha\right)$ for $\alpha > 0$,
iv. $P^\alpha = \left((d_P)^\alpha, \sqrt{1-(1-y_P^2)^\alpha},\right)$ for $\alpha > 0$ [12].

Therefore, Yager [5] proposed PFS characterized by a MD and a ND, which satisfies the condition that the square sum of its MD and ND is less than or equal to 1. Yager [10] gave an example to state this situation: a DMKR gives his support for membership of an alternative is $\frac{\sqrt{3}}{2}$ and his against membership is $\frac{1}{2}$. Owing to the sum of two values is bigger than 1, they are not available for IFS, but they are available for PFS since $(\frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2 \leq 1$. Obviously, PFS is more capable than IFS to model the vagueness in the practical multicriteria decision-making problems.

The main difference between PFNs and IFNs is their corresponding constraint conditions, which can be easily shown in Figure 1. Here, we observe that intuitionistic membership grades are all points under the line $m + n \leq 1$ and the Pythagorean membership grades are all points with $m^2 + n^2 \leq 1$.

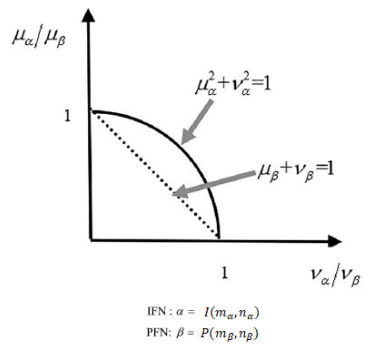


**Fig. 1**: The PFNs and the IFNs

## 3 Method

Input variables are taken from Cleveland dataset [13]. This data set contains 303 patients, 11 attributes and 5 outcomes.

### *3.1 PFSS method*

Choose a set of $k$ objects as $E = \{\epsilon_1, \epsilon_2, \cdots, \epsilon_k\}$, and a set of parameters $\{N(1), N(2), \cdots, N(i)\}$. Each parameter set $N(i)$ represent the $i$th class of parameters and the elements of $N(i)$ indicates a certain property set. Assumed that the property sets can be shown as FSs.

Let $\widetilde{T}_A, \widetilde{T}_B$ be the PFSSs on $E$ [8].

i. The operation $\widetilde{T}_A \wedge \widetilde{T}_B$ is called "AND" operator on $\widetilde{T}_A, \widetilde{T}_B$ such that $\widetilde{T}_A \wedge \widetilde{T}_B = \widetilde{T}_B(A \times B) = \{\min(d_A, d_B), \max(y_A, y_B)\}$.

ii. The operation $\widetilde{T}_A \vee \widetilde{T}_B$ is called "OR" operator on $\widetilde{T}_A, \widetilde{T}_B$ such that $\widetilde{T}_A \vee \widetilde{T}_B = \{\max(d_A, d_B), \min(y_A, y_B)\}$.

A square table with equal row numbers and column numbers is called a Comparison Table. This table contains $\epsilon_1, \epsilon_2, \dots \epsilon_n$ object names in both rows and columns. The entries in Comparison Table are denoted by $c_{ij}$, $(i, j = 1, 2, \dots, n)$. These entries are defined as is the number of parameters for which the membership value of $\epsilon_i$ greater than or equal to the membership value of $\epsilon_j$ and non-membership values of $\epsilon_i$ less than or equal to the non-membership value of $\epsilon_j$. If we take the number of parameters in PFSS as $k$, then it is clear that $0 \leq c_{ij} \leq k$, for all $i, j$. Further $c_{ii} = k$. From here it is understood that $c_{ij}$ is an integer number as a numerical measure. The formula $r_i = \sum_{j=1}^{n} c_{ij}$ calculates the row sum for an object $\epsilon_i$. In this calculations, $r_i$ indicates the total number of parameters in which $\epsilon_i$ dominates all the members of $E$. In the same way, the formula $t_j = \sum_{i=1}^{n} c_{ij}$ yields the column sum for an object $\epsilon_j$. In this summation, the integer $t_j$ indicates the total number of parameters in which $\epsilon_j$ is dominated by all the members of $E$. The formula $S_i = r_i^2 - t_j^2$ give the score of an object $\epsilon_i$.

Algorithm:

i. Input the PFSSs $\widetilde{T}_A$, $\widetilde{T}_B$ and $\widetilde{T}_C$,
ii. Input the parameter set $N$ obtained as a result of observations,,
iii. Compute the corresponding PFSSs $\widetilde{T}_A \wedge \widetilde{T}_B = \widetilde{T}_D$ from the FSSs $\widetilde{T}_A$, $\widetilde{T}_B$,
iv. Compute the corresponding resultant PFSS $\widetilde{T}_D \wedge \widetilde{T}_C = \widetilde{T}_H$
v. Set up the Comparison-table of PFSS $\widetilde{T}_G$ and compute $r_i$ and $t_i$ for $\epsilon_i$, for all $i$,
vi. Compute the score of $\epsilon_i$, for all $i$,
vii. If the obtain value of $S_k$ is maximum($S_k = \max_i S_i$), then signify that decision is $S_k$,
viii. If $k$ has more than one value then any one of $\epsilon_k$ may be chosen.

## 4 Application

In this section, we give an application of PFSS theory for medical decision making.

The PFSS $F_{\mathcal{G}}$ is defined with patients and attributes. The PFSS $F_{\mathcal{H}}$ is obtained from the measurements of Cleveland dataset. In the PFSS $F_{\mathcal{K}}$, there are predicted values of patients and disease degrees.

Take the PFSSs $F_{\mathcal{G}}$, $F_{\mathcal{H}}$ in Tables 1, 2, respectively and carry out "$(\mathcal{F}, R)$ AND $(\mathcal{G}, S)$" in form $e_{ij}$, where $e_{ij} = a_i \wedge b_j$. Then, we will have $11 \times 5 = 55$ parameters of the form $e_{ij}$, $(i = 1, 2, \cdots, 11; j = 1, 2, 3, 4, 5)$. For example, let the PFSS for the parameters

$$D = \{e_{11}, e_{15}, e_{23}, e_{34}, e_{42}, e_{53}, e_{72}, e_{95}, e_{(10,1)}, e_{(11,3)}\}.$$

Then, the resultant PFSS for the PFSSs $F_{\mathcal{G}}$, $F_{\mathcal{H}}$ will be $F_{\mathcal{R}}$ (Table 4).

Take the PFSSs $F_{\mathcal{G}}$, $F_{\mathcal{H}}$, $F_{\mathcal{K}}$ in Tables 1, 2, 3, respectively. Consider that

$$E = \{e_{11} \wedge c_1, e_{15} \wedge c_5, e_{23} \wedge c_2, e_{34} \wedge c_4, e_{42} \wedge c_3, e_{53} \wedge c_2, e_{72} \wedge c_5, e_{95} \wedge c_1, e_{(10,1)} \wedge c_3, e_{(11,3)} \wedge c_2\}.$$

The tabular representation of resultant PFSS $F_{\mathcal{R}}$ is depicted Table 5.

The Comparison Table of the above resultant PFSS is as Table 6. Later on compute the row-sum$(r_i)$, column-sum$(t_i)$, and the score$(S_i)$ for each $p_i$, as Table 7.

Now, we construct the tables for medical decision making by algorithm in previous section:

**Table 1** The FSS $F_{\mathcal{G}}$

| | $n_1$ | $n_2$ | $n_3$ | $n_4$ | $n_5$ | $n_6$ | $n_7$ | $n_8$ | $n_9$ | $n_{10}$ | $n_{11}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\epsilon_1$ | (0.9, 0.0) | (0.2, 0.5) | (0.5, 0.4) | (0.3, 0.7) | (0.1, 0.9) | (0.1, 0.8) | (0.6, 0.3) | (0.3, 0.5) | (0.7, 0.2) | (0.1, 0.6) | (0.5, 0.4) |
| $\epsilon_2$ | (0.9, 0.1) | (0.8, 0.0) | (0.6, 0.2) | (0.4, 0.5) | (0.1, 0.7) | (0.8, 0.1) | (0.4, 0.3) | (0.2, 0.5) | (0.5, 0.5) | (0.8, 0.1) | (0.1, 0.6) |
| $\epsilon_{24}$ | (0.7, 0.2) | (0.2, 0.6) | (0.4, 0.4) | (0.5, 0.3) | (0.1, 0.6) | (0.0, 0.9) | (0.2, 0.7) | (0.3, 0.6) | (0.7, 0.1) | (0.6, 0.5) | (0.9, 0.1) |
| $\epsilon_{25}$ | (0.1, 0.8) | (0.8, 0.0) | (0.5, 0.4) | (0.6, 0.3) | (0.1, 0.7) | (0.9, 0.1) | (0.3, 0.5) | (0.5, 0.4) | (0.5, 0.5) | (0.6, 0.3) | (0.9, 0.1) |
| $\epsilon_{75}$ | (0.5, 0.5) | (0.1, 0.8) | (0.2, 0.5) | (0.4,0.3) | (0.1, 0.5) | (0.5, 0.5) | (0.8, 0.2) | (0.0, 0.7) | (0.1, 0.7) | (0.5, 0.2) | (0.6, 0.1) |
| $\epsilon_{303}$ | (0.5, 0.4) | (0.6, 0.3) | (0.5, 0.4) | (0.3, 0.6) | (0.1, 0.8) | (0.1, 0.5) | (0.7, 0.2) | (0.0, 0.8) | (0.1, 0.5) | (0.4, 0.4) | (0.2, 0.6) |

**Table 2** The FSS $F_{\mathcal{H}}$

| | 1 | 2 | 3 | 4 | 0 |
|---|---|---|---|---|---|
| $\epsilon_1$ | (0.3, 0.6) | (0.2, 0.7) | (0.1, 0.7) | (0.0, 0.9) | (0.9, 0.0) |
| $\epsilon_2$ | (0.8, 0.0) | (0.9, 0.1) | (0.8, 0.1) | (0.5, 0.4) | (0.0, 0.9) |
| $\epsilon_{24}$ | (0.7, 0.1) | (0.8, 0.1) | (0.9, 0.2) | (0.8, 0.1) | (0.0, 0.8) |
| $\epsilon_{25}$ | (0.3, 0.6) | (0.5, 0.5) | (0.8, 0.1) | (0.9, 0.1) | (0.0, 0.8) |
| $\epsilon_{75}$ | (0.9, 0.3) | (0.8, 0.3) | (0.4, 0.6) | (0.2, 0.7) | (0.5, 0.5) |
| $\epsilon_{303}$ | (0.2, 0.6) | (0.1, 0.7) | (0.1, 0.6) | (0.0, 0.8) | (0.9, 0.1) |

 

**Table 3** The FSS $F_{\mathcal{K}}$

| | 1 | 2 | 3 | 4 | 0 |
|---|---|---|---|---|---|
| $\epsilon_1$ | (0.5, 0.6) | (0.3, 0.6) | (0.2, 0.7) | (0.1, 0.8) | (0.2, 0.9) |
| $\epsilon_2$ | (0.6, 0.2) | (0.3, 0.6) | (0.1, 0.7) | (0.0, 0.6) | (0.8, 0.1) |
| $\epsilon_{24}$ | (0.5, 0.3) | (0.6, 0.2) | (0.8, 0.4) | (0.9, 0.2) | (0.0, 0.9) |
| $\epsilon_{25}$ | (0.4, 0.5) | (0.5, 0.6) | (0.6, 0.3) | (0.7, 0.4) | (0.0, 0.9) |
| $\epsilon_{75}$ | (0.8, 0.3) | (0.8, 0.2) | (0.5, 0.4) | (0.3, 0.8) | (0.5, 0.4) |
| $\epsilon_{303}$ | (0.3, 0.6) | (0.2, 0.7) | (0.2, 0.5) | (0.1, 0.9) | (0.8, 0.1) |

**Table 4** The resultant FSS $F_{\mathcal{M}}$

| | $e_{11}$ | $e_{15}$ | $e_{23}$ | $e_{34}$ | $e_{42}$ | $e_{53}$ | $e_{72}$ | $e_{95}$ | $e_{(10,1)}$ | $e_{(11,3)}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\epsilon_1$ | (0.3, 0.6) | (0.9, 0.0) | (0.1, 0.7) | (0.0, 0.9) | (0.2, 0.7) | (0.1, 0.9) | (0.2, 0.7) | (0.7, 0.2) | (0.1, 0.6) | (0.1, 0.6) |
| $\epsilon_2$ | (0.8, 0.1) | (0.0, 0.9) | (0.8, 0.1) | (0.5, 0.4) | (0.4, 0.5) | (0.1, 0.7) | (0.4, 0.3) | (0.0, 0.9) | (0.8, 0.1) | (0.1, 0.6) |
| $\epsilon_{24}$ | (0.7, 0.2) | (0.0, 0.8) | (0.2, 0.6) | (0.4, 0.4) | (0.5, 0.3) | (0.1, 0.6) | (0.2, 0.7) | (0.0, 0.8) | (0.6, 0.5) | (0.9, 0.2) |
| $\epsilon_{25}$ | (0.1, 0.8) | (0.0, 0.8) | (0.8, 0.1) | (0.5, 0.4) | (0.5, 0.5) | (0.1, 0.7) | (0.3, 0.5) | (0.0, 0.8) | (0.3, 0.6) | (0.8, 0.1) |
| $\epsilon_{75}$ | (0.5, 0.5) | (0.5, 0.5) | (0.1, 0.8) | (0.2, 0.7) | (0.4, 0.3) | (0.1, 0.6) | (0.8, 0.3) | (0.1, 0.7) | (0.5, 0.3) | (0.4, 0.6) |
| $\epsilon_{303}$ | (0.2, 0.6) | (0.5, 0.4) | (0.1, 0.6) | (0.0, 0.8) | (0.1, 0.7) | (0.1, 0.8) | (0.1, 0.7) | (0.1, 0.5) | (0.2, 0.6) | (0.1, 0.6) |

**Table 5** The resultant FSS $F_{\mathcal{R}}$

| | $e_{11} \wedge c_1$ | $e_{15} \wedge c_3$ | $e_{23} \wedge c_4$ | $e_{34} \wedge c_2$ | $e_{42} \wedge c_5$ | $e_{53} \wedge c_1$ | $e_{72} \wedge c_3$ | $e_{95} \wedge c_2$ | $e_{(10,1)} \wedge c_4$ | $e_{(11,3)} \wedge c_1$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\epsilon_1$ | (0.3, 0.6) | (0.2, 0.7) | (0.1, 0.8) | (0.0, 0.9) | (0.2, 0.9) | (0.1, 0.9) | (0.2, 0.7) | (0.3, 0.6) | (0.1, 0.8) | (0.1, 0.7) |
| $\epsilon_2$ | (0.6, 0.2) | (0.0, 0.9) | (0.0, 0.6) | (0.3, 0.6) | (0.4, 0.5) | (0.1, 0.7) | (0.1, 0.7) | (0.0, 0.9) | (0.0, 0.6) | (0.1, 0.6) |
| $\epsilon_{24}$ | (0.5, 0.3) | (0.0, 0.8) | (0.2, 0.6) | (0.4, 0.5) | (0.0, 0.9) | (0.1, 0.6) | (0.2, 0.7) | (0.0, 0.8) | (0.6, 0.5) | (0.5, 0.3) |
| $\epsilon_{25}$ | (0.1, 0.8) | (0.0, 0.8) | (0.7, 0.4) | (0.5, 0.6) | (0.0, 0.9) | (0.1, 0.7) | (0.3, 0.5) | (0.0, 0.8) | (0.3, 0.6) | (0.4, 0.5) |
| $\epsilon_{75}$ | (0.5, 0.5) | (0.5, 0.5) | (0.1, 0.8) | (0.2, 0.7) | (0.4, 0.4) | (0.1, 0.6) | (0.5, 0.4) | (0.1, 0.7) | (0.3, 0.8) | (0.4, 0.6) |
| $\epsilon_{303}$ | (0.2, 0.6) | (0.2, 0.5) | (0.1, 0.9) | (0.0, 0.8) | (0.1, 0.7) | (0.1, 0.8) | (0.1, 0.7) | (0.1, 0.7) | (0.1, 0.9) | (0.1, 0.6) |

**Table 6** Comparison Table

| | $p_1$ | $p_2$ | $p_{24}$ | $p_{25}$ | $p_{75}$ | $p_{303}$ |
|---|---|---|---|---|---|---|
| $\epsilon_1$ | 10 | 3 | 4 | 3 | 2 | 5 |
| $\epsilon_2$ | 5 | 10 | 2 | 3 | 2 | 6 |
| $\epsilon_{24}$ | 7 | 8 | 10 | 7 | 6 | 7 |
| $\epsilon_{25}$ | 5 | 8 | 6 | 10 | 4 | 6 |
| $\epsilon_{75}$ | 9 | 6 | 5 | 6 | 10 | 10 |
| $\epsilon_{303}$ | 4 | 4 | 3 | 4 | 1 | 10 |

**Table 7** $r_i, t_i, S_i$

| | $r_i$ | $t_i$ | $S_i$ |
|---|---|---|---|
| $\epsilon_1$ | 28 | 40 | -816 |
| $\epsilon_2$ | 28 | 39 | -737 |
| $\epsilon_{24}$ | 45 | 30 | 1125 |
| $\epsilon_{25}$ | 39 | 34 | 365 |
| $\epsilon_{75}$ | 46 | 25 | 1491 |
| $\epsilon_{303}$ | 26 | 44 | -1260 |

From the Table 7, it is clear that the maximum score is 1491 and $p_{75}$ has the maximum score. Therefore, we can decide the accuracy of selection of $p_{75}$.

## 5 Conclusion

Since the emergence of IFS [2], it has received a lot of attention in field of science and technology. Unlike FS, IFS does not only have a membership function but also has a non-membership function. The sum of these functions is less than or equal to 1. Having two functions and having totals less than or equal to 1 makes IFS stronger and more decisive than FS. However, in some real-life situations, the total of membership and non-membership functions may be greater than 1 and this creates difficulties in solving problem. That is, IFS fails to cope with such a situation. For this reason, PFS, initiated by Yager to deal with uncertainty, has entered the literature as a very effective tool. Problems that cannot be solved in IFS are more easily solved with PFS and the necessary modelling can be made easier. Therefore, it is claimed that PFS, which is frequently used in the literature about decision-making problems, is a superior model. IFS is PFS, but the opposite does not have to be true (1).

The PFS was extended to the Pythagorean Fuzzy Soft Set(PFSS) [8] by the SS Theory presented by Molodtsov [3]. In this paper, a new decision-making method is given using the PFSS. The proposed algorithm for decision-making process has been successfully implemented with the help of a numerical example.

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# The Effect of Music on the Cognitive Development of Early Childhood Period: A Pythagorean Fuzzy Set Approach

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**Abstract:** Background: There are many genetic and environmental factors that affect cognitive development. Music education can also be considered as one of the environmental factors. Some researchers emphasize that Music is an action that requires meta-cognitive functions such as mathematics and chess, and supports spatial intelligence. The effect of music on cognitive development in early childhood was examined with the proposed decision-making method. Alternative methods and techniques can be used to support cognitive development. The Pythagorean Fuzzy Sets(PFS) method defined by Yager was used in this study. This method is used to manage uncertainty in real-world decision-making problems. In this study, PFS was created from experts' opinions on the subject. An algorithm was given according to PFS. This algorithm has been implemented. The effect of music on cognitive development in early childhood education was examined according to the opinions of the experts with the algorithm. The results of the algorithm supported the data of the experts on the development of spatial-temporal skills of music education given in early childhood. In algorithm, ranking is done with Expectation Score Function. Expert opinions were ranked according to real-life. The rankings obtained from the algorithm overlap with the experts' rankings.

**Keywords:** Cognitive development, Decision-making, Early childhood, Music education, Pythagorean fuzzy set.

## 1 Introduction

Uncertainty describes epistemic states including unknown and defective information. There is no uncertainty in the data, but our inferences about the data are uncertain. Uncertainty has always been an important problem for decision-makers. Creating solutions for uncertainties has become one of the most important problems in the world today. Various methods are used to understand the uncertainties in the inferences made based on the data. Many theories about uncertainty have emerged recently such as fuzzy set(FS) theory, intuitionistic fuzzy set(IFS) theory, soft set(SS) theory, probability theory etc. One of the theories that emerged as a solution to uncertainty problems is FS theory [1]. The idea of a membership function in FS theory is a good solution for uncertainties compared to previous methods. However, the membership function was not sufficient for some solutions in practice. Atanassov[2] developed IFS as a solution to the insufficiency in FS. IFS is a generalization of Fuzzy Set(FS). IFS includes non-membership function ($n$) along with membership function ($m$). In the IFS, each a IF number satisfies the condition m+n< 1. Yager [3] offered a new FS called Pythagorean fuzzy set(PFS). PFS fascinated the interest of many researchers in a little while. The formulation of the negation for IFSs and PFSs is examined by Yager [4].

The period in which the development of the individual is the fastest as holistic is the early childhood period. In this period, the child is in the process of development in terms of social, emotional, language, psycho-motor, self-care, and cognitive aspects. In this process, the child is affected by various environmental conditions with formal or informal experiences. In this context, Vygotsky [5] emphasizes that cognitive development is significantly affected by the child's environment in terms of socio-culture. However, it has been determined by many studies that cognitive development has made great progress in early childhood. There are many factors that affect this progress. It can be said that one of these factors is music. Cooper [6] stated that studies on the cognitive benefits of music education are arousing a global scale. Cooper [6] stated that the studies on the cognitive benefits of music education are arousing curiosity on a global scale.

Cognitive development can be defined as the process of learning, practising and controlling cognitive skills that an individual can do using his mind. Oakley [7] cited cognitive skills as all processes related to learning, organizing, using and developing knowledge. Solso [8] described the cognitive field as a science in which brain functions such as perception, attention, memory, thinking, language, problem-solving and reasoning are examined. In addition to those in these definitions, it can be stated that self-regulation skills that include reasoning, problem-solving and decision making related to one's emotions, impulses and thoughts are within the scope of cognitive development. Among the cognitive skills mentioned in the definitions, attention, memory, thinking and self-regulation skills were evaluated within the scope of this study.

 

It is known that it can benefit from different disciplines in order to support cognitive development. Katarzyna and Brenda [9] stated that the relationship between music and cognitive development has been studied by various researchers, including neuroscientists, psychologists, educational experts, and musicians. In addition, alternative methods and techniques are also used to support cognitive development outside of different disciplines. It can be said that some of these methods and techniques are methods such as play, drama, and music. In addition, different techniques such as finger game, rond, creative dance, and musicals have been created by combining these methods [11]. Perhaps the most important factor that supports cognitive development in music education is teacher qualifications. In addition to the knowledge of the needs of music education and the child's cognitive development, teachers should be well educated especially in the use of musical instruments and materials [10].

## 2 Preliminaries

Throughout the paper, the initial universe, parameters sets will denote $\mathcal{U}$, $\mathcal{P}$, respectively.

As an original idea, PFSs were created by Yager [3]. PFS is a very useful tool for uncertainty. PFS offers good results especially in solutions where IFS is insufficient. The differences between PFSs and IFSs can be mentioned as follows: For IFS, $m + n \leq 1$, $0 \leq m + n \leq 1$, $h = 1 - (m + n)$ and $h + m + n = 1$.

For PFS, $m + n \leq 1 or m + n \geq 1$, $0 \leq m^2 + n^2 \leq 1$, $h = \sqrt{1 - (m^2 + n^2)}$ and $h^2 + m^2 + n^2 = 1$.

The function $m_{\mathcal{A}}(x) : \mathcal{U} \to [0, 1]$ is called *FS* on $\mathcal{U}$. The FS can be indicated by

$$\mathcal{A} = \{(x_i, m_{\mathcal{A}}(x_i)) : m_{\mathcal{A}}(x_i) \in [0, 1]; \forall x_i \in \mathcal{U}\}.$$

The set

$$\mathcal{B} = \{(x, m_{\mathcal{B}}(x), n_{\mathcal{B}}(x)) : x \in \mathcal{U}\}$$

is called an IFS $\mathcal{B}$ on $\mathcal{U}$, where, $m_{\mathcal{B}} : \mathcal{U} \to [0, 1]$ and $n_{\mathcal{B}} : \mathcal{U} \to [0, 1]$ such that $0 \leq m_{\mathcal{B}}(x) + n_{\mathcal{B}}(x) \leq 1$ for any $x \in \mathcal{U}$ [2]. The degree of indeterminacy $h_{\mathcal{B}} = 1 - m_{\mathcal{B}}(x) - n_{\mathcal{B}}(x)$.

For $m_{\mathcal{C}} : \mathcal{U} \to [0, 1]$ and $n_{\mathcal{C}} : \mathcal{U} \to [0, 1]$, an PFS $\mathcal{C}$ in $\mathcal{U}$ is defined by

$$\mathcal{C} = \{(x, m_{\mathcal{C}}(x), n_{\mathcal{C}}(x)) : x \in \mathcal{U}\},$$

with the condition that $0 \leq [m_{\mathcal{C}}(x)]^2 + [n_{\mathcal{C}}(x)]^2 \leq 1$ [3, 4, 12]. The degree of indeterminacy $h_{\mathcal{C}} = \sqrt{1 - [m_{\mathcal{C}}(x)]^2 - [n_{\mathcal{C}}(x)]^2}$.

Let $\mathcal{E}$ be a PFS over $\mathcal{P}$. In this definition, $F_{\mathcal{E}}$ can be represented as

$$F_{\mathcal{E}} = \Big\{ \big(x, m_F(x), n_F(x), \big) : x \in \mathcal{P}, m_F(x) \in [0, 1], n_F(x) \in [0, 1] \Big\}.$$

Here, the values $m_F(x)$ and $n_F(x)$ are the degree of importance and unimportance of the parameter $x$. The set of all PFS on $\mathcal{U}$ will be denoted by $\Omega(\mathcal{U})$.

**Example 1.** *Let's choose four experts who work in Early School Education. The experts are studying research on the effect of music on the cognitive development of the early childhood period. Let's take the set $P = \{p_1, p_2, p_3, p_4\}$ as the set of experts. In this research, they examine the following situations:*

$(s_1)$ *Music education improves children's attention capacity.*

$(s_2)$ *Music education improves children's vocabulary knowledge.*

$(s_3)$ *Music education improves children's reasoning and logical thinking skills.*

$(s_4)$ *Music education improves children's self-regulation skills.*

*for the set $\mathcal{S} = \{s_1, s_2, s_3, s_4\}$. Then,*

$$\begin{aligned}
F(s_1) &= \{(p_1, 0.7, 0.7), (p_2, 0.5, 0.6), (p_3, 0.9, 0.4), (p_4, 0.7, 0.5)\} \\
F(s_2) &= \{(p_1, 0.6, 0.6), (p_2, 0.4, 0.9), (p_3, 0.8, 0.4), (p_4, 0.6, 0.5)\} \\
F(s_3) &= \{(p_1, 0.8, 0.2), (p_2, 0.8, 0.6), (p_3, 0.6, 0.7), (p_4, 0.5, 0.8)\} \\
F(s_4) &= \{(p_1, 0.4, 0.7), (p_2, 0.5, 0.6), (p_3, 0.7, 0.4), (p_4, 0.8, 0.3)\}.
\end{aligned}$$

*All this information can be represented in terms of the $F_{\mathcal{P}}$ as table in Table* 1. *The values given in the table are arranged according to the opinions of the experts given in the literature.*

 

**Table 1** $F_{\mathcal{P}}$

| P / $\mathcal{S}$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|
| $p_1$ | (0.7, 0.7) | (0.6, 0.6) | (0.8, 0.2) | (0.4, 0.7) |
| $p_2$ | (0.5, 0.6) | (0.4, 0.9) | (0.8, 0.6) | (0.5, 0.6) |
| $p_3$ | (0.9, 0.4) | (0.8, 0.4) | (0.6, 0.7) | (0.7, 0.4) |
| $p_4$ | (0.7, 0.5) | (0.6, 0.5) | (0.5, 0.8) | (0.8, 0.3) |

Let $(\mathcal{L}, \leq_{\mathcal{L}})$ be a complete lattice, where $\mathcal{L} = \{(u, v) : u, v \in [0, 1], u^2 + v^2 < 1\}$ and the corresponding partial order $\leq_{\mathcal{L}}$ is defined by $(u, v) \leq_{\mathcal{L}} (i, j) \quad \Leftrightarrow \quad u \leq i \quad and \quad v \geq j$, for all $(u, v), (i, j) \in \mathcal{L}$. Any ordered pair $(u, v) \in \mathcal{L}$ is called *Pythagorean fuzzy value*(PFV) or *Pythagorean fuzzy number*(PFN) [13].

Let Pythagorean fuzzy numbers (PFNs) are denoted by $R = (m_R, n_R)$ [14]. Choose three PFNs $\theta = (m, n), \theta_1 = \langle m_1, n_1 \rangle, \theta_2 = \langle m_2, n_2 \rangle$. We can give some basic operations as follows [3], [12]: For $\alpha > 0$,

- $\bar{\theta} = \langle n, m \rangle$;
- $\theta_1 \oplus \theta_2 = \langle \sqrt{m_1^2 + m_2^2 - m_1^2 m_2^2}, n_1 n_2 \rangle$;
- $\theta_1 \otimes \theta_2 = \langle m_1 m_2, \sqrt{n_1^2 + n_2 2 - n_1^2 n_2^2} \rangle$;
- $\theta_1 \wedge \theta_2 = \langle \min\{m_1, m_2\}, \max\{n_1, n_2\} \rangle$;
- $\theta_1 \vee \theta_2 = \langle \max\{m_1, m_2\}, \min\{n_1, n_2\} \rangle$;
- $\alpha.\theta = \langle \sqrt{1 - (1 - m^2)^\alpha}, n^\alpha \rangle$;
- $\theta^\alpha = \langle m^\alpha, \sqrt{1 - (1 - n^2)^\alpha}, \rangle$.

## 3 Method

### 3.1 PFS method

For PFNs, the mapping $\mathcal{SF} : \mathcal{L} \to [-1, 1]$ is called *score function*, if

$$\mathcal{SF}_R = m_R^2 - n_R^2 \tag{1}$$

for all $R = (m_R, n_R) \in \mathcal{L}$ [14], [15].

The mapping $\mathcal{AF} : \mathcal{L} \to [0, 1]$ is called *accuracy function*, if

$$\mathcal{AF}_R = m_R^2 + n_R^2 \tag{2}$$

for all $R = (m_R, n_R) \in \mathcal{L}$ [16].

The mapping $\mathcal{ES} : \mathcal{L} \to [0, 1]$ is called expectation score function [17] such that for all $R = (m_R, n_R) \in \mathcal{L}$

$$\mathcal{ES}_R = \frac{m_R^2 - n_R^2 + 1}{2}. \tag{3}$$

For two PFNs $R, T \in \mathcal{L}$ and the relation $\leq_{(m,\mathcal{ES})}$ on $\mathcal{L}$, we have

$$R \leq_{(m,\mathcal{ES})} T \quad \Leftrightarrow \quad (m_R < m_T) \vee (m_R = m_T \wedge \mathcal{ES}_R \leq \mathcal{ES}_T).$$

Let $R = (m_R, n_R), T = (m_T, n_T) \in \mathcal{L}$ be two PFVs. Then, for $\alpha > 0$, we have the following operations [14]:

i. $R \oplus T = \left(\sqrt{m_R^2 + m_T^2 - m_M^2 m_T^2}, n_R n_T\right)$,
ii. $R \otimes T = \left(m_R m_T, \sqrt{n_R^2 + n_T^2 - n_R^2 n_T^2}\right)$,
ii. $\alpha R = \left(\sqrt{1 - (1 - m_R^2)^\alpha}, (n_R)^\alpha\right)$,
iv. $R^\alpha = \left((m_R)^\alpha, \sqrt{1 - (1 - n_R^2)^\alpha},\right)$.

### 3.2 Algorithm

Let $F_{\mathcal{P}} \in \Omega(\mathcal{U})$. Then,

$$\mathcal{AP}_{F_{\mathcal{P}}} = \bigoplus_{p \in \mathcal{P}} \frac{\mathcal{ES}(s_i)}{\sum_{p \in \mathcal{P}} \mathcal{ES}(s_i)} F_{\mathcal{P}}$$

is called the aggregated Pythagorean fuzzy decision value(APFDV) [17], where

$$\mathcal{ES}(s_i) = \frac{\sum_{p_i \in P} m(s_i)^2 - \sum_{p_i \in P} n(s_i)^2 + 1}{2}.$$

**Algorithm:**

- **Step 1:** The set of experts $P$ and the set $S$ are recorded in PFS table.
- **Step 2:** The expectation values $\mathcal{ES}(s_i)$ are calculated.
- **Step 3:** The weights are found by

$$\omega = \frac{\mathcal{ES}(s_i)}{\sum \mathcal{ES}(s_i)}$$

- **Step 4:** For $k = 1, 2, \cdots, i$, the APFDVs are computed by

$$\mathcal{AP}_{F_{\mathcal{P}}}(p_k) = \bigoplus_{\ell=1}^{j} \frac{\mathcal{ES}(s_i)(x_\ell)}{\sum_{\ell=1}^{j} \mathcal{ES}(s_i)(x_\ell)} F_{\mathcal{P}(x_\ell)}(p_k)$$

- **Step 5:** Rank $\mathcal{AP}_F(p_k)$, $(k = 1, 2, \cdots, i)$ descending under the order $\leq_{(m,\mathcal{ES})}$.
- **Step 6:** Rank $p_j$, $(j = 1, 2, \cdots, k)$ correspondingly and output $p_k$ as the optimal decision, if $\mathcal{AP}_F(p_i)$ is the largest PFV under the order $\leq_{(m,\mathcal{ES})}$.

## 4 Application: The effect of music on cognitive development

Now, we will investigate the effect of music on cognitive development of early childhood education. In Example 1, four experts related to early childhood education were chosen and four opinions about the cognitive development of children were given. The experts are studying research on the effect of music on the cognitive development of the early childhood period.

The ranking method proposed with the given algorithm will be used for the effects of music on children's cognitive development. An assessment will be made with the opinions of the four experts. As decision-makers, these experts will make this assessment according to the criteria in Example 1. Experts' opinions regarding these criteria will be listed by the solving procedure. This ranking will show the importance given by experts to these criteria.

We consider the values of Table 1. We compute the expectation values $\mathcal{ES}$, that reveal the weight vector (Table 2)

$$\omega = \{0.32246377, 0.17028986, 0.24637681, 0.26086956\}^T$$

to be used for calculating the APFDVs. The $\mathcal{AP}_{F_{\mathcal{P}}}(p_k)$ is found as

$$\mathcal{AP}_{F_{\mathcal{P}}}(p_i) = \mathcal{PF}_\omega(\pi_i)\Big(F_{\mathcal{P}(s_1)}(p_i), F_{\mathcal{P}(s_2)}(p_i), F_{\mathcal{P}(s_3)}(p_i), F_{\mathcal{P}(s_5)}(p_i), F_{\mathcal{P}(s_6)}(p_i)\Big).$$

For example, $\mathcal{AP}_{F_{\mathcal{P}}}(p_1) = (0.668, 0.501)$.

From Table 3, we have,

$$\mathcal{AP}_F(p_3) \leq_{(m,\mathcal{ES})} \mathcal{AP}_F(p_1) \leq_{(m,\mathcal{ES})} \mathcal{AP}_F(p_4) \leq_{(m,\mathcal{ES})} \mathcal{AP}_F(p_2).$$

According to these results, the experts' opinion will be sorted as : $p_3 > p_1 > p_4 > p_2$.

**Table 2**

| | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|
| $\mathcal{ES}(s_i)$ | 0.89 | 0.47 | 0.68 | 0.72 |
| $\omega$ | 0.32246377 | 0.17028986 | 0.24637681 | 0.26086956 |

**Table 3** Measures

| | $APFDVs$ | $\mathcal{ES}(\mathcal{AP}_{F_{\mathcal{P}}}(p_k))$ | $\mathcal{SF}(\mathcal{AP}_{F_{\mathcal{P}}}(p_k))$ | $\mathcal{AF}(\mathcal{AP}_{F_{\mathcal{P}}}(p_k))$ |
|---|---|---|---|---|
| $p_1$ | (0.668, 0.501) | 0.598 | 0.195 | 0.7 |
| $p_2$ | (0.602, 0.641) | 0.476 | -0.048 | 0.77 |
| $p_3$ | (0.8, 0.46) | 0.7142 | 0.43 | 0.8516 |
| $p_4$ | (0.66, 0.494) | 0.595 | 0.2 | 0.68 |

 

## 5 Discussion and conclusion

Schellenberg and Weiss [18] stated that there is a strong relationship between music tendency and general cognitive abilities, especially in childhood. He also emphasized that cognitive performance may increase as a result of the improvement of the general mood with the effect of music. Schellenberg, Nakata, Hunter, and Tamoto, [19] showed that the cognitive performance of 5-year-old children with different music genres can be increased. In the longitudinal study by Costa-Giomi [20], it was revealed that children who started music education before the age of 5 got significantly higher scores in spatial skills than children who started later or did not receive an education. Ho, Cheung, and Chan [21] found that verbal memory significantly differs according to the control group in boys aged 6-15 who receive music education. Ho, Cheung, and Chan [21] concluded that music training systematically influenced memory processing according to possible neuroanatomical changes in the left temporal lobe in their study of brain waves. We applied, the entries in the PFS table, which is arranged according to the opinions of the experts, in the algorithm we obtained. Depending on the PFNs, the new method we proposed provides solutions to the decision analysis problems by the ranking of the PFNs. The results of the algorithm supported the data of the experts on the development of spatial-temporal skills of music education given in early childhood.

In this study, a new decision-making algorithm and method were given. We used PFS in the given method. PFS was preferred because it is known that PFS gives clearer results than IFS. The effect of music on cognitive development in early childhood was examined with this decision-making method. In practice, the opinions of the experts about cognitive development and the results of the method we proposed were compared. In this study, the expectation score function was used. Weights and thus APFDV are calculated with the values obtained from this function. The ranking is done with APFDV. Here, the values obtained from expert opinions are determined as follows: Whichever expert has given more opinion about the criteria, he has been in the ranking before. Again, whichever specialist has given fewer opinions remains behind the rankings. This is very suitable for real-life events. The rankings obtained from the algorithm of the study were the same as the rankings of the opinions of the experts.

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# Notes on Equalities of BLUPs under Linear Mixed Model and its Sub-sample Models

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**Abstract:** A linear mixed model $(\mathrm{LMM})$ $\mathcal{M}: \mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \boldsymbol{\varepsilon}$, and its two sub-sample $\mathrm{LMMs}$ $\mathcal{M}_i: \mathbf{y}_i = \mathbf{X}_i\boldsymbol{\beta} + \mathbf{Z}_i\mathbf{u} + \boldsymbol{\varepsilon}_i,\ i = 1,2$ are considered. This study concerns the problem of the equalities of linear predictors in $\mathcal{M}$ and $\mathcal{M}_i$ under general assumptions. We investigate the equality relations between the best linear unbiased predictors $(\mathrm{BLUPs})$ of unknown vectors by using various rank formulas of block matrices and elementary matrix operations.

**Keywords:** $\mathrm{BLUP}$, Equalities, Linear mixed model, Random vectors, Sub-sample model.

## 1 Introduction

Consider a linear mixed model (LMM) with the following divisions on its vectors and matrices:

$$\mathcal{M}: \mathbf{y} = \mathbf{X}\beta + \mathbf{Z}\mathbf{u} + \varepsilon$$
$$\text{with } \mathbf{y} = \begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{bmatrix},\ \mathbf{X} = \begin{bmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{bmatrix},\ \mathbf{Z} = \begin{bmatrix}\mathbf{Z}_1\\ \mathbf{Z}_2\end{bmatrix},\ \text{and } \boldsymbol{\varepsilon} = \begin{bmatrix}\boldsymbol{\varepsilon}_1\\ \boldsymbol{\varepsilon}_2\end{bmatrix}, \tag{1}$$

where $\mathbf{X} \in \mathbb{R}^{n\times k}$ with $\mathbf{X}_i \in \mathbb{R}^{n_i\times k}$ and $\mathbf{Z} \in \mathbb{R}^{n\times p}$ with $\mathbf{Z}_i \in \mathbb{R}^{n_i\times p}$ are known matrices of arbitrary ranks, $\boldsymbol{\beta} \in \mathbb{R}^{k\times 1}$ is a vector of fixed but unknown parameters, $\mathbf{u} \in \mathbb{R}^{p\times 1}$ is a vector of unobservable random effects, and $\boldsymbol{\varepsilon} \in \mathbb{R}^{n\times 1}$ with $\boldsymbol{\varepsilon}_i \in \mathbb{R}^{n_i\times 1}$ is an unobservable vector of random errors, $i = 1, 2$, $n_1 + n_2 = n$, where the symbol $\mathbb{R}^{m\times n}$ stands for the collection of all $m \times n$ real matrices. Two sub-sample $\mathrm{LMMs}$ of $\mathcal{M}$ are obtained as follows from pre-multiplying $\mathcal{M}$ by the matrices $\begin{bmatrix}\mathbf{I}_{n_1}, & \mathbf{0}\end{bmatrix}$ and $\begin{bmatrix}\mathbf{0}, & \mathbf{I}_{n_2}\end{bmatrix}$, respectively,

$$\begin{aligned}\mathcal{M}_1 &: \mathbf{y}_1 = \mathbf{X}_1\boldsymbol{\beta} + \mathbf{Z}_1\mathbf{u} + \boldsymbol{\varepsilon}_1,\\ \mathcal{M}_2 &: \mathbf{y}_2 = \mathbf{X}_2\boldsymbol{\beta} + \mathbf{Z}_2\mathbf{u} + \boldsymbol{\varepsilon}_2.\end{aligned} \tag{2}$$

In order to establish some general results on simultaneous predictions of all unknown vectors under $\mathcal{M}$ and $\mathcal{M}_i$, we can consider the following vector

$$\boldsymbol{\phi}_i = \mathbf{K}\boldsymbol{\beta} + \mathbf{G}\mathbf{u} + \mathbf{H}_i\mathbf{T}_i\boldsymbol{\varepsilon} \tag{3}$$

for given matrices $\mathbf{K} \in \mathbb{R}^{s\times k}$, $\mathbf{G} \in \mathbb{R}^{s\times p}$ and $\mathbf{H}_i \in \mathbb{R}^{s\times n_i}$, $i = 1, 2$, where $\mathbf{T}_1 = \begin{bmatrix}\mathbf{I}_{n_1}, & \mathbf{0}\end{bmatrix}$ and $\mathbf{T}_2 = \begin{bmatrix}\mathbf{0}, & \mathbf{I}_{n_2}\end{bmatrix}$.

We will assume some assumptions for the models $\mathcal{M}$ and $\mathcal{M}_i$. The assumptions on the expectations and covariance matrices of random vectors in these models are given as follows

$$\mathrm{E}\begin{bmatrix}\mathbf{u}\\ \boldsymbol{\varepsilon}_1\\ \boldsymbol{\varepsilon}_2\end{bmatrix} = \mathbf{0} \ \text{ and } \ \mathrm{cov}\left\{\begin{bmatrix}\mathbf{u}\\ \boldsymbol{\varepsilon}_1\\ \boldsymbol{\varepsilon}_2\end{bmatrix}, \begin{bmatrix}\mathbf{u}\\ \boldsymbol{\varepsilon}_1\\ \boldsymbol{\varepsilon}_2\end{bmatrix}\right\} = \mathrm{D}\begin{bmatrix}\mathbf{u}\\ \boldsymbol{\varepsilon}_1\\ \boldsymbol{\varepsilon}_2\end{bmatrix} = \begin{bmatrix}\boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} & \boldsymbol{\Sigma}_{13}\\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} & \boldsymbol{\Sigma}_{23}\\ \boldsymbol{\Sigma}_{31} & \boldsymbol{\Sigma}_{32} & \boldsymbol{\Sigma}_{33}\end{bmatrix} := \boldsymbol{\Sigma}, \tag{4}$$

where $\boldsymbol{\Sigma} \in \mathbb{R}^{(n+p)\times(n+p)}$ is a nonnegative definite matrix and all the entries of $\boldsymbol{\Sigma}$ are known. Under (4), we obtain

$$\mathrm{E}(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta},\ \ \mathrm{E}(\mathbf{y}_i) = \mathbf{X}_i\boldsymbol{\beta},\ \ \mathrm{E}(\boldsymbol{\phi}_i) = \mathbf{K}\boldsymbol{\beta}, \tag{5}$$

$$\mathrm{D}(\mathbf{y}) = \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}',\ \ \mathrm{D}(\mathbf{y}_i) = \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i',\ \ \mathrm{cov}(\mathbf{y}_i, \mathbf{y}) = \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}', \tag{6}$$

$$\mathrm{D}(\boldsymbol{\phi}_i) = \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{J}_i',\ \ \mathrm{cov}(\boldsymbol{\phi}_i, \mathbf{y}) = \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}',\ \ \mathrm{cov}(\boldsymbol{\phi}_i, \mathbf{y}_i) = \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}_i', \tag{7}$$

where $\mathbf{B} = \begin{bmatrix}\mathbf{Z}, & \mathbf{I}_n\end{bmatrix}$, $\mathbf{B}_i = \begin{bmatrix}\mathbf{Z}_i, & \mathbf{T}_i\end{bmatrix}$, $\mathbf{J}_i = \begin{bmatrix}\mathbf{G}, & \mathbf{H}_i\mathbf{T}_i\end{bmatrix}$, $i = 1, 2$. Another assumption on the models $\mathcal{M}$ and $\mathcal{M}_i$ is consistency of these models. The consistency condition of $\mathcal{M}$ is holding $\mathbf{y} \in \mathscr{C}\begin{bmatrix}\mathbf{X}, & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}'\end{bmatrix}$ with probability 1; see [13]. If $\mathcal{M}$ is consistent, then $\mathcal{M}_i$ is consistent, i.e., $\mathbf{y}_i \in \mathscr{C}\begin{bmatrix}\mathbf{X}_i, & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i'\end{bmatrix}$ holds with probability 1; see [17]. Here, $\mathscr{C}(.)$ refers to the column space of a matrix.

 

The vector $\phi_i$ in (3) is said to be predictable under $\mathcal{M}$ if there exists $\mathbf{Ly}$ with $\mathbf{L} \in \mathbb{R}^{s\times n}$ such that

$$\mathrm{E}(\mathbf{Ly} - \phi_i) = \mathbf{0}, \text{ i.e., } \mathscr{C}(\mathbf{K}') \subseteq \mathscr{C}(\mathbf{X}') \tag{8}$$

holds. If there exists $\mathbf{Ly}$ such that

$$\mathrm{D}(\mathbf{Ly} - \phi_i) = \min \quad \text{s.t.} \quad \mathrm{E}(\mathbf{Ly} - \phi_i) = \mathbf{0}, \quad i = 1, 2, \tag{9}$$

holds in the Löwner partial ordering, the linear statistic $\mathbf{Ly}$ is defined to be the best linear unbiased predictor (BLUP) of $\phi_i$ under $\mathcal{M}$ and is denoted by $\mathbf{Ly} = \mathrm{BLUP}_{\mathcal{M}}(\phi_i)$. This is a well-known definition of the BLUP of $\phi_i$ which is originated from [4]. If $\mathbf{G} = \mathbf{0}$ and $\mathbf{H}_i = \mathbf{0}$ in $\phi_i$, BLUP of $\phi_i$ reduces the best linear unbiased estimator (BLUE) of $\mathbf{K}\beta$ under $\mathcal{M}$ and denoted by $\mathrm{BLUE}_{\mathcal{M}}(\mathbf{K}\beta)$.

We can face the LMMs $\mathcal{M}$ and $\mathcal{M}_i$ in some cases where adding or deleting observations exist in a given LMMs. Further, these models may occur in two periods of observation. Although observed random vectors, $\mathbf{X}$ and $\mathbf{Z}$ matrices, and error vectors in $\mathcal{M}$ and $\mathcal{M}_i$ are different, the vectors $\beta$ and $\mathbf{u}$ are the same. Therefore, it is natural to take these models separately or simultaneously for establishing results on $\beta$ and $\mathbf{u}$. Making comparisons between predictors and analyzing relations between them are the main issues in linear regression analysis since predictors under $\mathcal{M}$ and $\mathcal{M}_i$ have different properties and performances. Studies on relations between predictors under different LMMs, we may refer [2], [5]-[10], and [18], among others. In this study, we investigate the equality relations between the BLUPs of unknown vectors under the LMMs $\mathcal{M}$ and $\mathcal{M}_i$. For doing this, we use the following expression on equality of random vectors.

$$\mathbf{F}_1\mathbf{u} = \mathbf{F}_2\mathbf{u} \text{ holds definitely if } \mathbf{F}_1 = \mathbf{F}_2, \tag{10}$$

where $\mathbf{u}$ is a random vector. (10) is one of the equality criteria for random vectors in statistical point of view, for detail see, e.g., [3]. If coefficient matrices $\mathbf{F}_1$ and $\mathbf{F}_2$ in (10) are not unique, then the equality $\mathbf{F}_1 = \mathbf{F}_2$ can be divided into following four possible situations

$$(i)\ \{\mathbf{F}_1\} \cap \{\mathbf{F}_2\} \neq \emptyset, \quad (ii)\ \{\mathbf{F}_1\} \subset \{\mathbf{F}_2\}, \quad (iii)\ \{\mathbf{F}_1\} \supset \{\mathbf{F}_2\}, \quad (iv)\ \{\mathbf{F}_1\} = \{\mathbf{F}_2\}, \tag{11}$$

where $\{\mathbf{F}_1\}$ and $\{\mathbf{F}_2\}$ stand for the collections of all solutions of the equations. We use the following lemma, related to the characterizations in (11) based on (10); see, [15], for establishing the results on relations between BLUPs under the models $\mathcal{M}$ and $\mathcal{M}_i$.

**Lemma 1.** *Let $\mathbf{A} \in \mathbb{R}^{m\times n_1}$, $\mathbf{B} \in \mathbb{R}^{p\times n_1}$, $\mathbf{C} \in \mathbb{R}^{m\times n_2}$, and $\mathbf{D} \in \mathbb{R}^{p\times n_2}$ be given. Then,*

*(a) The pair of matrix equations $\mathbf{XA} = \mathbf{B}$ and $\mathbf{XC} = \mathbf{D}$ has a common solution if and only if $\mathscr{C}\begin{bmatrix}\mathbf{B}'\\ \mathbf{D}'\end{bmatrix} \subseteq \mathscr{C}\begin{bmatrix}\mathbf{A}'\\ \mathbf{C}'\end{bmatrix}$, or equivalently, $r\begin{bmatrix}\mathbf{A} & \mathbf{C}\\ \mathbf{B} & \mathbf{D}\end{bmatrix} = r\begin{bmatrix}\mathbf{A}, & \mathbf{C}\end{bmatrix}$.*

*(b) Any solution of the matrix equation $\mathbf{XC} = \mathbf{D}$ is a solution of $\mathbf{XA} = \mathbf{B}$ if and only if $r\begin{bmatrix}\mathbf{A} & \mathbf{C}\\ \mathbf{B} & \mathbf{D}\end{bmatrix} = r(\mathbf{C})$.*

We also use some known rank formulas of block matrices, collected in the following lemma; see [11], and use elementary matrix operations to give the results on BLUPs under the considered models.

**Lemma 2.** *Let $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times k}$, and $\mathbf{C} \in \mathbb{R}^{l\times n}$. Then,*

$$r\begin{bmatrix}\mathbf{A}, & \mathbf{B}\end{bmatrix} = r(\mathbf{A}) + r(\mathbf{E_A B}) = r(\mathbf{B}) + r(\mathbf{E_B A}), \tag{12}$$

$$r\begin{bmatrix}\mathbf{A}\\ \mathbf{C}\end{bmatrix} = r(\mathbf{A}) + r(\mathbf{C E_{A'}}) = r(\mathbf{C}) + r(\mathbf{A E_{C'}}). \tag{13}$$

*Here $\mathbf{E_A} = \mathbf{A}^{\perp} = \mathbf{I}_n - \mathbf{AA}^{+}$ represents the orthogonal projector and $\mathbf{A}^{+}$ denote the Moore–Penrose generalized inverse of $\mathbf{A}$.*

## 2 Preliminary results on BLUPs

In this section, we briefly review the fundamental equations on BLUPs of $\phi_i$ and related properties under models $\mathcal{M}$ and $\mathcal{M}_i$. For the following lemmas see, e.g., [12] and [16].

**Lemma 3.** *Let $\mathcal{M}$ be as given in (1). Assume that $\phi_i$ in (3) is predictable under $\mathcal{M}$, i.e., (8) holds. Then*

$$\mathrm{E}(\mathbf{Ly} - \phi_i) = \mathbf{0} \text{ and } \mathrm{D}(\mathbf{Ly} - \phi_i) = \min \Leftrightarrow \mathbf{L}\begin{bmatrix}\mathbf{X}, & \mathbf{B\Sigma B'X^{\perp}}\end{bmatrix} = \begin{bmatrix}\mathbf{K}, & \mathbf{J}_i\mathbf{\Sigma B'X^{\perp}}\end{bmatrix}. \tag{14}$$

*This equation is always consistent. The general solution of (14) and $\mathrm{BLUP}_{\mathcal{M}}(\phi_i)$ are written as*

$$\mathbf{L} = \begin{bmatrix}\mathbf{K}, & \mathbf{J}_i\mathbf{\Sigma B'X^{\perp}}\end{bmatrix}\begin{bmatrix}\mathbf{X}, & \mathbf{B\Sigma B'X^{\perp}}\end{bmatrix}^{+} + \mathbf{U}\begin{bmatrix}\mathbf{X}, & \mathbf{B\Sigma B'X^{\perp}}\end{bmatrix}^{\perp}, \tag{15}$$

$$\mathrm{BLUP}_{\mathcal{M}}(\phi_i) = \mathbf{Ly} = \left(\begin{bmatrix}\mathbf{K}, & \mathbf{J}_i\mathbf{\Sigma B'X^{\perp}}\end{bmatrix}\begin{bmatrix}\mathbf{X}, & \mathbf{B\Sigma B'X^{\perp}}\end{bmatrix}^{+} + \mathbf{U}\begin{bmatrix}\mathbf{X}, & \mathbf{B\Sigma B'X^{\perp}}\end{bmatrix}^{\perp}\right)\mathbf{y}, \tag{16}$$

*respectively, where $\mathbf{U} \in \mathbb{R}^{s\times n}$ is an arbitrary matrix. Further, $\mathbf{L}$ in (15) is unique $\Leftrightarrow r\begin{bmatrix}\mathbf{X}, & \mathbf{B\Sigma B'X^{\perp}}\end{bmatrix} = n$. $\mathrm{BLUP}_{\mathcal{M}}(\phi_i)$ in (16) is unique with probability 1 $\Leftrightarrow \mathcal{M}$ is consistent.*

Let consider sub-sample LMM $\mathcal{M}_i, i = 1, 2$. The predictability requirement of $\boldsymbol{\phi}_i$ under $\mathcal{M}_i$ is expressed as

$$\mathscr{C}(\mathbf{K}') \subseteq \mathscr{C}(\mathbf{X}_i'). \tag{17}$$

It is obvious that the predictability of $\boldsymbol{\phi}_i$ under $\mathcal{M}_i$ shows predictability of $\boldsymbol{\phi}_i$ under $\mathcal{M}$.

**Lemma 4.** *Let $\mathcal{M}_i$ be as given in* (2)*. Assume that $\boldsymbol{\phi}_i$ in* (3) *is predictable under $\mathcal{M}_i$, i.e.,* (17) *holds. Then*

$$\mathrm{E}(\mathbf{L}_i\mathbf{y}_i - \boldsymbol{\phi}_i) = \mathbf{0} \ \textit{and} \ \ \mathrm{D}(\mathbf{L}_i\mathbf{y}_i - \boldsymbol{\phi}_i) = \textit{min} \ \Leftrightarrow \mathbf{L}_i \begin{bmatrix}\mathbf{X}_i, & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\end{bmatrix} = \begin{bmatrix}\mathbf{K}, & \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\end{bmatrix}. \tag{18}$$

*This equation is always consistent. The general solution of* (18) *and* $\mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\phi}_i)$ *are written as*

$$\mathbf{L}_i = \begin{bmatrix}\mathbf{K}, & \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\end{bmatrix}\begin{bmatrix}\mathbf{X}_i, & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\end{bmatrix}^{+} + \mathbf{U}_i\begin{bmatrix}\mathbf{X}_i, & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\end{bmatrix}^{\perp}, \tag{19}$$

$$\begin{aligned}\mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\phi}_i) = \mathbf{L}_i\mathbf{y}_i &= \left(\begin{bmatrix}\mathbf{K}, & \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\end{bmatrix}\begin{bmatrix}\mathbf{X}_i, & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\end{bmatrix}^{+} + \mathbf{U}_i\begin{bmatrix}\mathbf{X}_i, & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\end{bmatrix}^{\perp}\right)\mathbf{y}_i \\ &= \mathbf{L}_i\mathbf{T}_i\mathbf{y} = \left(\begin{bmatrix}\mathbf{K}, & \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\end{bmatrix}\begin{bmatrix}\mathbf{X}_i, & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\end{bmatrix}^{+}\mathbf{T}_i + \mathbf{U}_i\begin{bmatrix}\mathbf{X}_i, & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\end{bmatrix}^{\perp}\mathbf{T}_i\right)\mathbf{y},\end{aligned} \tag{20}$$

*respectively, where $\mathbf{U}_i \in \mathbb{R}^{s\times n_i}$ is an arbitrary matrix. Further, $\mathbf{L}_i$ in* (19) *is unique $\Leftrightarrow \boldsymbol{r}\begin{bmatrix}\mathbf{X}_i, & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\end{bmatrix} = n_i$. $\mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\phi}_i)$ in* (20) *is unique with probability 1 $\Leftrightarrow \mathcal{M}_i$ is consistent.*

The requirements in (8) and (17) correspond to the estimability of vector $\mathbf{K}\boldsymbol{\beta}$ under $\mathcal{M}$ and $\mathcal{M}_i$, respectively; see, e.g., [1]. We also note that $\mathbf{X}_i\boldsymbol{\beta}$ is always estimable under $\mathcal{M}_i$ and also is estimable under $\mathcal{M}$. Let $\mathbf{K}\boldsymbol{\beta}$ be estimable under $\mathcal{M}_i$ (also estimable under $\mathcal{M}$). Then, from Lemmas 3 and 4, we obtain

$$\mathbf{L}_{\mathbf{K}\boldsymbol{\beta}} = \begin{bmatrix}\mathbf{K}, & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{X}, & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}'\mathbf{X}^{\perp}\end{bmatrix}^{+} + \mathbf{U}\begin{bmatrix}\mathbf{X}, & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}'\mathbf{X}^{\perp}\end{bmatrix}^{\perp}, \ \mathbf{U} \in \mathbb{R}^{s\times n}, \tag{21}$$

$$\mathrm{BLUE}_{\mathcal{M}}(\mathbf{K}\boldsymbol{\beta}) = \mathbf{L}_{\mathbf{K}\boldsymbol{\beta}}\mathbf{y} = \left(\begin{bmatrix}\mathbf{K}, & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{X}, & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}'\mathbf{X}^{\perp}\end{bmatrix}^{+} + \mathbf{U}\begin{bmatrix}\mathbf{X}, & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}'\mathbf{X}^{\perp}\end{bmatrix}^{\perp}\right)\mathbf{y}, \tag{22}$$

$$\mathbf{L}_{\mathbf{X}_i\boldsymbol{\beta}} = \begin{bmatrix}\mathbf{X}_i, & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{X}, & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}'\mathbf{X}^{\perp}\end{bmatrix}^{+} + \mathbf{U}\begin{bmatrix}\mathbf{X}, & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}'\mathbf{X}^{\perp}\end{bmatrix}^{\perp}, \ \mathbf{U} \in \mathbb{R}^{n_i\times n}, \tag{23}$$

$$\mathrm{BLUE}_{\mathcal{M}}(\mathbf{X}_i\boldsymbol{\beta}) = \mathbf{L}_{\mathbf{X}_i\boldsymbol{\beta}}\mathbf{y} = \left(\begin{bmatrix}\mathbf{X}_i, & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{X}, & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}'\mathbf{X}^{\perp}\end{bmatrix}^{+} + \mathbf{U}\begin{bmatrix}\mathbf{X}, & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}'\mathbf{X}^{\perp}\end{bmatrix}^{\perp}\right)\mathbf{y}, \tag{24}$$

$$\mathbf{L}_{i\mathbf{K}\boldsymbol{\beta}} = \begin{bmatrix}\mathbf{K}, & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{X}_i, & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\end{bmatrix}^{+} + \mathbf{U}_i\begin{bmatrix}\mathbf{X}_i, & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\end{bmatrix}^{\perp}, \ \mathbf{U}_i \in \mathbb{R}^{s\times n_i}, \tag{25}$$

$$\mathrm{BLUE}_{\mathcal{M}_i}(\mathbf{K}\boldsymbol{\beta}) = \mathbf{L}_{i\mathbf{K}\boldsymbol{\beta}}\mathbf{y}_i = \left(\begin{bmatrix}\mathbf{K}, & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{X}_i, & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\end{bmatrix}^{+} + \mathbf{U}_i\begin{bmatrix}\mathbf{X}_i, & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\end{bmatrix}^{\perp}\right)\mathbf{y}_i, \tag{26}$$

$$\mathbf{L}_{i\mathbf{X}_i\boldsymbol{\beta}} = \begin{bmatrix}\mathbf{X}_i, & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{X}_i, & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\end{bmatrix}^{+} + \mathbf{U}_i\begin{bmatrix}\mathbf{X}_i, & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\end{bmatrix}^{\perp}, \ \mathbf{U}_i \in \mathbb{R}^{n_i\times n_i}, \tag{27}$$

$$\mathrm{BLUE}_{\mathcal{M}_i}(\mathbf{X}_i\boldsymbol{\beta}) = \mathbf{L}_{i\mathbf{X}_i\boldsymbol{\beta}}\mathbf{y}_i = \left(\begin{bmatrix}\mathbf{X}_i, & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{X}_i, & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\end{bmatrix}^{+} + \mathbf{U}_i\begin{bmatrix}\mathbf{X}_i, & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\end{bmatrix}^{\perp}\right)\mathbf{y}_i. \tag{28}$$

## 3 Characterization of equalities of BLUPs under $\mathcal{M}$ and $\mathcal{M}_i$

In this section, we give the results on the equalities between BLUPs under models $\mathcal{M}$ and $\mathcal{M}_i$ related to the characterizations in (11).

**Theorem 1.** *Let consider the models $\mathcal{M}$ in* (1) *and $\mathcal{M}_i$ in* (2)*. Assume that the $\boldsymbol{\phi}_i$ in* (3) *is predictable under $\mathcal{M}_i$ (also predictable under $\mathcal{M}$). Let the coefficient matrices $\mathbf{L}$ and $\mathbf{L}_i\mathbf{T}_i$ be as given in* (15) *and* (20)*, respectively. Then*

$$\{\mathbf{L}\} \cap \{\mathbf{L}_i\mathbf{T}_i\} \neq \emptyset \Leftrightarrow \boldsymbol{r}\begin{bmatrix}\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{0} & \mathbf{0} & \mathbf{I}_n \\ \mathbf{0} & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i' & \mathbf{X}_i & \mathbf{T}_i \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_i' & \mathbf{0} & \mathbf{0} \\ -\mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}_i' & \mathbf{K} & \mathbf{0}\end{bmatrix} = \boldsymbol{r}\begin{bmatrix}\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{0} & \mathbf{0} & \mathbf{I}_n \\ \mathbf{0} & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i' & \mathbf{X}_i & \mathbf{T}_i \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_i' & \mathbf{0} & \mathbf{0}\end{bmatrix}. \tag{29}$$

*In this case, $\{\mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\phi}_i)\} \cap \{\mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\phi}_i)\} \neq \emptyset$ holds definitely.*

*Proof:* From (15) and (20),

$$\mathbf{L}-\mathbf{L}_i\mathbf{T}_i=\left[\mathbf{K},\quad \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}'\mathbf{X}^{\perp}\right]\mathbf{W}^{+}-\left[\mathbf{K},\quad \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\right]\mathbf{W}_i^{+}\mathbf{T}_i+\mathbf{U}\mathbf{W}^{\perp}-\mathbf{U}_i\mathbf{W}_i^{\perp}\mathbf{T}_i, \tag{30}$$

where $\mathbf{W}=\left[\mathbf{X},\quad \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}'\mathbf{X}^{\perp}\right]$ and $\mathbf{W}_i=\left[\mathbf{X}_i,\quad \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\right]$. Then applying the formula $\min_{\mathbf{U}} r(\mathbf{C}+\mathbf{U}\mathbf{D})=r\begin{bmatrix}\mathbf{C}\\ \mathbf{D}\end{bmatrix}-r(\mathbf{D})$, given in [14], to (30) and using Lemma 2, we obtain

$$\begin{aligned}
&=\min_{\mathbf{U},\mathbf{U}_i} r\left(\left[\mathbf{K},\quad \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}'\mathbf{X}^{\perp}\right]\mathbf{W}^{+}-\left[\mathbf{K},\quad \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\right]\mathbf{W}_i^{+}\mathbf{T}_i+\left[\mathbf{U},\quad -\mathbf{U}_i\right]\begin{bmatrix}\mathbf{W}^{\perp}\\ \mathbf{W}_i^{\perp}\mathbf{T}_i\end{bmatrix}\right)\\
&=r\begin{bmatrix}\left[\mathbf{K},\quad \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}'\mathbf{X}^{\perp}\right]\mathbf{W}^{+}-\left[\mathbf{K},\quad \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\right]\mathbf{W}_i^{+}\mathbf{T}_i\\ \mathbf{W}^{\perp}\\ \mathbf{W}_i^{\perp}\mathbf{T}_i\end{bmatrix}-r\begin{bmatrix}\mathbf{W}^{\perp}\\ \mathbf{W}_i^{\perp}\mathbf{T}_i\end{bmatrix}\\
&=r\begin{bmatrix}\left[\mathbf{K},\quad \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}'\mathbf{X}^{\perp}\right]\mathbf{W}^{+}-\left[\mathbf{K},\quad \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\right]\mathbf{W}_i^{+}\mathbf{T}_i & \mathbf{0} & \mathbf{0}\\ \mathbf{I}_n & \mathbf{W} & \mathbf{0}\\ \mathbf{T}_i & \mathbf{0} & \mathbf{W}_i\end{bmatrix}-r\begin{bmatrix}\mathbf{I}_n & \mathbf{W} & \mathbf{0}\\ \mathbf{T}_i & \mathbf{0} & \mathbf{W}_i\end{bmatrix}\\
&=r\begin{bmatrix}\mathbf{0} & -\left[\mathbf{K},\quad \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}'\mathbf{X}^{\perp}\right] & \left[\mathbf{K},\quad \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\right]\\ \mathbf{I}_n & \mathbf{W} & \mathbf{0}\\ \mathbf{T}_i & \mathbf{0} & \mathbf{W}_i\end{bmatrix}-r\begin{bmatrix}\mathbf{I}_n & \mathbf{W} & \mathbf{0}\\ \mathbf{T}_i & \mathbf{0} & \mathbf{W}_i\end{bmatrix}\\
&=r\begin{bmatrix}\mathbf{0} & -\mathbf{K} & -\mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{K} & \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}_i'\\ \mathbf{I}_n & \mathbf{X} & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{0} & \mathbf{0}\\ \mathbf{T}_i & \mathbf{0} & \mathbf{0} & \mathbf{X}_i & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i'\\ \mathbf{0} & \mathbf{0} & \mathbf{X}' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_i'\end{bmatrix}-r\begin{bmatrix}\mathbf{I}_n & \mathbf{X} & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{0} & \mathbf{0}\\ \mathbf{T}_i & \mathbf{0} & \mathbf{0} & \mathbf{X}_i & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i'\\ \mathbf{0} & \mathbf{0} & \mathbf{X}' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_i'\end{bmatrix}\\
&=r\begin{bmatrix}\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{0} & \mathbf{0} & \mathbf{I}_n\\ \mathbf{0} & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i' & \mathbf{X}_i & \mathbf{T}_i\\ \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_i' & \mathbf{0} & \mathbf{0}\\ -\mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}_i' & \mathbf{K} & \mathbf{0}\end{bmatrix}-r\begin{bmatrix}\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{0} & \mathbf{0} & \mathbf{I}_n\\ \mathbf{0} & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i' & \mathbf{X}_i & \mathbf{T}_i\\ \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_i' & \mathbf{0} & \mathbf{0}\end{bmatrix}.
\end{aligned} \tag{31}$$

The required result is seen from (31). □

**Corollary 1.** *Let consider the models $\mathcal{M}$ in* (1) *and $\mathcal{M}_i$ in* (2).

*(a) Let $\mathbf{K}\boldsymbol{\beta}$ be estimable under $\mathcal{M}_i$ (also estimable under $\mathcal{M}$). Let the coefficients $\mathbf{L}_{\mathbf{K}\boldsymbol{\beta}}$ and $\mathbf{L}_{i\mathbf{K}\boldsymbol{\beta}}$ be as given in* (21) *and* (25)*, respectively. Then*

$$\{\mathbf{L}_{\mathbf{K}\boldsymbol{\beta}}\}\cap\{\mathbf{L}_{i\mathbf{K}\boldsymbol{\beta}}\mathbf{T}_i\}\neq\emptyset\Leftrightarrow r\begin{bmatrix}\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{0} & \mathbf{0} & \mathbf{I}_n\\ \mathbf{0} & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i' & \mathbf{X}_i & \mathbf{T}_i\\ \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_i' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{K} & \mathbf{0}\end{bmatrix}=r\begin{bmatrix}\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{0} & \mathbf{0} & \mathbf{I}_n\\ \mathbf{0} & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i' & \mathbf{X}_i & \mathbf{T}_i\\ \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_i' & \mathbf{0} & \mathbf{0}\end{bmatrix}. \tag{32}$$

*In this case, $\{\mathrm{BLUE}_{\mathcal{M}}(\mathbf{K}\boldsymbol{\beta})\}\cap\{\mathrm{BLUP}_{\mathcal{M}_i}(\mathbf{K}\boldsymbol{\beta})\}\neq\emptyset$ holds definitely.*
*(b) Let the coefficients $\mathbf{L}_{\mathbf{X}_i\boldsymbol{\beta}}$ and $\mathbf{L}_{i\mathbf{X}_i\boldsymbol{\beta}}$ be as given in* (23) *and* (27)*, respectively. Then*

$$\{\mathbf{L}_{\mathbf{X}_i\boldsymbol{\beta}}\}\cap\{\mathbf{L}_{i\mathbf{X}_i\boldsymbol{\beta}}\mathbf{T}_i\}\neq\emptyset\Leftrightarrow r\begin{bmatrix}\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{0} & \mathbf{I}_n\\ \mathbf{0} & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i' & \mathbf{T}_i\\ \mathbf{X}' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_i' & \mathbf{0}\end{bmatrix}+r(\mathbf{X}_i)=r\begin{bmatrix}\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{0} & \mathbf{0} & \mathbf{I}_n\\ \mathbf{0} & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i' & \mathbf{X}_i & \mathbf{T}_i\\ \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_i' & \mathbf{0} & \mathbf{0}\end{bmatrix}. \tag{33}$$

*In this case, $\{\mathrm{BLUE}_{\mathcal{M}}(\mathbf{X}_i\boldsymbol{\beta})\}\cap\{\mathrm{BLUE}_{\mathcal{M}_i}(\mathbf{X}_i\boldsymbol{\beta})\}\neq\emptyset$.*

**Theorem 2.** *Let consider the models $\mathcal{M}$ in* (1) *and $\mathcal{M}_i$ in* (2)*. Assume that the $\boldsymbol{\phi}_i$ in* (3) *is predictable under $\mathcal{M}_i$ (also predictable under $\mathcal{M}$). Let the coefficients $\mathbf{L}$ and $\mathbf{L}_i\mathbf{T}_i$ be as given in* (15) *and* (20)*, respectively. Then*

$$\{\mathbf{L}_i\mathbf{T}_i\}\subset\{\mathbf{L}\}\Leftrightarrow r\begin{bmatrix}\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}_i' & \mathbf{X}\\ \mathbf{X}' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_i' & \mathbf{0}\\ \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}_i' & \mathbf{K}\end{bmatrix}=r\begin{bmatrix}\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}_i' & \mathbf{X}\\ \mathbf{X}_i' & \mathbf{0}\end{bmatrix}+r(\mathbf{X}). \tag{34}$$

*In this case, $\{\mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\phi}_i)\}\subset\{\mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\phi}_i)\}$ holds.*

*Proof:* The equation in (18) can be written as

$$\mathbf{L}_i\left[\mathbf{T}_i\mathbf{X},\quad \mathbf{T}_i\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\right]=\left[\mathbf{K},\quad \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\right]. \tag{35}$$

Then from Lemma 1 (b), all solutions of the equation given in (20) are the solutions of the equation in (16) if and only if

$$\boldsymbol{r}\begin{bmatrix}\mathbf{X} & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}'\mathbf{X}^{\perp} & \mathbf{X} & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\\ \mathbf{K} & \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}'\mathbf{X}^{\perp} & \mathbf{K} & \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\end{bmatrix}=\boldsymbol{r}\left[\mathbf{X},\quad \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\right]. \tag{36}$$

(36) equivalently written as

$$\boldsymbol{r}\begin{bmatrix}\mathbf{X} & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{X} & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}_i'\\ \mathbf{K} & \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{K} & \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}_i'\\ \mathbf{0} & \mathbf{X}' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_i'\end{bmatrix}-\boldsymbol{r}(\mathbf{X})-\boldsymbol{r}(\mathbf{X}_i)=\boldsymbol{r}\begin{bmatrix}\mathbf{X} & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}_i'\\ \mathbf{0} & \mathbf{X}_i'\end{bmatrix}-\boldsymbol{r}(\mathbf{X}_i), \tag{37}$$

which is equivalent to (34). □

**Corollary 2.** *Let consider the models $\mathcal{M}$ in* (1) *and $\mathcal{M}_i$ in* (2)*. Let $\mathbf{K}\boldsymbol{\beta}$ be estimable under $\mathcal{M}_i$ (also estimable under $\mathcal{M}$). Let the coefficients $\mathbf{L}_{\mathbf{K}\boldsymbol{\beta}}$ and $\mathbf{L}_{i\mathbf{K}\boldsymbol{\beta}}$ be as given in* (21) *and* (25)*, respectively. Then the following holds.*

$$\{\mathbf{L}_{i\mathbf{K}\boldsymbol{\beta}}\mathbf{T}_i\}\subset\{\mathbf{L}_{\mathbf{K}\boldsymbol{\beta}}\}\Leftrightarrow\boldsymbol{r}\begin{bmatrix}\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}_i' & \mathbf{X}\\ \mathbf{X}' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_i' & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{K}\end{bmatrix}=\boldsymbol{r}\begin{bmatrix}\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}_i' & \mathbf{X}\\ \mathbf{X}_i' & \mathbf{0}\end{bmatrix}+\boldsymbol{r}(\mathbf{X}). \tag{38}$$

*In this case,* $\{\mathrm{BLUE}_{\mathcal{M}}(\mathbf{K}\boldsymbol{\beta})\}\subset\{\mathrm{BLUE}_{\mathcal{M}_i}(\mathbf{K}\boldsymbol{\beta})\}$ *holds.*

**Theorem 3.** *Let consider the models $\mathcal{M}$ in* (1) *and $\mathcal{M}_i$ in* (2)*. Assume that the $\boldsymbol{\phi}_i$ in* (3) *is predictable under $\mathcal{M}_i$ (also predictable under $\mathcal{M}$). Let the coefficients $\mathbf{L}$ and $\mathbf{L}_i\mathbf{T}_i$ be as given in* (15) *and* (20)*, respectively. Then*

$$\mathbf{L}_i\mathbf{T}_i\in\{\mathbf{L}\}\Leftrightarrow\boldsymbol{r}\begin{bmatrix}\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}_i' & \mathbf{X}\\ \mathbf{X}' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_i' & \mathbf{0}\\ \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}_i' & \mathbf{K}\end{bmatrix}=\boldsymbol{r}\begin{bmatrix}\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}_i' & \mathbf{X}\\ \mathbf{X}' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_i' & \mathbf{0}\end{bmatrix}. \tag{39}$$

*Then,* $\mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\phi}_i)\in\{\mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\phi}_i)\}$ *holds.*

*Proof:* According to (35) and from Lemma 1 (a), the pair of the equations given in (16) and (20) have a common solution if and only if

$$\boldsymbol{r}\begin{bmatrix}\mathbf{X} & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}'\mathbf{X}^{\perp} & \mathbf{X} & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\\ \mathbf{K} & \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}'\mathbf{X}^{\perp} & \mathbf{K} & \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\end{bmatrix}=\boldsymbol{r}\left[\mathbf{X}\quad \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}'\mathbf{X}^{\perp}\quad \mathbf{X}\quad \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}_i'\mathbf{X}_i^{\perp}\right]. \tag{40}$$

(40) equivalently written as

$$\boldsymbol{r}\begin{bmatrix}\mathbf{X} & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{X} & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}_i'\\ \mathbf{K} & \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{K} & \mathbf{J}_i\boldsymbol{\Sigma}\mathbf{B}_i'\\ \mathbf{0} & \mathbf{X}' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_i'\end{bmatrix}-\boldsymbol{r}(\mathbf{X})-\boldsymbol{r}(\mathbf{X}_i)=\boldsymbol{r}\begin{bmatrix}\mathbf{X} & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{X} & \mathbf{B}\boldsymbol{\Sigma}\mathbf{B}_i'\\ \mathbf{0} & \mathbf{X}' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_i'\end{bmatrix}-\boldsymbol{r}(\mathbf{X})-\boldsymbol{r}(\mathbf{X}_i), \tag{41}$$

which is equivalent to (39). □

**Corollary 3.** *Let consider the models $\mathcal{M}$ in* (1) *and $\mathcal{M}_i$ in* (2)*. Let $\mathbf{K}\boldsymbol{\beta}$ be estimable under $\mathcal{M}_i$ (also estimable under $\mathcal{M}$). Let the coefficients $\mathbf{L}_{\mathbf{K}\boldsymbol{\beta}}$ and $\mathbf{L}_{i\mathbf{K}\boldsymbol{\beta}}$ be as given in* (21) *and* (25)*, respectively. Then the following holds.*

$$\mathbf{L}_{i\mathbf{K}\boldsymbol{\beta}}\mathbf{T}_i\in\{\mathbf{L}_{\mathbf{K}\boldsymbol{\beta}}\}\Leftrightarrow\boldsymbol{r}\begin{bmatrix}\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i' & \mathbf{X}\\ \mathbf{X}' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_i' & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{K}\end{bmatrix}=\boldsymbol{r}\begin{bmatrix}\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}' & \mathbf{B}_i\boldsymbol{\Sigma}\mathbf{B}_i' & \mathbf{X}\\ \mathbf{X}' & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_i' & \mathbf{0}\end{bmatrix}. \tag{42}$$

*In this case,* $\mathrm{BLUE}_{\mathcal{M}}(\mathbf{K}\boldsymbol{\beta})\in\{\mathrm{BLUE}_{\mathcal{M}_i}(\mathbf{K}\boldsymbol{\beta})\}$ *holds.*

 

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# Qualitative Analysis for the p-Laplacian Equation with Logarithmic Source Term

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**Abstract:** This work is concerned with p-Laplacian equation with logarithmic source term. Under the suitable conditions, we established stability of the problem.

**Keywords:** Asymptotic behavior, Pseudohyperbolic equations, Logarithmic nonlinearity.

## 1 Introduction

In this work, we investigate the following p-Laplacian hyperbolic type equation with logarithmic nonlinearity

$$\begin{cases} u_{tt} - \nabla\left(|\nabla u|^{p-2}\nabla u\right) - \Delta u_t = |u|^{p-2}u\ln|u|, & x\in\Omega,\ t>0, \\ u(x,0)=u_0(x),\ \ u_t(x,0)=u_1(x), & x\in\Omega, \\ u(x,t)=0, & x\in\partial\Omega,\ t\geq 0, \end{cases} \tag{1}$$

where $u\in W_0^{1,p}(\Omega)\setminus\{0\}$ and $u_1\in H_0^1(\Omega)$ are given initial data and $\Omega\subset R^n$ $(n\geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. The exponent $p$, satisfy $2<p$.

The logarithmic nonlineraity is encountered naturally in quantum mechanics, inflation cosmolog, supersymmetric field theories, and a lot of different areas of physics such as, optics, geophysics and nuclear physics [2, 4, 5]. These special applications in physics allow many mathematicians to work with logarithmic nonlinear equations. In recent years, many authors considered the mathematical behavior for nonlinear hyperbolic equation with logartihmic nonlinearity and obtained some remarkable achievements (see[7, 9, 18])

Without the logartihmic source term the problem (1) becomes

$$u_{tt} - \nabla\left(|\nabla u|^{p-2}\nabla u\right) + f(u_t) = g(u). \tag{2}$$

In mathematics, the study of this class of equations originated from the work of MacCamy and Mizel [10]. Many authors have studied the properties of the problem (1) see([1, 3, 11, 12, 15]).

In recent years, existence and asymptotic behavior of solutions for a class of p-Laplacian parabolic type equations with logarithmic nonlinearity have been intensively studied in the litarature ([6, 8, 17]). Fewer results are, at the present time, known for the logarithmic p-Laplacian hyperbolic type equations with logarithmic nonlinearity (see [13, 14, 16]). For this reason, we study the decay rate of solution of the problem (1) in this paper.

## 2 Preliminaries

In order to state the main results to problem (1) more clearly, we start to our work by introducing some notations, lemmas and definitions which will be used in this paper. Throughout this paper, we denote

$$\|u\|_m = \|u\|_{L^m(\Omega)},\quad \|u\|_{1,m} = \|u\|_{W_0^{1,m}(\Omega)} = \left(\|u\|_m^m + \|\nabla u\|_m^m\right)^{\frac{1}{m}}$$

for $1<m<\infty$. We consider $W_0^{-1,m'}(\Omega)$ to denote the dual space of $W_0^{1,m'}(\Omega)$ where $m'$ is Hölder conjugate exponent for $m>1$.

 

We define energy function as follows

$$E(t)=\frac{1}{2}\left\|u_t\right\|^2+\frac{1}{p}\left\|\nabla u\right\|_p^p-\frac{1}{p}\int\limits_{\Omega}|u|^p\ln|u|\,dx+\frac{1}{p^2}\left\|u\right\|_p^p. \tag{3}$$

Let us define some useful funcionals as follows

$$J\left(u\right)=\frac{1}{p}\left\|\nabla u\right\|_p^p-\frac{1}{p}\int\limits_{\Omega}|u|^p\ln|u|\,dx+\frac{1}{p^2}\left\|u\right\|_p^p, \tag{4}$$

and

$$I\left(u\right)=\left\|\nabla u\right\|_p^p-\int\limits_{\Omega}|u|^p\ln|u|\,dx. \tag{5}$$

By the Gagliardo-Nirenberg multiplicative embedding inequality that $J\left(u\right)$ and $I\left(u\right)$ are continuous. Then, by (4) and (5), it tells us that

$$J\left(u\right)=\frac{1}{p}I\left(u\right)+\frac{1}{p^2}\left\|u\right\|_p^p \tag{6}$$

and

$$E\left(t\right)=\frac{1}{2}\left\|u_t\right\|^2+J\left(u\right). \tag{7}$$

We can define the mountain-pass level

$$d=\inf_{u\in\aleph}J\left(t\right), \tag{8}$$

where $\aleph$ is the Nehari manifold, which is defined by

$$\aleph=\left\{u\in W_0^{1,p}\left(\Omega\right)\setminus\{0\}:I\left(u\right)=0\right\}.$$

We define the potential well depth

$$0<d=\inf_{u}\left\{\sup_{\lambda\geq0}J\left(\lambda u\right):u\in W_0^{1,p}\left(\Omega\right),\left\|u\right\|_p^p\neq0\right\}. \tag{9}$$

Now, we introduce the potential well $W$ and its corresponding set $V$

$$W=\left\{u\in W_0^{1,p}\left(\Omega\right):I\left(u\right)>0,\ J\left(u\right)<d\right\}\cup\{0\},$$

$$V=\left\{u\in W_0^{1,p}\left(\Omega\right):I\left(u\right)<0,\ J\left(u\right)<d\right\}.$$

**Lemma 1.** *[16] (Logarithmic Sobolev Inequality). Let $u$ be any function $u\in W_0^{1,p}(\Omega)$ and $a>0$ be any number. Then,*

$$\int\limits_{\Omega}\ln|u|\,u^p dx<\left\|u\right\|_p^p\ln\left\|u\right\|_p+\frac{(p-2)\,a^2}{4\pi}\left\|u\right\|_p^p+\frac{a^2}{2\pi}\left\|\nabla u\right\|_p^p-\frac{n}{p}(1+\ln a)\left\|u\right\|_p^p.$$

**Lemma 2.** *$E(t)$ is a nonincreasing function, for $t\geq0$*

$$E\left(0\right)=E\left(t\right)+\int\limits_0^t\left\|\nabla u_t\right\|^2dt. \tag{10}$$

*Proof:* Multiplying the equation (1) by $u_t$ and integrating on $\Omega$, we have

$$E'\left(t\right)=-\left\|\nabla u_t\right\|^2.$$

□

**Lemma 3.** *For any $u\in W_0^{1,p}\left(\Omega\right),\ \left\|u\right\|_p^p\neq0$ and let $g\left(\sigma\right)=J\left(\sigma u\right)$. Then, there exists a unique $\sigma^*>0$ such that*

$$I\left(\sigma u\right)=\sigma g'\left(\sigma\right)\begin{cases}>0, & 0\leq\sigma<\sigma^*,\\ =0, & \sigma=\sigma^*,\\ <0, & \sigma<\sigma^*<\infty.\end{cases}$$

*Proof:* By the definition of $J(u)$, we obtain

$$\begin{aligned} g(\sigma) &= J(\sigma u) \\ &= \frac{1}{p}\|\sigma\nabla u\|_p^p - \frac{1}{q}\int\limits_\Omega |\sigma u|^q \ln|\sigma u|\,dx + \frac{1}{q^2}\|\sigma u\|_q^q \\ &= \frac{\sigma^p}{p}\|\nabla u\|_p^p - \frac{\sigma^q}{q}\int\limits_\Omega |u|^q \ln|u|\,dx - \ln|\sigma|\frac{\sigma^q}{q}\|u\|_q^q + \frac{\sigma^q}{q^2}\|u\|_q^q. \end{aligned}$$

Since $2<p$ and $\|u\|_p^p \neq 0$, $\lim\limits_{\sigma\to 0} g(\sigma)=0$, $\lim\limits_{\sigma\to\infty} g(\sigma)=-\infty$ hold. Now, differentiating $g(\sigma)$ with respect to $\sigma$, we have

$$g'(\sigma) = \sigma^{p-1}\left(\|\nabla u\|_p^p - \int\limits_\Omega |u|^p \ln|u|\,dx - \ln|\sigma|\,\|u\|_p^p\right).$$

Let $g'(\sigma)=0$. So that we denote that

$$\sigma^* = \exp\frac{\|\nabla u\|_p^p - \int\limits_\Omega |u|^p \ln|u|\,dx}{\|u\|_p^p}.$$

It follows from definition of that $I(u)$

$$I(u) = \sigma^p\left(\|\nabla u\|_p^p - \int\limits_\Omega |u|^p \ln|u|\,dx - \ln\sigma\,\|u\|_p^p\right)$$

□

Thus, we have

$$I(\sigma u) = \sigma g'(\sigma)\begin{cases} >0, & 0\le\sigma<\sigma^*, \\ =0, & \sigma=\sigma^*, \\ <0, & \sigma<\sigma^*<\infty. \end{cases}$$

So the proof is completed.

**Lemma 4.** *Let $u\in W_0^{1,p}(\Omega)$ and $l=(2\pi)^{\frac{n}{2}}e^{\frac{2(n+p)-p^2}{2}}$,*

i) if $0<\|u\|_p^p<l$, then $I(u)>0$;
ii) if $I(u)=0$ and $\|u\|_p^p\neq 0$, then $\|u\|_p^p>l$;
iii) The constant $d$ in (9) satisfies

$$d \ge \frac{1}{p^2}(2\pi)^{\frac{n}{2}}e^{\frac{2(n+p)-p^2}{2}}.$$

*Proof:* Thanks to Logarithmic Sobolev Inequality to the last term of the $I(u)$ function, we have

$$\begin{aligned} I(u) &= \|\nabla u\|_p^p - \int\limits_\Omega |u|^p \ln|u|\,dx. \\ &\ge \|\nabla u\|_p^p - \left(\|u\|_p^p \ln\|u\|_p + \frac{(p-2)a^2}{4\pi}\|u\|_p^p\right. . \\ &\quad \left.+\frac{a^2}{2\pi}\|\nabla u\|_p^p - \frac{n}{p}(1+\ln a)\|u\|_p^p\right), \\ &\ge \left(1-\frac{a^2}{2\pi}\right)\|\nabla u\|_p^p + \left(\frac{n}{p}(1+\ln a) - \ln\|u\|_p - \frac{(p-2)a^2}{4\pi}\right)\|u\|_p^p \end{aligned} \tag{11}$$

Taking any $a=\sqrt{2\pi}$ satisfying in (11), we have

$$I(u) \ge \left(\frac{n}{p}(1+\ln a) - \ln\|u\|_p - \frac{(p-2)a^2}{4\pi}\right)\|u\|_p^p. \tag{12}$$

i) If $0<\|u\|_p^p<l$, then $I(u)>0$ from the above inequality.

ii) if $I(u)=0$ and $\|u\|_p^p \neq 0$, then

$$\|u\|_p^p \geq (2\pi)^{\frac{n}{2}} e^{\frac{2}{k}+2n+2p-p^2} = l.$$

iii) Because of (9), we write

$$\sup_{\lambda\geq 0} J(\lambda u) = J(\lambda^* u) = \frac{1}{p} I(\lambda^* u) + \frac{1}{p^2} (\lambda^*)^p \|u\|_p^p \tag{13}$$

By the Lemma 3 and (12), we obtain

$$0 = I(\lambda^* u) \geq \left(\frac{n}{p}(1+\ln\sqrt{2\pi}) - \ln\|\lambda^* u\|_p - \frac{(p-2)}{2\pi}\right)\|\lambda^* u\|_p^p.$$

Therefore; we have

$$0 \geq \frac{n}{p}(1+\ln\sqrt{2\pi}) - \ln\|\lambda^* u\|_p - \frac{(p-2)}{2\pi},$$

$$\|\lambda^* u\|^p \geq (2\pi)^{\frac{n}{2}} e^{\frac{2(n+p)-p^2}{2}} = l. \tag{14}$$

Thus by combination of the (9), (13) and (14), we obtain

$$d \geq \sup_{\lambda\geq 0} J(\lambda u) \geq \frac{1}{p^2}(2\pi)^{\frac{n}{2}} e^{\frac{2(n+p)-p^2}{2}}.$$

□

## 3 Local existence

In this section we state and prove the local existence result for the problem (1). The proof is based Faedo-Galerkin method.

**Definition 5.** *A function $u$ defined on $[0,T]$ is called a weak solution of (1) if*

$$u \in C\left([0,T);\left(W_0^{1,p}(\Omega)\right)\cap L^p(\Omega)\right),\ u_t \in C\left([0,T);H_0^1(\Omega)\right)$$

*and $u$ satisfies*

$$\begin{cases} \int_\Omega u_{tt}(x,t)\,w(x)\,dx + \int_\Omega |\nabla u(x,t)|^{p-2}\nabla u(x,t)\nabla w dx \\ +\int_\Omega \nabla u_t(x,t)\nabla w(x)\,dx = \int_\Omega u^{p-2}(x,t)\,u(x,t)\ln|u(x,t)|\,w(x)\,dx, \end{cases}$$

*for $w \in W_0^{1,p}(\Omega)$.*

**Theorem 6.** *Let $(u_0,u_1) \in W_0^{1,p}(\Omega)\times H_0^1(\Omega)$, then the problem (1) has a weak solution on $[0,T]$.*

*Proof:* The proof method of this lemma can refer to [16]. Here, we omit it. □

## 4 Decay of solution

In this section, we will prove decay of solutions to problem (1).
For this purpose, we define the functional

$$L(t) = E(t) + \varepsilon\int_\Omega u u_t dx + \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2 dx \tag{15}$$

where $\varepsilon$ is a positive constant. We will show that the $L(t)$ and $E(t)$ are equivalent:

**Lemma 7.** *For $\varepsilon > 0$ small enough, the relation*

$$\beta_1 L(t) \leq E(t) \leq \beta_2 L(t) \tag{16}$$

*holds for two positive constants $\beta_1$ and $\beta_2$.*

We can choose $\varepsilon$ small enough such that $L \sim E$.

**Theorem 8.** *Let* $u_0 \in W, u_1 \in H_0^1(\Omega)$ *. Assume further* $0 < E(0) < \alpha l < d$ *, where*

$$l = (2\pi)^{\frac{n}{2}} e^{\frac{2}{k}+2n+2p-p^2} \quad \textit{and } 0 < \alpha < a^{\frac{n}{p}} p^{-2} (2\pi)^{\frac{-n}{2}} e^{-\frac{pn+p^2-p^3-n}{p}}$$

*then there exist two positive constants* $r_1$ *and* $r_2$ *such that*

$$0 < E(t) \leq r_1 e^{-r_2 t}, \ t \geq 0.$$

*Proof:* By derivativing of the (15) and using equation (1), we have

$$\begin{aligned} L'(t) &= E'(t) + \varepsilon \int_\Omega \left(u_{tt}u + u_t^2\right) dx + \varepsilon \int_\Omega \nabla u \nabla u_t dx \\ &= -\|\nabla u_t\|^2 + \varepsilon \int_\Omega \left(\nabla\left(|\nabla u|^{p-2}\nabla u\right) + \Delta u_t + |u|^{p-2} u \ln|u|\right) u dx \\ &\quad + \varepsilon \int_\Omega \nabla u \nabla u_t dx + \varepsilon \|u_t\|^2, \\ &= -\|\nabla u_t\|^2 + \varepsilon\|u_t\|^2 - \varepsilon\|\nabla u\|_p^p + \varepsilon \int_\Omega |u|^p \ln|u| \, dx \end{aligned} \tag{17}$$

Adding and subtracting $\varepsilon\delta E(t)$ into (17) where $\delta$ is a positive constant and using the following equality

$$\delta E(t) = \frac{\delta p^2}{2}\|u_t\|^2 + p\delta\|\nabla u\|_p^p - p\delta \int_\Omega |u|^p \ln|u| \, dx + \delta\|u\|_p^p,$$

we get

$$\begin{aligned} L'(t) &= -\|\nabla u_t\|^2 + \varepsilon\left(\frac{p^2\delta}{2}+1\right)\|u_t\|^2 + \varepsilon(p\delta-1)\|\nabla u\|_p^p \\ &\quad + \varepsilon(1-p\delta)\int_\Omega |u|^p \ln|u| \, dx + \delta\|u\|_p^p - \varepsilon\delta p^2 E(t). \end{aligned} \tag{18}$$

By using the embedding theorem and the Logarithmic Sobolev inequality where $C_p$ is the positive constant, it becomes

$$\begin{aligned} L'(t) &\leq -\|\nabla u_t\|^2 + \varepsilon\left(\frac{p^2\delta}{2}+1\right)\|u_t\|^2 + \varepsilon(p\delta-1)\|\nabla u\|_p^p + \delta\|u\|_p^p - \varepsilon\delta p^2 E(t) \\ &\quad + \varepsilon(1-p\delta)\left[\|u\|_p^p \ln\|u\|_p + \frac{(p-2)a^2}{4\pi}\|u\|_p^p + \frac{a^2}{2\pi}\|\nabla u\|_p^p - \frac{n}{p}(1+\ln a)\|u\|_p^p\right] \\ &\leq \left(\frac{\varepsilon C_p\delta}{2} + \varepsilon C_p - 1\right)\|\nabla u_t\|^2 - \varepsilon\delta p^2 E(t) \\ &\quad - \varepsilon(1-p\delta).\left(1-\frac{a^2}{2\pi}\right)\|\nabla u\|_p^p \\ &\quad + \varepsilon\left[\delta + (1-p\delta)\frac{(p-2)a^2}{4\pi} + (1-p\delta)\left(\ln\|u\|_p - \frac{n}{p}(1+\ln a)\right)\right]\|u\|_p^p. \end{aligned} \tag{19}$$

By using (3), (10), assumption in the Theorem 8 and definition of $l$, we get

$$\begin{aligned} \ln\|u\|_p^p &\leq \ln\left(p^2 E(t)\right) \\ &\leq \ln\left(p^2 E(0)\right) \\ &\leq \ln\left(p^2 \alpha l\right) \\ &= \ln\left(p^2 \alpha (2\pi)^{\frac{n}{2}} e^{\frac{2(n+p)-p^2}{2}}\right). \end{aligned} \tag{20}$$

Using (20) and taking $\alpha$ satisfying

$$\left(\left(p^2\alpha\right)^{\frac{1}{n}}\sqrt{2\pi}\right)^p e^{\frac{pn+p^2-p^3}{n}-1} < a \leq \sqrt{2\pi}$$

we guarantee

$$\left(\ln\|u\|_p - \frac{n}{p}(1+\ln a)\right) \leq 0.$$

Finally, Noting $(1-p\delta)$ since by picking $0<\delta<\min\left\{\frac{(p-2)}{(p-1)^2+3},\frac{1}{p}\right\}$ small enough such that

$$(1-p\delta)\,.\left(1-\frac{a^2}{2\pi}\right) > 0,$$

and

$$\delta + (1-p\delta)\frac{(p-2)\,a^2}{4\pi} + (1-p\delta)\left(\ln\|u\|_p - \frac{n}{p}(1+\ln a)\right) < 0,$$

then we obtain

$$L'(t) \leq \left(\frac{\varepsilon C_p\delta}{2}+\varepsilon C_p - 1\right)\|\nabla u_t\|^2 - \varepsilon\delta p^2 E(t)\,.$$

Now, we choose $\varepsilon > 0$ small enough such that

$$\frac{\varepsilon C_p\delta}{2}+\varepsilon C_p - 1 < 0.$$

Consequently, inequlatiy (19) becomes

$$L'(t) \leq -\varepsilon\delta p^2 E(t)\,. \tag{21}$$

By using (16), (21) can be replaced

$$L'(t) \leq -\varepsilon\delta\beta_2 L(t)\,. \tag{22}$$

Setting $c_2 = \varepsilon\beta\beta_2 > 0$ and integrating (22) between $(0,t)$ gives the following estimate

$$L(t) \leq r_1 e^{-r_2 t}.$$

□

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# A New Approach to the Fixed-Circle Problem on $S$-Metric Spaces

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**Abstract:** In this paper, we give new solutions to the fixed-circle problem on $S$-metric spaces. To do this, we define the notions of Moradi type $x_0$-$S$-contraction, Geraghty type $x_0$-$S$-contraction and Skof type $x_0$-$S$-contraction. Using these new notions, we prove some fixed-circle theorems on $S$-metric spaces. Also, we give an example to show the validity of the obtained results.

**Keywords:** Fixed-circle problem, S-metric space.

## 1 Introduction and preliminaries

Recently, the notion of an $S$-metric space was introduced as a generalization of a metric space [12] as follows:

**Definition 1.** *[12] Let $X$ be a nonempty set and $S : X \times X \times X \to [0, \infty)$ be a function satisfying the following conditions for all $x, y, z, a \in X$ :*
$(S1)$ $S(x, y, z) = 0$ *if and only if* $x = y = z$,
$(S2)$ $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.
*Then $S$ is called an $S$-metric on $X$ and the pair $(X, S)$ is called an $S$-metric space.*

Note that there is no symmetry condition in the $S$-metric definition. But the following lemma can be considered as a symmetry property for $S$-metric spaces.

**Lemma 1.** *[12] Let $(X, S)$ be an $S$-metric space and $x, y \in X$. Then we have*

$$S(x, x, y) = S(y, y, x).$$

After then, many researchers have studied some fixed-point theorems on this space (for example, see [3], [5], [6], [7], [13], [15] and the references therein).

"Fixed-Circle Problem" was presented as a geometric generalization of the fixed-point theory in [8]. Also, some solutions to the this problem have been investigated on $S$-metric spaces using the following basic notions:

In [10] and [12], a circle and a disc are defined on an $S$-metric space as follows, respectively:

$$C^S_{x_0,\mu} = \{x \in X : S(x, x, x_0) = \mu\}$$

and

$$D^S_{x_0,\mu} = \{x \in X : S(x, x, x_0) \leq \mu\}.$$

Let $(X, S)$ be an $S$-metric space, $C^S_{x_0,\mu}$ be a circle and $f : X \to X$ be a self-mapping. If $fx = x$ for every $x \in C^S_{x_0,\mu}$ (resp. $x \in D^S_{x_0,\mu}$) then the circle $C^S_{x_0,\mu}$ (the disc $D^S_{x_0,\mu}$) is called as the fixed circle (the fixed disc) of $f$ (see [2] and [10] for more details).

The fixed-circle problem has been studied on $S$-metric spaces with various aspects (for example, see [2], [9], [10], [15], [16] and the references therein).

In this paper, we define the notions of Moradi type $x_0$-$S$-contraction, Geraghty $x_0$-$S$-contraction and Skof type $x_0$-$S$-contraction on $S$-metric spaces modifying some known contractions (see [1], [4], [11] and [14]). Also, we give an illustrative example to show the validity of the obtained results.

## 2 Main results

In this section, we prove new fixed-circle results using different contractive conditions.

**Definition 2.** *Let $(X,S)$ be an $S$-metric space and $f:X\to X$ a self-mapping. If there exists $x_0\in X$ such that*

$$S(x,x,fx)>0 \Longrightarrow \varphi\left(S(x,x,fx)\right)\leq \alpha\left(\varphi\left(S(x,x,x_0)\right)\right),$$

*for all $x\in X$, where the function $\varphi:[0,\infty)\to[0,\infty)$ is such that $\varphi$ is nondecreasing with $\varphi(0)=0$ and $0<\varphi(t)<t$ for all $t>0$ and the function $\alpha:[0,\infty)\to[0,\infty)$ is such that $\alpha$ is a function with $\alpha(0)=0$ and $0<\alpha(t)<t$ for all $t>0$, then $f$ is called Moradi type $x_0$-$S$-contraction.*

**Proposition 1.** *Let $(X,S)$ be an $S$-metric space and $f:X\to X$ Moradi type $x_0$-$S$-contraction with $x_0\in X$. Then we have $fx_0=x_0$.*

*Proof:* Let us assume that $fx_0\neq x_0$, that is, $S(x_0,x_0,fx_0)>0$. Using the hypothesis, we get

$$\varphi\left(S(x_0,x_0,fx_0)\right)\leq \alpha\left(\varphi\left(S(x_0,x_0,x_0)\right)\right)=\alpha\left(\varphi\left(0\right)\right)\alpha(0)=0,$$

a contradiction. It should be $fx_0=x_0$. □

**Theorem 1.** *Let $(X,S)$ be an $S$-metric space, $f:X\to X$ Moradi type $x_0$-$S$-contraction with $x_0\in X$ and the number $\mu$ defined as*

$$\mu=\inf\left\{S(x,x,fx):x\neq fx,\ x\in X\right\}. \quad (1)$$

*Then $C^S_{x_0,\mu}$ is a fixed circle of $f$.*

*Proof:* At first, assume that $\mu=0$. Then $C^S_{x_0,\mu}=\{x_0\}$. Using Proposition 1, we have $fx_0=x_0$, that is, $C^S_{x_0,\mu}$ is a fixed circle of $f$.
Let $\mu>0$ and $x\in C^S_{x_0,\mu}$ be any point such that $S(x,x,fx)>0$. Using the Moradi type $x_0$-$S$-contraction property, we obtain

$$\varphi\left(S(x,x,fx)\right)\leq \alpha\left(\varphi\left(S(x,x,x_0)\right)\right)=\alpha\left(\varphi\left(\mu\right)\right)<\varphi\left(\mu\right)\leq\varphi\left(S(x,x,fx)\right),$$

a contradiction. So it should be $x=fx$. Consequently, $C^S_{x_0,\mu}$ is a fixed circle of $f$. □

As a consequence of Theorem 1, we get the following result:
Let $(X,S)$ be an $S$-metric space, $f:X\to X$ Moradi type $x_0$-$S$-contraction with $x_0\in X$ and the number $\mu$ defined as in (1). Then $D^S_{x_0,\mu}$ is a fixed disc of $f$.

**Definition 3.** *Let $(X,S)$ be an $S$-metric space and $f:X\to X$ a self-mapping. If there exists $x_0\in X$ such that*

$$S(x,x,fx)>0 \Longrightarrow \varphi\left(S(x,x,fx)\right)\leq \beta\left(S(x,x,x_0)\right)\varphi\left(S(x,x,x_0)\right),$$

*for all $x\in X$, where the function $\varphi:(0,\infty)\to(0,\infty)$ is such that $\varphi$ is nondecreasing and $\beta:(0,\infty)\to(0,1)$ is a function, then $f$ is called Geraghty type $x_0$-$S$-contraction.*

**Proposition 2.** *Let $(X,S)$ be an $S$-metric space and $f:X\to X$ Geraghty type $x_0$-$S$-contraction with $x_0\in X$. Then we have $fx_0=x_0$.*

*Proof:* Let $S(x_0,x_0,fx_0)>0$. Then using the hypothesis, we get

$$\varphi\left(S(x_0,x_0,fx_0)\right)\leq \beta\left(S(x_0,x_0,x_0)\right)\varphi\left(S(x_0,x_0,x_0)\right)=\beta(0)\varphi(0),$$

a contradiction. So it should be $fx_0=x_0$. □

**Theorem 2.** *Let $(X,S)$ be an $S$-metric space, $f:X\to X$ Geraghty type $x_0$-$S$-contraction with $x_0\in X$ and the number $\mu$ defined as in* (1). *Then $C^S_{x_0,\mu}$ is a fixed circle of $f$.*

*Proof:* Let $\mu=0$. Then we have $C^S_{x_0,\mu}=\{x_0\}$. Using Proposition 2, we get $fx_0=x_0$.
Let $\mu>0$ and $x\in C^S_{x_0,\mu}$ be an arbitrary point such that $x\neq fx$. Using the Geraghty type $x_0$-$S$-contraction property, we find

$$\begin{aligned}\varphi\left(S(x,x,fx)\right) &\leq \beta\left(S(x,x,x_0)\right)\varphi\left(S(x,x,x_0)\right)=\beta\left(\mu\right)\varphi\left(\mu\right)\\ &< \varphi\left(\mu\right)\leq\varphi\left(S(x,x,fx)\right),\end{aligned}$$

a contradiction. Hence it should be $x=fx$. Consequently, $C^S_{x_0,\mu}$ is a fixed circle of $f$. □

We obtain the following result as a consequence of Theorem 2:
Let $(X,S)$ be an $S$-metric space, $f:X\to X$ Geraghty type $x_0$-$S$-contraction with $x_0\in X$ and the number $\mu$ defined as in (1). Then $D^S_{x_0,\mu}$ is a fixed disc of $f$.

**Definition 4.** *Let $(X,S)$ be an $S$-metric space and $f:X\to X$ a self-mapping. If there exists $x_0\in X$ such that*

$$\begin{aligned} S(x,x,fx) &> 0 \Longrightarrow \varphi\left(S(x,x,fx)\right)\leq a\varphi\left(S(x,x,x_0)\right) \\ & +b\varphi\left(S(x,x,fx)\right)+c\varphi\left(S(x_0,x_0,fx_0)\right), \end{aligned}$$

*for all $x\in X$, where $a,b,c\in[0,1)$ with $0\leq a+b+c<1$ and the function $\varphi:[0,\infty)\to[0,\infty)$ is such that $\varphi$ is nondecreasing with*

$$\varphi(t)=0 \Longleftrightarrow t=0,$$

*then $f$ is called Skof type $x_0$-$S$-contraction.*

**Proposition 3.** *Let $(X,S)$ be an $S$-metric space and $f:X\to X$ Skof type $x_0$-$S$-contraction with $x_0\in X$. Then we have $fx_0=x_0$.*

*Proof:* Let $S(x_0,x_0,fx_0)>0$. Then using the hypothesis, we get

$$\begin{aligned} \varphi\left(S(x_0,x_0,fx_0)\right) &\leq a\varphi\left(S(x_0,x_0,x_0)\right)+b\varphi\left(S(x_0,x_0,fx_0)\right)+c\varphi\left(S(x_0,x_0,fx_0)\right) \\ &= (b+c)\varphi\left(S(x_0,x_0,fx_0)\right), \end{aligned}$$

a contradiction with $b+c<1$. Thereby, it should be $fx_0=x_0$. □

**Theorem 3.** *Let $(X,S)$ be an $S$-metric space, $f:X\to X$ Skof type $x_0$-$S$-contraction with $x_0\in X$ and the number $\mu$ defined as in (1). Then $C^S_{x_0,\mu}$ is a fixed circle of $f$.*

*Proof:* Let $\mu=0$. Then we have $C^S_{x_0,\mu}=\{x_0\}$ and using Proposition 3, we obtain $fx_0=x_0$.

Suppose that $\mu>0$ and $x\in C^S_{x_0,\mu}$ is any point with $x\neq fx$. Using the Skof type $x_0$-$S$-contraction property and Proposition 3, we get

$$\begin{aligned} \varphi\left(S(x,x,fx)\right) &\leq a\varphi\left(S(x,x,x_0)\right)+b\varphi\left(S(x,x,fx)\right)+c\varphi\left(S(x_0,x_0,fx_0)\right) \\ &= a\varphi\left(\mu\right)+b\varphi\left(S(x,x,fx)\right) \\ &\leq a\varphi\left(S(x,x,fx)\right)+b\varphi\left(S(x,x,fx)\right) \\ &= (a+b)\varphi\left(S(x,x,fx)\right), \end{aligned}$$

a contradiction with $a+b<1$. So it should be $x=fx$. Consequently, $C^S_{x_0,\mu}$ is a fixed circle of $f$. □

Now we get the following result as a consequence of Theorem 3:

Let $(X,S)$ be an $S$-metric space, $f:X\to X$ Skof type $x_0$-$S$-contraction with $x_0\in X$ and the number $\mu$ defined as in (1). Then $D^S_{x_0,\mu}$ is a fixed disc of $f$.

Finally, we give the following example.

**Example 1.** *Let $X=\mathbb{R}$ and the $S$-metric defined as*

$$S(x,y,z)=|x-z|+|x+z-2y|,$$

*for all $x,y,z\in\mathbb{R}$ [7]. Let us define the function $f:\mathbb{R}\to\mathbb{R}$ as*

$$f(x)=\begin{cases} x & , \quad x\in[-4,4] \\ x+2 & , \quad x\in(-\infty,-4)\cup(4,\infty) \end{cases},$$

*for all $x\in\mathbb{R}$. Then the function $f$ is Moradi type $x_0$-$S$-contraction with $x_0=0$, the function $\varphi:[0,\infty)\to[0,\infty)$ is defined by*

$$\varphi(t)=\begin{cases} 0 & , \quad t=0 \\ \frac{t}{24} & , \quad t>0 \end{cases}$$

*and the function $\alpha:[0,\infty)\to[0,\infty)$ is defined by*

$$\alpha(t)=\begin{cases} 0 & , \quad t=0 \\ \frac{t}{2} & , \quad t>0 \end{cases}.$$

*The function $f$ is also Geraghty type $x_0$-$S$-contraction with $x_0=0$, the function $\varphi:(0,\infty)\to(0,\infty)$ defined by $\varphi(t)=3t$ and the function $\beta:(0,\infty)\to(0,1)$ defined by $\beta(t)=\frac{1}{2}$. Finally, the function $f$ is Skof type $x_0$-$S$-contraction with $x_0=0$, $a=\frac{1}{2}$, $b=\frac{1}{4}$ and $c\in\mathbb{R}$. Consequently, we have $\mu=4$ and so $f$ fixes the circle $C^S_{0,4}=\{-2,2\}$ and the disc $D^S_{0,4}=[-2,2]$.*

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# Hyperbolic Padovan and Perrin Numbers

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**Abstract:** The main idea of this paper is to investigate the *hyperbolic Padovan and Perrin numbers* as indicated $\mathbb{H}$PPN from now on. Then, by taking into account the properties of the hyperbolic numbers, we try to show some properties of $\mathbb{H}$PPN. Moreover, we present interesting relationships between $\mathbb{H}$PPN.

**Keywords:** Hyperbolic number, Padovan number, Perrin number.

## 1 Introduction

There are so many articles in the literature that concern about the special numbers like Fibonacci, Lucas, Pell, Jacobsthal, Padovan and Perrin (see, for example [1, 3, 8], [10]-[12]).

In Fibonacci numbers, there clearly exists the term Golden ratio which is defined as the ratio of two consecutive of Fibonacci numbers that converges to $\frac{1+\sqrt{5}}{2}$. It is also clear that this ratio is used in so many applications such as Physics, Engineering, Architecture, etc. In a similar manner, the ratio of two consecutive Padovan and Perrin numbers converges to

$$\alpha=\sqrt[3]{\frac{1}{2}+\frac{1}{6}\sqrt{\frac{23}{3}}}+\sqrt[3]{\frac{1}{2}-\frac{1}{6}\sqrt{\frac{23}{3}}}$$

that is named as *Plastic constant* and was firstly defined in 1924 by Gérard Cordonnier. He presented applications to architecture; in 1958 he gave a lecture tour that pictured the use of the Plastic constant in many buildings and monuments. The smallest Pisot number is the positive root of the characteristic equation $X^3-X-1=0$ known as the Plastic constant. This is also the characteristic equation of the recurrence equations (1) and (2) in below, and the Plastic constant is one of its roots which is the unique real root.

Although the study of Perrin numbers started in the beginning of 19. century under different names, the master study was published in 2006 by Shannon et al. in [8]. In this reference, the authors defined the Perrin $\{R_n\}_{n\in\mathbb{N}}$ and Padovan $\{P_n\}_{n\in\mathbb{N}}$ sequences as in the forms

$$R_{n+3}=R_{n+1}+R_n,\ \text{where}\quad R_0=3,\ R_1=0,\ R_2=2 \tag{1}$$

and

$$P_{n+3}=P_{n+1}+P_n,\ \text{where}\quad P_0=P_1=P_2=1\,, \tag{2}$$

respectively. Also, the Padovan and Perrin numbers with negative subscripts are defined by

$$P_{-n}=P_{-n+3}-P_{-n+1} \text{ and } R_{-n}=R_{-n+3}-R_{-n+1}. \tag{3}$$

Therefore, recurrences (1) and (2) hold for all integer $n$. The general form of these numbers are

$$P_n=a\alpha^n+b\beta^n+c\gamma^n \text{ and } R_n=\alpha^n+\beta^n+\gamma^n, \tag{4}$$

where $\alpha=\sqrt[3]{\frac{1}{2}+\frac{1}{6}\sqrt{\frac{23}{3}}}+\sqrt[3]{\frac{1}{2}-\frac{1}{6}\sqrt{\frac{23}{3}}}$,

$\beta=-\sqrt[3]{\frac{1}{16}+\frac{1}{48}\sqrt{\frac{23}{3}}}-\sqrt[3]{\frac{1}{16}-\frac{1}{48}\sqrt{\frac{23}{3}}}+i\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{1}{2}+\frac{1}{6}\sqrt{\frac{23}{3}}}-\sqrt[3]{\frac{1}{2}-\frac{1}{6}\sqrt{\frac{23}{3}}}\right),\gamma=\overline{\beta}$

and $a=\dfrac{(\beta-1)(\gamma-1)}{(\alpha-\beta)\,(\alpha-\gamma)}, b=\dfrac{(\alpha-1)(\gamma-1)}{(\beta-\alpha)\,(\beta-\gamma)}, c=\dfrac{(\alpha-1)(\beta-1)}{(\gamma-\alpha)\,(\gamma-\beta)}$.

In [13], the most well-known the relationships between $\{R_n\}$ and $\{P_n\}$ given by

$$R_n = 3P_{n-5} + 2P_{n-4} \text{ and } P_n = \frac{1}{23}(10R_n + 8R_{n-1} + R_{n-2})\,. \tag{5}$$

On the other hand, hyperbolic numbers have applications in different areas of mathematics and theoretical physics. A hyperbolic number (or split complex number, also perplex number, double number) has two real number components $a$ and $b$, and the set of hyperbolic numbers is

$$\mathbb{H} = \left\{x = a + \mathbf{h}b : \ \mathbf{h}^2 = 1, a, b \in \mathbb{R}\right\}.$$

The hyperbolic ring $\mathbb{H}$ is a bidimensional Clifford algebra, look at [6] for details. Also, hyperbolic numbers are helpful for measuring distances in the Lorentz space-time plane (you can examine [9]). The addition and multiplication of any two hyperbolic numbers $x = a + \mathbf{h}b, y = c + \mathbf{h}d$ are defined by

$$x + y = a + c + \mathbf{h}(b + d) \text{ and } xy = ac + bd + \mathbf{h}(ad + bc)\,.$$

It is clear that this algebra of hyperbolic number is commutative. The conjugate and norm of $x$ are enounced by

$$\overline{x} = a - \mathbf{h}b, \quad x\overline{x} = a^2 - b^2\,. \tag{6}$$

For more information on hyperbolic numbers, see for example, [2],[4]-[7] and [9].

In recent years so many researchs activities can be seen on hyperbolic Fibonacci, Lucas, Jacobsthal and Tribonacci numbers (see [1], [3], [10], [12]). For example, in [1], it was investigated some properties of the hyperbolic Fibonacci numbers as defined $\tilde{F}_n = F_n + \mathbf{h}F_{n+1}$ .

It is natural to marvel whether there exits a connection between the parameters hyperbolic numbers and Padovan, Perrin numbers. Motivated by [1], the goal of this paper is to define *hyperbolic Padovan and Perrin numbers($\mathbb{H}$PPN)* with a different viewpoint. Then, we acquire the Binet formulas, the generating functions, the summations of the $\mathbb{H}$PPN. We also actually research the relations between the hyperbolic Padovan and hyperbolic Perrin numbers.

## 2 The hyperbolic Padovan and Perrin numbers

In this section, we introduce the $\mathbb{H}$PPN. Also, we present some properties of these numbers such as the Binet formulas, the generating functions and the relationships between the hyperbolic Padovan and Perrin numbers.

**Definition 1.** *The hyperbolic Padovan $(\tilde{P}_n)$ and Perrin $(\tilde{R}_n)$ numbers are defined by*

$$\tilde{P}_n = P_n + \mathbf{h}P_{n+1}, \tag{7}$$

*and*

$$\tilde{R}_n = R_n + \mathbf{h}R_{n+1}, \tag{8}$$

*where $n \in \mathbb{N}, \mathbf{h}^2 = 1$ and $P_n$, $R_n$ are the Padovan and Perrin numbers, respectively.*

It can be easily shown that

$$\tilde{P}_n = \tilde{P}_{n-2} + \tilde{P}_{n-3} \tag{9}$$

and

$$\tilde{R}_n = \tilde{R}_{n-2} + \tilde{R}_{n-3}. \tag{10}$$

From the Equations (3), the $\mathbb{H}$PPN with negative subscripts are defined by

$$\tilde{P}_{-n} = -\tilde{P}_{-n+1} + \tilde{P}_{-n+3}, \tag{11}$$

and

$$\tilde{R}_{-n} = -\tilde{R}_{-n+1} + \tilde{R}_{-n+3}, \tag{12}$$

where $n \in \mathbb{N}$.

After all, we give the following Table 1. This table show that the first few $\mathbb{H}$PPN with positive and negative subscripts.

| $n$ | ... | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tilde{P}_n$ | ... | 0 | $\mathbf{h}$ | 1 | $\mathbf{h}$ | 1+$\mathbf{h}$ | 1+$\mathbf{h}$ | 1+2$\mathbf{h}$ | 2+2$\mathbf{h}$ | 2+3$\mathbf{h}$ | ... |
| $\tilde{R}_n$ | ... | -3+2$\mathbf{h}$ | 2+$\mathbf{h}$ | 1-$\mathbf{h}$ | -1+3$\mathbf{h}$ | 3 | 2$\mathbf{h}$ | 2+3$\mathbf{h}$ | 3+2$\mathbf{h}$ | 2+5$\mathbf{h}$ | ... |

**Table 1** The $\mathbb{H}$PPN with positive and negative subscripts

Now, we give the Binet formulas for the $\mathbb{H}$PPN and so find some well-known mathematical properties.

**Theorem 1.** *For any integer $n$, the Binet formulas for the $\mathbb{H}$PPN are*

$$\tilde{P}_n = a\tilde{a}\alpha^n + b\tilde{b}\beta^n + c\tilde{c}\gamma_n \tag{13}$$

*and*

$$\tilde{R}_n = \tilde{a}\alpha^n + \tilde{b}\beta^n + \tilde{c}\gamma_n, \tag{14}$$

*where $\alpha, \beta, \gamma, a, b, c$ are as the Equations (4) and $\tilde{a} = 1 + \mathbf{h}\alpha, \tilde{b} = 1 + \mathbf{h}\beta, \tilde{c} = 1 + \mathbf{h}\gamma$.*

*Proof:* It can easily established by using the Definition 1 and the Equations (4). □

The generating functions of the $\mathbb{H}$PPN are given in the following theorem.

**Theorem 2.** *For the $\mathbb{H}$PPN, we have the generating functions*

$i)$ $\sum\limits_{i=0}^{\infty} \tilde{P}_i x^i = \dfrac{1 + \mathbf{h} + (1 + \mathbf{h})x + \mathbf{h}x^2}{1 - x^2 - x^3}$,
$ii)$ $\sum\limits_{i=0}^{\infty} \tilde{R}_i x^i = \dfrac{3 + 2\mathbf{h}x + (3\mathbf{h} - 1)x^2}{1 - x^2 - x^3}$.

*Proof:*

$i)$ Let $f(x) = \sum\limits_{i=0}^{\infty} \tilde{P}_i x^i$. Then we have

$$f(x) = \tilde{P}_0 + \tilde{P}_1 x + \tilde{P}_2 x^2 + \cdots + \tilde{P}_n x^n + \cdots. \tag{15}$$

Multiplying both sides of the Equation (15) by $-x^2$ and $-x^3$, then we get

$$-x^2 f(x) = -\tilde{P}_0 x^2 - \tilde{P}_1 x^3 - \tilde{P}_2 x^4 - \cdots - \tilde{P}_n x^{n+2} - \cdots, \tag{16}$$

$$-x^3 f(x) = -\tilde{P}_0 x^3 - \tilde{P}_1 x^4 - \tilde{P}_2 x^5 - \cdots - \tilde{P}_n x^{n+3} - \cdots. \tag{17}$$

By considering the Equations (15), (16), (17) and Definition 1, it is obtained the equation

$$(1 - x^2 - x^3) f(x) = \tilde{P}_0 + x\tilde{P}_1 + x^2(\tilde{P}_2 - \tilde{P}_0),$$

as needed.
$ii)$ Similarly, we obtain equation in $ii)$.

□

Next, we present the formulas which give the summation of the $\mathbb{H}$PPN.

**Theorem 3.** *For $n \geq 0$, the following formulas are true:*

$i)$ $\sum_{r=0}^{n} \tilde{P}_r = \tilde{P}_{n+5} - 2 - 3\mathbf{h}$, $\sum_{r=0}^{n} \tilde{P}_{2r} = \tilde{P}_{2n+3} - 1 - \mathbf{h}$, $\sum_{r=0}^{n} \tilde{P}_{2r+1} = \tilde{P}_{2n+4} - 1 - 2\mathbf{h}$,
$ii)$ $\sum_{r=0}^{n} \tilde{R}_r = \tilde{R}_{n+5} - 2 - 5\mathbf{h}$, $\sum_{r=0}^{n} \tilde{R}_{2r} = \tilde{R}_{2n+3} - 2\mathbf{h}$, $\sum_{r=0}^{n} \tilde{R}_{2r+1} = \tilde{R}_{2n+4} - 2 - 3\mathbf{h}$.

*Proof:* We will establish the first part of $i)$, since the proof of the others can be done similarly with it. The proof will be contacted just the result of Theorem 1. Thus, we consider:

$$\begin{aligned}\sum_{r=0}^{n} \tilde{P}_r &= \sum_{r=0}^{n} (a\tilde{a}\alpha^r + b\tilde{b}\beta^r + c\tilde{c}\gamma^r) \\ &= a\tilde{a}\frac{\alpha^{n+1} - 1}{\alpha - 1} + b\tilde{b}\frac{\beta^{n+1} - 1}{\beta - 1} + c\tilde{c}\frac{\gamma^{n+1} - 1}{\gamma - 1}.\end{aligned}$$

At this point, by simplifying the last equality, we give first part of the equality in $i)$ as required. □

**Theorem 4.** *The relations of between the $\mathbb{H}$PPN are*

$i)$ $3\tilde{P}_{n-5} + 2\tilde{P}_{n-4} = \tilde{R}_n$,
$ii)$ $10\tilde{R}_n + 8\tilde{R}_{n-1} + \tilde{R}_{n-2} = 23\tilde{P}_n$,
$iii)$ $\tilde{P}_{n+r} = P_{r-3}(P_{n-2} + \mathbf{h}P_{n-1}) + P_{r-2}(P_n + \mathbf{h}P_{n+1}) + P_{r-1}(P_{n-1} + \mathbf{h}P_n)$,
$iv)$ $\tilde{R}_{n+r} = P_{r-3}(R_{n-2} + \mathbf{h}R_{n-1}) + P_{r-2}(R_n + \mathbf{h}R_{n+1}) + P_{r-1}(R_{n-1} + \mathbf{h}R_n)$,

 

*where* $n, r \in \mathbb{Z}^+$.

*Proof:*

$i)$ The result is obtained by using Definition 1 and Equation (5). That is, we have

$$\begin{aligned} 3\tilde{P}_{n-5} + 2\tilde{P}_{n-4} &= 3(P_{n-5} + \mathbf{h}P_{n-4}) + 2(P_{n-4} + \mathbf{h}P_{n-3}) \\ &= R_n + \mathbf{h}R_{n+1}\,, \end{aligned}$$

as required.
$ii)$ The proof can be done quite similarly as the part $i)$ by using Definition 1 and Equations (5).
$iii)$ Using the Equation (2), Definition 1 and Table 1, it was obtained
$\tilde{P}_{n+1} = P_{n+1} + \mathbf{h}P_n + \mathbf{h}P_{n-1}$,
$\tilde{P}_{n+2} = \mathbf{h}P_{n+1} + (1+\mathbf{h})P_n + P_{n-1}$,
$\tilde{P}_{n+3} = (1+\mathbf{h})P_{n+1} + (1+\mathbf{h})P_n + \mathbf{h}P_{n-1}$,
$\vdots$
$\tilde{P}_{n+r} = P_{r-3}(P_{n-2} + \mathbf{h}P_{n-1}) + P_{r-2}(P_n + \mathbf{h}P_{n+1}) + P_{r-1}(P_{n-1} + \mathbf{h}P_n)$.
$iv)$ Using the Equation (1), Definition 1 and Table 1, it was obtained
$\tilde{R}_{n+1} = R_{n+1} + \mathbf{h}R_n + \mathbf{h}R_{n-1}$,
$\tilde{R}_{n+2} = \mathbf{h}R_{n+1} + (1+\mathbf{h})R_n + R_{n-1}$,
$\tilde{R}_{n+3} = (1+\mathbf{h})R_{n+1} + (1+\mathbf{h})R_n + \mathbf{h}R_{n-1}$,
$\vdots$
$\tilde{R}_{n+r} = P_{r-3}(R_{n-2} + \mathbf{h}R_{n-1}) + P_{r-2}(R_n + \mathbf{h}R_{n+1}) + P_{r-1}(R_{n-1} + \mathbf{h}R_n)$.

□

**Theorem 5.** *For* $n \in \mathbb{N}$, *we obtain the identities relation with the* $\mathbb{H}$*PPN:*

$i)$ $\tilde{P}_n + \overline{\tilde{P}_n} = 2P_n$ *and* $\tilde{R}_n + \overline{\tilde{R}_n} = 2R_n$,
$ii)$ $\tilde{P}_n\overline{\tilde{P}_n} = -P_{n+3}P_{n-4}$ *and* $\tilde{R}_n\overline{\tilde{R}_n} = -R_{n+3}R_{n-4}$,
$iii)$ $\tilde{P}_n^2 + \tilde{P}_n\overline{\tilde{P}_n} = 2P_n\tilde{P}_n$ *and* $\tilde{R}_n^2 + \tilde{R}_n\overline{\tilde{R}_n} = 2R_n\tilde{R}_n$.

*Proof:*

$i)$ By using Definition 1, we acquire

$$\tilde{P}_n + \overline{\tilde{P}_n} = P_n + \mathbf{h}P_{n+1} + P_n - \mathbf{h}P_{n+1} = 2P_n$$

and

$$\tilde{R}_n + \overline{\tilde{R}_n} = R_n + \mathbf{h}R_{n+1} + R_n - \mathbf{h}R_{n+1} = 2R_n\,.$$

$ii)$ It is easily showed by using the Equations (1), (2) and (6).
$iii)$ By considering $i)$, we get

$$\tilde{P}_n^2 = \tilde{P}_n(2P_n - \overline{\tilde{P}_n})$$

and

$$\tilde{R}_n^2 = \tilde{R}_n(2R_n - \overline{\tilde{R}_n})\,.$$

□

## 3 Conclusion

In this paper, we defined the hyperbolic Padovan and Perrin numbers. In addition, Binet's formulas for the hyperbolic Padovan and Perrin numbers are given. Furthermore, the relationships, the generating functions and the summations of these numbers are presented.

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# Some Necessary Conditions for Rough Wijsman Convergence

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**Abstract:** Wijsman [11] gave two necessary conditions for the Wijsman convergence of a sequence of sets, depending on the elements of the limit set of this sequence. In this paper, we generalize these conditions for rough Wijsman convergence. Since we have to take into consider the points of space in three different sets, we obtain three conditions for the rough Wijsman convergence.

**Keywords:** Rough convergence, Sequences of sets, Wijsman convergence .

## 1 Introduction

Wijsman [10] introduced the famous concept of "Wijsman convergence" by using the pointwise convergence of distance functions. Many authors have worked on this concept and its generalizations. The notion of Wijsman statistical convergence was first given by Nuray and Rhoades [4]. Kişi and Nuray [3] gave the definitions of Wijsman $I$-convergence and Wijsman $I^*$-convergence by using the concept of ideal. Hazarika and Esi [2] defined the idea of asymptotically equivalent sequences of sets in the sense of ideal Wijsman convergence.

As for the concept of rough convergence, it was first introduced by Phu [7] in finite dimensional normed spaces. Defining the rough limit set, he gave some essential results related to this set. Aytar [1] examined the relation between core and rough limit set of a sequence.

The ideas of rough convergence and Wijsman convergence has been combined by many authors. In this sense, Ölmez and Aytar [5] first defined the notion of rough Wijsman convergence. Ölmez et al. [6] gave the equivalent definition of rough Wijsman convergence. Subramanian and Esi [8, 9] defined the concepts of rough Wijsman convergence and rough Wijsman statistical convergence for a triple sequence of sets, respectively.

Wijsman [11] gave two necessary conditions for the Wijsman convergence of a sequence of sets, depending on the elements of the limit set of this sequence. In this paper, we generalized these necessary conditions for rough Wijsman convergence. In this context, we gave first necessary condition as Theorem 2 in case the point belongs to the limit set. Then we proved another necessary condition as Theorem 3 in case the point does not belong to the $r$ expansion of the limit set. As the last necessary condition, we have given Theorem 4 whenever the point is between the $r$ expansion of the limit set and the limit set.

## 2 Rough Wijsman convergence

Throughout this paper, we assume that $X$ is a nonempty set and $\rho_X$ is a metric on $X$ and $A$, $A_n$ are nonempty closed subsets of $X$ for each $n \in \mathbb{N}$.

Let $r$ be a nonnegative real number. The sequence $\{x_n\}$ is said to be *rough convergent* to $x$ with the roughness degree $r$, denoted by $x_n \xrightarrow{r} x$, if for each $\varepsilon > 0$ there exists an $n(\varepsilon) \in \mathbb{N}$ such that $\rho_X(x_n, x) < r + \varepsilon$ for each $n \geq n(\varepsilon)$ [7].

The *r-limit set* of the sequence $\{x_n\}$ is denoted by

$$LIM^r x_n = \{x \in X : x_n \xrightarrow{r} x\} \text{ [7].}$$

The *distance function* $d(\cdot, A) : X \to [0, \infty)$ is defined by the formula

$$d(x, A) = \inf\{\rho_X(x, y) : y \in A\} \text{ [10].}$$

The *closed ball* with centre $a \in X$ and radius $r > 0$ is the set

$$\overline{B}(a, r) = \{x \in X : \rho_X(a, x) \leq r\}.$$

It is clear that $\overline{B}(A, r) = \{x \in X : d(x, A) \leq r\}$.

 

We say that the sequence $\{A_n\}$ is *Wijsman convergent* to the set $A$ if

$$\lim_{n\to\infty} d(x, A_n) = d(x, A) \text{ for all } x \in X.$$

In this case, we write $A_n \overset{W}{\to} A$, as $n \to \infty$ [10].

The set $A$ is called *Wijsman cluster point* of the sequence $\{A_n\}$ provided that there is a subsequence that Wijsman converges to $A$. In this case $L_{\{A_n\}}$ denotes the set of all cluster points of the sequence $\{A_n\}$.

A sequence $\{A_n\}$ is said to be $r-$*Wijsman convergent* to the set $A$ if for every $\varepsilon > 0$ and each $x \in X$ there is an $N(x, \varepsilon) \in \mathbb{N}$ such that

$$|d(x, A_n) - d(x, A)| < r + \varepsilon \text{ for all } n \geq N(x, \varepsilon)$$

and we write $d(x, A_n) \overset{r}{\to} d(x, A)$ or $A_n \overset{r-W}{\to} A$ as $n \to \infty$ [5].

Now define

$$1 - LIM^r A_n = \{A \subset X : d(x, A_n) \overset{r}{\to} d(x, A)\} \tag{1}$$

and

$$2 - LIM^r A_n = \bigcap_{c \in L_{\{A_n\}}} \overline{B}(c, r) = \{A \in X : L_{\{A_n\}} \subseteq \overline{B}(A, r)\} \text{ [6]}. \tag{2}$$

Then, for the usual rough convergence of a sequence of elements in any metric space, the sets defined by the equalities (1) and (2) coincide with each other. But these sets are not equal for an arbitrary sequence of sets. The following example compares the sets $1 - LIM^r A_n$ and $2 - LIM^r A_n$.

**Example 1.** *Let $X = \mathbb{R}^2$ and define a sequence $\{A_n\}$ as follows:*

$$A_n := \begin{cases} \left[-4, -1 - \frac{1}{n}\right] \times [-1, 1] & \text{, if } n \text{ is an odd integer} \\ \left[1 + \frac{1}{n}, 4\right] \times [-1, 1] & \text{, if } n \text{ is an even integer} \end{cases}.$$

*This sequence is not Wijsman convergent to the set $A = [-1, 1] \times [-1, 1]$ . But, this sequence is $r-$Wijsman convergent to the set $A$ for $r \geq 3$. Moreover, we have $A \in 1 - LIM^r A_n$ and*

$$\begin{aligned} 2 - LIM^r A_n &= \overline{B}\left([1, 4] \times [-1, 1], 3\right) \cap \overline{B}\left([-4, -1] \times [-1, 1], 3\right) \\ &= [-2, 2] \times [-4, 4] \end{aligned}$$

*for $r = 3$. Hence, the definition of $1 - LIM^r A_n$ does not coincide with that of $2 - LIM^r A_n$.*

Ölmez et al. [6] gave the following equivalent definition of rough Wijsman convergence.

**Proposition 1** ([6]). *A sequence $\{A_n\}$ is $r-$Wijsman convergent to the set $A$ if and only if*

$$\limsup_{n\to\infty} |d(x, A_n) - d(x, A)| \leq r$$

*for each $x \in X$.*

If a sequence of sets is Wijsman convergent, then this sequence $r$-Wijsman converges to the same set for each $r$. However, there are some sequences of sets which are $r$-Wijsman convergent, but not Wijsman convergent as can be seen in the following Example 2.

**Example 2.** *Define*

$$A_n := \left\{(x, y) \in \mathbb{R}^2 : |y| - n \leq x \leq -|y|\right\}$$

*and $A = \left\{(x, y) \in \mathbb{R}^2 : |y| \geq x\right\} \cup \{(3, 0)\}$. Firstly we show that the sequence $\{A_n\}$ is not Wijsman convergent to the set $A$. Let $\varepsilon > 0$ and $(x^*, y^*) = (5, 0) \in \mathbb{R}^2$. Since we have*

$$d((x^*, y^*), A_n) = 5 \text{ and } d((x^*, y^*), A) = 2,$$

*there exists an $n_1 = n_1\left((x^*, y^*), \varepsilon\right)$ such that*

$$\left|d((x^*, y^*), A_n) - d((x^*, y^*), A)\right| = 3 \nless \varepsilon$$

*for each $n \geq n_1$. Then we obtain $A_n \overset{W}{\nrightarrow} A$. But, this sequence is rough Wijsman convergent to the same set for $r \geq 3$.*

We will end this section recalling the Wijsman's theorem for classical convergence.

**Theorem 1** ([11]). *If the sequence $\{A_n\}$ of sets is Wijsman convergent to the set $A$ then $d(x, A_n) \to 0$ as $n \to \infty$ for every $x \in A$ and $\liminf_{n\to\infty} d(x, A_n) > 0$ for every $x \notin \overline{A}$.*

 

## 3 Necessary conditions for Rough Wijsman convergence

In this section, we will generalize the Theorem 1 given by Wijsman [11] for classical convergence to the rough convergence theory. First, we will give a necessary condition for the points that are elements of the limit set.

**Theorem 2.** *If the sequence $\{A_n\}$ is rough Wijsman convergent to the set $A$ with the roughness degree $r$ then the sequence $\{d(x, A_n)\}$ rough Wijsman converges to the function $0$ with the same roughness degree $r$ for each $x \in A$.*

*Proof:* Assume that $A_n \overset{r-W}{\to} A$. Then we have $d(x, A_n) \overset{r}{\to} d(x, A)$ for all $x \in X$. Since $A \subset X$ and $d(x, A) = 0$ for each $x \in A$, we have $d(x, A_n) \overset{r}{\to} d(x, A) = 0$. □

The second condition of the Theorem 1 for rough Wijsman convergence does not in general. In other words, as can be seen from Example 3 below, there exists a sequence $\{A_n\}$ that are rough Wijsman convergent and that does't satisfy the condition $\liminf_{n\to\infty} d(x, A_n) > r$ for all $x \notin \overline{A}$.

**Example 3.** *Define $A_n := [0, 1] \times \left\{\frac{1}{n}\right\}$ and $A = [-1, 1] \times \{(0, 0)\}$ in the space $\mathbb{R}^2$ equipped with the Euclid metric.*
*First we show that the sequence $\{A_n\}$ is not Wijsman convergent to the set $A$.*
*Let $\varepsilon > 0$ and $(x^*, y^*) \in \mathbb{R}^2$. Then we calculate*

$$d\left(\left(x^*, y^*\right), A\right) = \begin{cases} |y^*| & , \text{ if } -1 \le x^* \le 1 \text{ and } y^* \in \mathbb{R} \\ \sqrt{(x^*-1)^2 + (y^*-0)^2} & , \text{ if } x^* > 1 \text{ and } y^* \in \mathbb{R} \\ \sqrt{(x^*+1)^2 + (y^*-0)^2} & , \text{ if } x^* < -1 \text{ and } y^* \in \mathbb{R} \end{cases}.$$

*Similarly, $d\left(\left(x^*, y^*\right), A_n\right)$ can be easily calculated. Then there exists an $n_1 = n_1\left(\left(x^*, y^*\right), \varepsilon\right)$ such that it can be easily obtained*

$$\left|d\left(\left(x^*, y^*\right), A_n\right) - d\left(\left(x^*, y^*\right), A\right)\right| \le 1 + \varepsilon$$

*for each $n \ge n_1$ using the inequality $\sqrt{(x^* - x)^2 + (y^* - y)^2} \le |x^* - x| + |y^* - y|$. Hence, it is proved that $A_n \overset{r-W}{\longrightarrow} A$ for every $r \ge 1$.*

*Now if we take $\left(\frac{3}{2}, 0\right) \notin \overline{A}$, then we have*

$$d\left(\left(\frac{3}{2}, 0\right), A_n\right) = \sqrt{\left(\frac{3}{2} - 1\right)^2 + \left(0 - \frac{1}{n}\right)^2}.$$

*Since $\liminf_{n\to\infty} d\left(\left(\frac{3}{2}, 0\right), A_n\right) = \frac{1}{2} \ngtr r = 1$, it follows that the condition is not hold.*

At this point, the readers may have the following question: Is it possible to add an additional condition to the hypothesis of Theorem 1 or to modify its hypothesis so that the Theorem 1 is valid for the rough Wijsman convergence? We answered this question in the following Theorems 2 and 3.

**Theorem 3.** *If the sequence $\{A_n\}$ is rough Wijsman convergent to the set $A$ with the roughness degree $r$ then we have $\liminf_{n\to\infty} d(x, A_n) > 0$ for each $x \notin \overline{B}(A, r)$.*

*Proof:* Suppose $A_n \overset{r-W}{\to} A$. Then, for every $\varepsilon > 0$ and each $x \in X$ there exists an $N(x, \varepsilon) \in \mathbb{N}$ such that $|d(x, A_n) - d(x, A)| < r + \varepsilon$ for all $n \ge N(x, \varepsilon)$. If $x \notin \overline{B}(A, r)$, then we have $d(x, A) > r$, that is $d(x, A) - r > 0$. Take $\widetilde{\varepsilon} := \frac{d(x,A)-r}{3}$. We also have $|d(x, A_n) - d(x, A)| < r + \widetilde{\varepsilon}$ for this $\widetilde{\varepsilon}$. Then we get

$$-r - \widetilde{\varepsilon} < d(x, A_n) - d(x, A) < r + \widetilde{\varepsilon}.$$

Hence we have

$$0 < d(x, A) - r - \widetilde{\varepsilon} < d(x, A_n).$$

It is clear that

$$0 < \liminf_{n\to\infty} \left(d(x, A) - r - \widetilde{\varepsilon}\right) < \liminf_{n\to\infty} d(x, A_n).$$

Since $x$ is an arbitrary point, we have $\liminf_{n\to\infty} d(x, A_n) > 0$ for each $x \notin \overline{B}(A, r)$. □

Finally, we will end our work by giving the following necessary condition for rough Wijsman convergence.

**Theorem 4.** *If the sequence $\{A_n\}$ is rough Wijsman convergent to the set $A$ with the roughness degree $r$ then we have $d(x, A_n) \overset{2r}{\to} 0$, as $n \to \infty$ for each $x \in \overline{B}(A, r) - A$.*

*Proof:* If $x \in \overline{B}(A,r) - A$, then we have $x \in \overline{B}(A,r)$ and $x \notin A$. Thus we get $d(x,A) \leq r$. Assume that $A_n \overset{r-W}{\rightarrow} A$. Let $\varepsilon > 0$. Then, for every $x \in X$ there exists an $N(x,\varepsilon) \in \mathbb{N}$ such that $|d(x,A_n) - d(x,A)| < r + \varepsilon$ for all $n \geq N(x,\varepsilon)$. Then we have

$$-r - \varepsilon < d(x,A_n) - d(x,A) < r + \varepsilon.$$

Thus we get

$$\begin{aligned} d(x,A_n) &< d(x,A) + r + \varepsilon \\ &\leq r + r + \varepsilon \\ &= 2r + \varepsilon. \end{aligned}$$

If we apply limit superior both sides of the inequality, then we have

$$\begin{aligned} \limsup_{n\to\infty} d(x,A_n) &\leq \limsup_{n\to\infty} (2r + \varepsilon) \\ &\leq 2r + \varepsilon. \end{aligned}$$

Since $\varepsilon$ is arbitrarily, we have $\limsup_{n\to\infty} d(x,A_n) \leq 2r$. By Proposition 1, we conclude $d(x,A_n) \overset{2r}{\rightarrow} 0$. □

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# Fučik Spectrum for the $(p,q)$-Laplacian Operator

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**Abstract:** For a bounded open subset $\Omega \subset \mathbb{R}^N, N \geq 1$ with smooth boundary, we study the Fučik spectrum of the $(p,q)$-Laplacian operator through a Dirichlet boundary value problem with positive weights. Using the Ljusternick-Schnirelmann approach and the Mountain Pass theorem in a suitable manifold, the existence of eigenvalues pairs is ensured.

**Keywords:** Critical value, Fučik spectrum, Ljusternick-Schnirelmann Theorem, Mountain Pass Theorem, $(p,q) - Laplacian$.

## 1 Introduction

The objective of this paper is to investigate the existence of nontrivial solutions of the nonlinear elliptic problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda P(x)\left(u^+\right)^{p-1} - \mu Q(x)\left(u^-\right)^{q-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\Delta_r$ represents the r-Laplace operator defined by $\Delta_r u := \text{div}(|\nabla u|^{r-2}\nabla u)$ with $r \in \{p,q\}$ and $1 < q \leq p < \infty$. The weights $P, Q \in L^\infty(\Omega)$ and $u = u^+ - u^-$ is the solution of the problem (1) where $u^\pm = \max\{0, \pm u\}$.
Hereinafter, the sign $W_0^{1,p}(\Omega)$ denotes the standard Sobolev space equipped with the norm $||.||_{1,r}$ and $||.||_r$ will denote the norm in $L^r(\Omega)$.
The Fučik spectrum of the $(p,q)$-Laplacian operator with the Dirichlet boundary condition is defined as the set $\Sigma_{p,q}$ of those $(\lambda,\mu) \in \mathbb{R}^2$ such that the problem (1) has a non-trivial solution in the Sobolev space $W_0^{1,p}(\Omega)$.
The notion of Fučik spectrum was introduced for $p = 2$ in the 1970's by Fučik [5] and Dancer [2] in connection with the study of the jumping nonlinearities. The set $\Sigma_2$ itself has attracted an enormous interest among mathematicians. For the linear case, we refer to [2] where it is proved that the two lines $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$ are isolated in $\Sigma_2$. We also refer to [4] where the authors characterized and constructed variationnally the first curve in $\Sigma_2$ through $(\lambda_2, \lambda_2)$.
In the quasilinear case, only the ODE situation $(N = 1)$ seems to have been investigated in [3].

The Fučik spectrum as a notion, can be extended to nonlinear differential operators. We refer to [1] for the p-Laplacian operator where the authors have constructed the curve in $\Sigma_p$ and have shown that this is the first nonlinear curve.

This paper is devoted to study equation (1) as a constrained problem to which an appropriate min-max approach is applied to establish the existence of non trivial solutions.
We have organized this work in two sections. The next section contains some necessary preliminaries which will be often used throughout the paper. The last section contains the main result and its proof.

## 2 Preliminaries

We start this section by introducing the constrained case of the Palais-Smale condition. Let $X$ a Banach space, we consider the manifold

$$S = \{v \in X : F(v) = \alpha\}, \alpha \neq 0$$

where $F \in C^1(X,\mathbb{R})$ such that for all $v \in S, F'(v) \neq 0$.
Let now $J \in C^1(X,\mathbb{R})$ and $c \in \mathbb{R}$. We say that $J|_S$ satisfies the Palais-Smale condition (in the level c) if any sequence $(u_n, b_n) \in S \times \mathbb{R}$ such that

$$J(u_n) \to c \text{ in } \mathbb{R} \text{ and } J'(u_n) - b_n F'(u_n) \to 0 \text{ in } X'$$

contains a sub-sequence $(u_{n_k}, b_{n_k})_k$ that converges strongly in $S \times \mathbb{R}$.

Now we have the Ljusternick-Schnirelmann Theorem [6].

 

**Theorem 1.** *Suppose that $F$ and $J$ are even, that $J$ is not constant, and satisfy the Palais-Smale condition on $S$ and that $0$ does not belong to $S$. For any integer $k \geq 1$ we put:*

$$c_k = \inf_{A \in B_k} \sup_{u \in A} J(v)$$

*where*

$$B_k = \{A \in S(X); A \subset S, \gamma(A) \geq k\}$$

.

*The $S(X)$ designs the set of all closed symmetric subsets $A$ of $X$ such that $0 \notin A$. Then we have.*

*(i) For $k \geq 1$ such that $B_k \neq \emptyset$ and $c_k \in \mathbb{R}$, $c_k$ is the critical value of $j$ on $S$. Moreover $c_k \leq c_{k+1}$, and for all integer $j \geq 1$ we have $B_{k+j} \neq \emptyset$ and $c_k \leq c_{k+j} \in \mathbb{R}$, then*

$$\gamma(k(c_k)) \geq j+1$$

*where*

$$k(c_k) = \{u \in S; J(u) = c_k, \text{ there exist } \lambda \in \mathbb{R} \text{ such that } E'(u) = \lambda F'(u)\}.$$

*(ii) If for any $k \geq 1$ we have $B_k \neq \emptyset$ and $c_k \in \mathbb{R}$ then*

$$\lim_{k \to +\infty} c_k = +\infty.$$

The following theorem is called the Mountain-Pass Theorem.

**Theorem 2.** *Let $g, f \in C^1(E, \mathbb{R})$, $M = \{u \in E : g(u) = 1\}$ and $u_0, u_1 \in M$. Assume that $1$ is a regular value of $g$ and that $\| u_1 - u_0 \|_X > \epsilon$ for some $\epsilon > 0$.*
*Also assume that $f$ satisfies the Mountain-Pass geometry, that is*

$$\inf \{f(u) : u \in M et \| u - u_0 \|_X = \epsilon\} > \max \{f(u_0), f(u_1)\}$$

*and We also assume that $f$ satisfies the Palais-Smale condition on $M$.*
*Then, the quantity*

$$c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[-1,1]} f(u)$$

*where*

$$\Gamma = \{\gamma \in C([-1,1], M) : \gamma(-1) = u_0, \gamma(1) = u_1\}$$

*is a critical value of $f|_M$.*

## 3 Main results

Let $\alpha > 0, \beta > 0$ be fixed. We define the following manifold

$$M_{\alpha,\beta} = \left\{ u \in W_0^{1,p}(\Omega) : \frac{\alpha}{p} \int_\Omega P(x)|u|^p dx + \frac{\beta}{q} \int_\Omega Q(x)|u|^q dx = 1 \right\}.$$

The variational approach of problem (1) is relying on the following functionals

$$I_{s,s_0,t,t_0}, G_{\alpha,\beta} : W_0^{1,p}(\Omega) \mapsto \mathbb{R}$$

$$I_{s,s_0,t,t_0} = \frac{1}{p} \int_\Omega |\nabla u|^p dx + \frac{1}{q} \int_\Omega |\nabla u|^q dx - \frac{s}{p} \int_\Omega P(x)|u^+|^p dx + \frac{s_0}{p} \int_\Omega P(x)|u^-|^p dx$$
$$- \frac{t}{q} \int_\Omega Q(x)|u^-|^q dx + \frac{t_0}{q} \int_\Omega Q(x)|u^+|^q dx$$

and

$$G_{\alpha,\beta}(u) = \frac{\alpha}{p} \int_\Omega P(x)|u|^p dx + \frac{\beta}{q} \int_\Omega Q(x)|u|^q dx.$$

Thus, $I_{s,s_0,t,t_0}, G_{\alpha,\beta} \in C^1\left(W_0^{1,p}(\Omega), \mathbb{R}\right)$. Let us define

$$\overline{I} = I_{s,s_0,t,t_0}|_{M_{\alpha,\beta}}.$$

The set $M_{\alpha,\beta}$ is a smooth sub-manifold of $W_0^{1,p}(\Omega)$ and thus $\overline{I}$ is $C^1$. By Lagrange multipliers rule, $u \in M_{\alpha,\beta}$ is a critical point of $\overline{I}$ if and only if there exists $\lambda \in \mathbb{R}$ such that

$$\overline{I}'(u)v = \lambda G'_{\alpha,\beta}(u)v, \forall v \in W_0^{1,p}(\Omega).$$

Let us describe the relationship between the critical points of $\overline{I}$ and the Fučik spectrum of problem (1). Given $s > 0$ and $t > 0$, one has that $(\alpha c + s, \beta c + t)$ belongs to the spectrum $\Sigma_{p,q}$ if and only if there exits a critical point $u \in M_{\alpha,\beta}$ of $\overline{I}$ such that $c = \overline{I}(u)$.

 

In order to construct a critical point of $\overline{I}$, let us first check the Palais-Smale condition.

**Lemma 3.** *$\overline{I}$ satisfies the Palais-Smale condition on the sub-manifold $M_{\alpha,\beta}$.*

*Proof:* Let $\{u_n\}_{n\in\mathbb{N}} \subset M_{\alpha,\beta}$ and $\{c_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be sequences such that for some constant $K > 0$ we have

$$|I_{s,s_0,t,t_0}(u_n)| \leq K \tag{2}$$

and

$$\begin{aligned} |\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla v dx + \int_\Omega |\nabla u_n|^{q-2}\nabla u_n \nabla v dx - (\alpha c_n + s)\int_\Omega P(x)\left(u_n^+\right)^{p-1} v dx \\ -(\beta c_n + t)\int_\Omega Q(x)\left(u_n^-\right)^{q-1} v dx - (\alpha c_n - s_0)\int_\Omega P(x)\left(u_n^-\right)^{p-1} v dx \\ -(\beta c_n - t_0)\int_\Omega Q(x)\left(u_n^+\right)^{q-1} v dx| \leq \xi_n \parallel v \parallel_{W_0^{1,p}(\Omega)} \end{aligned} \tag{3}$$

for all $v \in W_0^{1,p}(\Omega)$, where $\epsilon_n \longrightarrow 0$.
From (2) it follows that the sequence $u_n$ remains bounded in $W_0^{1,p}(\Omega)$. Consequently, for a subsequence, $u_n$ converges strongly in $L^p(\Omega)$ and weakly in $W_0^{1,p}(\Omega)$. Note this limit by $u$.
In order to show that $u_n \to u$ in $W_0^{1,p}(\Omega)$ we remind that

$$-\Delta_r : W_0^{1,r}(\Omega) \longrightarrow W_0^{1,r}((\Omega))^*$$

with $r = p$ or $q$, owns the $(S_+)$ property. It is to say that if $u_n \rightharpoonup u$ in $W_0^{1,r}(\Omega)$ and $\limsup\limits_{n\to\infty} \int_\Omega |\nabla u_n|^{r-2}\nabla u_n \nabla(u_n - u) \leq 0$, then $u_n \longrightarrow u$ in $W_0^{1,r}(\Omega)$.
Putting $v = u_n - u$ in (3), we get

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u) dx + \int_\Omega |\nabla u_n|^{q-2}\nabla u_n \nabla(u_n - u) dx$$

$$= (\alpha c_n + s)\int_\Omega P(x)\left(u_n^+\right)^{p-1}(u_n - u) dx + (\beta c_n + t)\int_\Omega Q(x)\left(u_n^-\right)^{q-1}(u_n - u) dx$$

$$+(\alpha c_n - s_0)\int_\Omega P(x)\left(u_n^-\right)^{p-1}(u_n - u) dx + (\beta c_n - t_0)\int_\Omega Q(x)\left(u_n^+\right)^{q-1}(u_n - u) dx.$$

Since

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u) \underset{n\to+\infty}{\longrightarrow} 0$$

and

$$\int_\Omega |\nabla u_n|^{q-2}\nabla u_n \nabla(u_n - u) \underset{n\to+\infty}{\longrightarrow} 0$$

and according to the $(S_+)$ property we obtain that $u_n \longrightarrow u$ in $W_0^{1,p}(\Omega)$. □

In the next step, we will look for local minimizers of the functional

$$J : W_0^{1,p}(\Omega) \mapsto \mathbb{R}$$

defined by

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{q}\int_\Omega |\nabla u|^q dx$$

to fulfill the Mountain-Pass geometry of the functional $\overline{I}$.

**Lemma 4.** *For any integer $k \in \mathbb{N}$, the set*

$$B_k = \{A \in S(W_0^{1,p}(\Omega)); A \subset S, \gamma(A) \geq k\}$$

*is not empty. In particular if $X_k \subset W_0^{1,p}(\Omega)$ is a sub-space of dimension, then $\gamma(M_{\alpha,\beta} \cap X_k) = k$.*

*Proof:* Let $X_k$ a sub-space of $W_0^{1,p}(\Omega)$ such that $\dim X_k = k$. We can show easily that $(X_k \cap M_{\alpha,\beta})$ is a symmetrical and closed set that does not contain the origin, so $\gamma(M_{\alpha,\beta} \cap X_k)$ is well defined.

Let now $S$ be the unit sphere in $W_0^{1,p}(\Omega)$. Denote by $P : u \mapsto \dfrac{1}{||u||_{1,p}}u, u \neq 0$ the radial projection in $W_0^{1,p}(\Omega)$. Then $P$ is a bijection between $M_{\alpha,\beta}$ and $S$. We have

$$P(X_k \cap M_{\alpha,\beta}) = X_k \cap P(M_{\alpha,\beta}) = X_k \cap S.$$

So $P$ is a homeomorphism between $X_k \cap M_{\alpha,\beta}$ and $X_k \cap S$. Since $P$ is odd we get

$$\gamma(X_k \cap M_{\alpha,\beta}) = \gamma(X_k \cap S).$$

According to the genus properties we have

$$\gamma(X_k \cap M_{\alpha,\beta}) = k.$$

□

Similar arguments as those used in Lemma (3) show that $J$ satisfies the Palais-Smale condition on $M_{\alpha,\beta}$. Combing this fact and the previous Lemma, one can get by the Ljusternick-Schnirelmann theorem that for any $k \in \mathbb{N}$ the quantity

$$c_k := \inf_{A \in B_k} \sup_{u \in A} J(u)$$

is a critical value of the functional $J$ with respect to the manifold $M_{\alpha,\beta}$. Hence, a sequence of critical points that we note by $\{u_k^1\}_{k\in\mathbb{N}} \subset M_{\alpha,\beta}$ also exists.
Next we give the main result of the paper.

**Theorem 5.** *For $s > 0$, $t > 0$*

1.

$$c_n(s,t) = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[-1,1]} I_{s,s_0,t,t_0}(u)$$

*is a sequence of critical value of $I_{s,s_0,t,t_0}$, where*

$$\Gamma = \{\gamma \in C([-1,1], M_{\alpha,\beta}) : \gamma(-1) = -u_k^1, \gamma(1) = u_k^1\}.$$

2. *The curve $(s + c_n(s,t), t + c_n(s,t)) \in \Sigma_{p,q}$.*

*Proof:*

1. First we have: $u_k^1, (-u_k^1) \in M_{\alpha,\beta}$, then for any $\epsilon > 0$ we have

$$\| u_k^1 - \left(-u_k^1\right) \|_{1,p} = 2 \| u_k^1 \|_{1,p} > \epsilon.$$

Now we show that
$\inf\{I_{s,s_0,t,t_0}(u) : u \in M_{\alpha,\beta}, \| u - \left(-u_k^1\right) \|_{1,p} = \epsilon\} > \max\{I_{s,s_0,t,t_0}\left(-u_k^1\right), I_{s,s_0,t,t_0}\left(u_k^1\right)\}$.
Since $c_k$ is a critical value of $J$, there exists a Lagrange multiplier $\vartheta_k \in \mathbb{R}$ and $u \in M_{\alpha,\beta}$ such that

$$J'(u) = \vartheta_k G'_{\alpha,\beta}(u).$$

In other words, we have

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx + \int_\Omega |\nabla u|^{q-2} \nabla u \nabla v dx = \alpha \vartheta_k \int_\Omega P(x)|u|^{p-2} uv$$

$$+\beta \vartheta_k \int_\Omega Q(x)|u|^{q-2} uv dx.$$

Taking $u = v$ in the last equation, we get

$$\frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{q}\int_\Omega |\nabla u|^q dx = \vartheta_k (\frac{\alpha}{p}\int_\Omega P(x)|u|^p dx + \frac{\beta}{q}\int_\Omega Q(x)|u|^q dx.$$

Since $u \in M_{\alpha,\beta}$, we obtain

$$J(u) = \vartheta_k.$$

So $c_k = \vartheta_k$ and

$$\max\{I^{\sim}_{s,s_0,t,t_0}\left(-u_k^1\right), I^{\sim}_{s,s_0,t,t_0}\left(u_k^1\right)\} = c_k.$$

In the other hand, we have

$\frac{1}{p}\int_{\Omega}|\nabla u|^{p}dx+\frac{1}{q}\int_{\Omega}|\nabla u|^{q}dx<\frac{1}{p}\int_{\Omega}|\nabla u|^{p}dx+\frac{1}{q}\int_{\Omega}|\nabla u|^{q}dx-\frac{s}{p}\int_{\Omega}P\left(x\right)|u^{+}|^{p}dx+\frac{s_{0}}{p}\int_{\Omega}P\left(x\right)|u^{-}|^{p}dx-\frac{t}{q}\int_{\Omega}Q\left(x\right)|u^{-}|^{q}dx+\frac{t_{0}}{q}\int_{\Omega}Q\left(x\right)|u^{+}|^{q}dx$
for all $u\in M_{\alpha,\beta}$. Then, it results

$$\inf_{A\in B_k}\sup_{w\in A}\overline{I}(w)<\overline{I}(u)$$

wich apply that

$$\inf_{A\in B_k}\sup_{w\in A}\overline{I}(w)<\inf\overline{I}(u)$$

for any $u\in M_{\alpha,\beta}$. Consequently,

$$\inf_{A\in B_k}\sup_{w\in A}\overline{I}(w)<\inf\{\overline{I}(u);u\in M_{\alpha,\beta},\parallel u-(-u_k^1)\parallel_{1,p}=\epsilon\}$$

and this provides the following estimate
$\inf\{I\left(u\right);u\in M_{\alpha,\beta},\parallel u-\left(-u_k^1\right)\parallel_{1,p}=\epsilon\}>c_k$.

Since $\overline{I}$ verifies the Palais-Smale condition and 1 is a regular value of $G_{\alpha,\beta}$, then

$$c_n\left(s,t\right)=\inf_{\gamma\in\Gamma}\max_{u\in\gamma[-1,1]}I_{s,s_0,t,t_0}\left(u\right)$$

is a critical value of $I_{s,s_0,t,t_0}$.
2. $\left(s+c_n\left(s,t\right),t+c_n\left(s,t\right)\right)\in\Sigma_{p,q}$ if and only if there exist a critical point $u\in M_{\alpha,\beta}$ such that $c_n=I_{s,s_0,t,t_0}\left(u\right)$, and since 1 is satisfaite then the curve

$$\left(s+c_n\left(s,t\right),t+c_n\left(s,t\right)\right)\in\Sigma_{p,q}.$$

□

**Lemma 6.** *If* $c_n\left(s,t\right)=\inf_{\gamma\in\Gamma}\max_{u\in\gamma[-1,1]}I_{s,s_0,t,t_0}\left(u\right)$ *is a critical value of* $I_{s,s_0,t,t_0}$ *then* $s_0=\alpha c_n$ *and* $t_0=\beta c_n$.

*Proof:* We have $c_n=c_n\left(s,t\right)$ is a critical value of $I_{s,s_0,t,t_0}$ then

$$I'_{s,s_0,t,t_0}(u_n)=c_nG'_{\alpha,\beta}(u_n)$$

where $u_n$ is the critical point associated to $c_n$.

For any $v\in W_0^{1,p}(\Omega)$, we have

$$I'_{s,s_0,t,t_0}(u_n)v=G'_{\alpha,\beta}(u_n)v$$

that is

$$\int_{\Omega}|\nabla u_n|^{p-2}\nabla u_n\nabla vdx+\int_{\Omega}|\nabla u_n|^{q-2}\nabla u_n\nabla vdx-s\int_{\Omega}P\left(x\right)|u_n^{+}|^{p-1}vdx+s_0\int_{\Omega}P\left(x\right)|u_n^{-}|^{p-1}vdx$$
$$-t\int_{\Omega}Q\left(x\right)|u_n^{-}|^{q-1}vdx+t_0\int_{\Omega}Q\left(x\right)|u_n^{+}|^{q-1}vdx$$
$$=c_n\left(\alpha\int_{\Omega}P\left(x\right)|u_n|^{p-2}u_nvdx+\beta\int_{\Omega}Q\left(x\right)|u_n|^{q-2}u_nvdx\right).$$

Consequently

$$\int_{\Omega}|\nabla u_n|^{p-2}\nabla u_n\nabla vdx+\int_{\Omega}|\nabla u_n|^{q-2}\nabla u_n\nabla vdx$$
$$=(\alpha c_n+s)\int_{\Omega}P\left(x\right)|u_n^{+}|^{p-1}vdx+(\beta c_n+t)\int_{\Omega}Q\left(x\right)|u_n^{-}|^{q-1}vdx$$
$$+(\alpha c_n-s_0)\int_{\Omega}P\left(x\right)|u_n^{-}|^{p-1}vdx+(\beta c_n-t_0)\int_{\Omega}Q\left(x\right)|u_n^{+}|^{q-1}vdx.$$

Taking

$$(\alpha c_n-s_0)\int_{\Omega}P\left(x\right)|u_n^{-}|^{p-1}vdx+(\beta c_n-t_0)\int_{\Omega}Q\left(x\right)|u_n^{+}|^{q-1}vdx=0$$

we get as required

$$\alpha c_n=s_0 \text{ and } \beta c_n=t_0.$$

□

## 4 Conclusion

We conclude that the Fučik sepectrum of the $(p, q) - Laplacian$ operator is essentially made up by a groupe of curves $C_n$ given by

$$C_n = (s + c_n (s, t), t + c_n (s, t))$$

where $c_n$ is a sequence of critical value.

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# Global Existence and Exponential Decay of Solutions for Higher-Order Parabolic Equation with Logarithmic Nonlinearity

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**Abstract:** In this paper, we consider the inital boundary-value problem for a higher-order parabolic equation with logarithmic non-linearity. By using the potential wells method, we obtain the existence of global weak solution. In addition, we also obtain the exponential decay for the weak solutions.

**Keywords:** Higher-order parabolic equation, Global existence, Logarithmic nonlinearity.

## 1 Introduction

In this paper, we consider the global existence and decay of weak solutions for the higher-order parabolic equation with logarithmic nonlinearity

$$\begin{cases} u_t + Pu = u^{r-2}u\ln|u|, & x \in \Omega,\ t > 0, \\ D^\gamma u(x,t) = 0,\quad |\gamma| \le m-1, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

where $P = (-\Delta)^m,\ m \geq 1$ a positive integer, $\Omega$ is a bound domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)$ is multi-index, $\gamma_i$ $(i = 1, 2, ..., n)$ are positive integers, $|\gamma| = \gamma_1 + \gamma_2 + ... + \gamma_n$, $D^\gamma = \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1}\partial x_2^{\gamma_2}...\partial x_n^{\gamma_n}}$ are derivative operator, $\Delta = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and

$$\begin{cases} 2 \le r \le +\infty, & n = 1, 2, \\ 2 \le r \le \frac{2n}{n-2}, & n \geq 3. \end{cases}$$

When $m = 1$, the equation (1) becomes a heat equation as follows

$$u_t - \Delta u = u^{r-2}u\ln|u|,$$

where $2 \le r$, was considered by many authors ([1], [3], [8]). In the case of $r = 2$, Chen et al. [1] obtained under some suitable conditions for the existence of global, decay estimate and blow-up at $+\infty$ of weak solutions, by using the logarithmic Sobolev inequality and potential well method. In the case of $2 < k$, Peng and Zhou [8] studied the existence of the unique global weak solutions and blow-up in the finite time of weak solutions, by using potential well method and energy method.

When $m = 2$, Li and Liu [5] considered the following equation

$$u_t + \Delta^2 u = u^{r-2}u\ln|u|,$$

where $2 < r$. They studied the existence of global solutions, by using potential well technique. In addition, they also studied result of decay and blow-up in the finite time for weak solutions.

Recently some authors studied higher-order parabolic and hyperbolic type equation ([2], [4], [9], [11], [12]).

This paper is organized as follows. In Section2, some important Lemmas are given. In Section 3, the main result is proved.

## 2 Preliminaries

In this section, we material needed for proving the main result is introduced. Let $\|.\|_r$ and $\|.\|$ denote the usual $L^r(\Omega)$ norm and $L^2(\Omega)$ norm.

For $u \in H_0^m(\Omega)\backslash\{0\}$, we define

$$J(u) = \frac{1}{2}\left\|P^{\frac{1}{2}}u\right\|^2 - \frac{1}{r}\int_\Omega |u|^r \ln|u|\,dx + \frac{1}{r^2}\|u\|_r^r, \tag{2}$$

$$I(u) = \left\|P^{\frac{1}{2}}u\right\|^2 - \int_\Omega |u|^r \ln|u|\,dx, \tag{3}$$

$$J(u) = \frac{1}{r}I(u) + \left(\frac{1}{2} - \frac{1}{r}\right)\left\|P^{\frac{1}{2}}u\right\|^2 + \frac{1}{r^2}\|u\|_r^r. \tag{4}$$

Further, we define the potential depth by

$$d = \inf_{u\in\mathcal{N}} J(u), \tag{5}$$

$$\mathcal{N} = \{u \in H_0^m(\Omega)\backslash\{0\} : I(u) = 0\}.$$

**Lemma 1.** *Let $k$ be a number with $2 \le k < +\infty$, $n \le 2m$ and $2 \le k \le \frac{2n}{n-2m}$, $n > 2m$. Then there is a constant $C$ depending*

$$\|u\|_k \le C\left\|P^{\frac{1}{2}}u\right\|,\ \ \forall u \in H_0^m(\Omega).$$

**Lemma 2.** *Let $u \in H_0^m(\Omega)\backslash\{0\}$. we consider the function $j : \lambda \to J(\lambda u)$ for $\lambda > 0$. Then we get*
*(i) $\lim_{\lambda\to0^+} j(\lambda) = 0$ and $\lim_{\lambda\to+\infty} j(\lambda) = -\infty$;*
*(ii) there is a unique $\lambda^* > 0$ such that $j'(\lambda^*) = 0$;*
*(iii) $j(\lambda)$ is increasing on $(0,\lambda^*)$, decreasing on $(\lambda^*,+\infty)$ and taking the maximum at $\lambda^*$;*
*(iv) $I(\lambda u) > 0$ for $\lambda \in (0,\lambda^*)$, $I(\lambda u) < 0$ for $\lambda \in (\lambda^*,+\infty)$ and $I(\lambda^* u) = 0$.*

*Proof:* For $u \in H_0^1(\Omega)\backslash\{0\}$, by the definition of $j$, we get

$$j(\lambda) = \frac{\lambda^2}{2}\left\|P^{\frac{1}{2}}u\right\|^2 - \frac{\lambda^r}{r}\int_\Omega |u|^r \ln|u|\,dx - \frac{\lambda^r}{r}\ln\lambda\,\|u\|_r^r + \frac{\lambda^r}{r^2}\|u\|_r^r. \tag{6}$$

We see that *(i)* holds due to $\int_\Omega |u|^r\,dx \ne 0$. We have

$$\frac{d}{d\lambda}j(\lambda) = \lambda\left(\left\|P^{\frac{1}{2}}u\right\|^2 - \lambda^{r-2}\int_\Omega |u|^r \ln|u|\,dx - \lambda^{r-2}\ln\lambda\,\|u\|_r^r\right).$$

Let $\phi(\lambda) = \lambda^{-1}j'(\lambda)$, thus we obtain

$$\phi(\lambda) = \left\|P^{\frac{1}{2}}u\right\|^2 - \lambda^{r-2}\int_\Omega |u|^r \ln|u|\,dx - \lambda^{r-2}\ln\lambda\,\|u\|_r^r.$$

Then

$$\phi'(\lambda) = -(r-2)\lambda^{r-3}\int_\Omega |u|^r \ln|u|\,dx - (r-2)\lambda^{r-3}\ln\lambda\,\|u\|_r^r - \lambda^{r-3}\|u\|_r^r,$$

which yields that there exists a $\lambda^* > 0$ such that $\phi'(\lambda) > 0$ on $(0,\lambda^*)$, $\varphi'(\lambda) < 0$ on $(\lambda^*,+\infty)$ and $\varphi'(\lambda) = 0$. So, $\phi(\lambda)$ is increasing on $(0,\lambda^*)$, decreasing on $(\lambda^*,+\infty)$. Since $\lim_{\lambda\to0^+}\phi(\lambda) = \left\|P^{\frac{1}{2}}u\right\|^2 > 0$, $\lim_{\lambda\to+\infty}\phi(\lambda) = -\infty$, there exists a unique $\lambda^* > 0$ such that $\varphi(\lambda^*) = 0$, i.e., $j'(\lambda^*) = 0$. So, *(ii)* holds. Then, $j'(\lambda) = \lambda\phi(\lambda)$ is positive on $(0,\lambda^*)$, negative on $(\lambda^*,+\infty)$. Thus, $j(\lambda)$ is increasing on $(0,\lambda^*)$, decreasing on $(\lambda^*,+\infty)$ and taking the maximum at $\lambda^*$. So, *(iii)* holds. By (3), we get

$$\begin{aligned} I(\lambda u) &= \lambda^2\left\|P^{\frac{1}{2}}u\right\|^2 - \lambda^r\int_\Omega |u|^r \ln|u|\,dx - \lambda^r\ln\lambda\,\|u\|_r^r \\ &= \lambda j'(\lambda). \end{aligned}$$

Thus, $I(\lambda u) > 0$ for $\lambda \in (0,\lambda^*)$, $I(\lambda u) < 0$ for $\lambda \in (\lambda^*,+\infty)$ and $I(\lambda^* u) = 0$. So, *(iv)* holds. Therefore, the proof is completed. □

**Lemma 3.** *([6]) Let $g : R^+ \to R^+$ be a nonincreasing function and $\sigma$ is a nonnegative constant such that*

$$\int_t^{+\infty} g^{1+\sigma}(s)ds \le \frac{1}{\omega}g^\sigma(0)g(t),\ \ \textit{for all } t \ge 0.$$

*Hence*
*(a) $g(t) \le g(0)e^{1-\omega t}$, $\forall t \ge 0$, whenever $\sigma = 0$,*
*(b) $g(t) \le g(0)\left(\frac{1+\sigma}{1+\omega\sigma t}\right)^{\frac{1}{\sigma}}$, $\forall t \ge 0$, whenever $\sigma > 0$.*

 

## 3 Main results

Now as in ([7]), we consider the following notations:

$$\begin{aligned}
\mathcal{W}_1 &= \{u \in H_0^m(\Omega)\backslash\{0\} : J(u) < d\},\ \ \mathcal{W}_2 = \{u \in H_0^m(\Omega)\backslash\{0\} : J(u) = d\},\ \ \mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2, \\
\mathcal{W}_1^+ &= \{u \in \mathcal{W}_1 : I(u) > 0\},\ \ \mathcal{W}_2^+ = \{u \in \mathcal{W}_2 : I(u) > 0\},\ \ \mathcal{W}^+ = \mathcal{W}_1^+ \cup \mathcal{W}_2^+, \\
\mathcal{W}_1^- &= \{u \in \mathcal{W}_1 : I(u) < 0\},\ \ \mathcal{W}_2^- = \{u \in \mathcal{W}_2 : I(u) < 0\},\ \ \mathcal{W}^- = \mathcal{W}_1^- \cup \mathcal{W}_2^-.
\end{aligned}$$

We call the set $\mathcal{W}$ the potential well and $d$ the depth of $\mathcal{W}$.

**Definition 1.** *(Maximal Existence Time). Suppose that $u(t)$ be weak solutions of problem (1). We define the maximal existence time $T_{\max}$ as follows*

$$T_{\max} = \sup\{T > 0 : u(t) \text{ exists on } [0, T]\}.$$

*Then*
*(i) If $T_{\max} < \infty$, we say that $u(t)$ blows up in finite time and $T_{\max}$ is the blow-up time;*
*(ii) If $T_{\max} = \infty$, we say that $u(t)$ is global.*

**Theorem 1.** *(Global Existence). Let $u_0 \in \mathcal{W}_1^+$. Then the problem of (1) admits a unique global weak solution such that*

$$u(t) \in \mathcal{W}_1^+,\ \ 0 \le t < +\infty,$$

*and the energy estimate*

$$\int_0^t \|u_s(s)\|^2\, ds + J(u(t)) \le J(u_0),\quad 0 \le t < +\infty.$$

*Also, the solution decay exponential provided $u_0 \in \mathcal{W}_1^+$.*

*Proof:* The Faedo-Galerkin's methods is used. In the space $H_0^m(\Omega)$, we take a bases $\{w_j\}_{j=1}^{\infty}$ and define the finite orthogonal space

$$V_s = span\{w_1, w_2, ..., w_s\}.$$

Let $u_{0s}$ be an element of $V_s$ such that

$$u_{0s} = \sum_{j=1}^{s} a_{sj} w_j \to u_0,\ \text{ in } H_0^m(\Omega), \tag{7}$$

as $s \to \infty$. We find the approximate solution $u_s(x,t)$ of the problem (1) in the form

$$u_s(x,t) = \sum_{j=1}^{s} a_{sj}(t) w_j(x), \tag{8}$$

where the coefficients $a_{sj}$ $(1 \le j \le s)$ satisfy the ordinary differential equations

$$\int_\Omega u_{st} w_i dx + \int_\Omega P u_s w_i dx = \int_\Omega |u_s|^{r-2} u_s \ln|u_s|\, w_i dx, \tag{9}$$

for $i \in \{1, 2, ..., s\}$, with the initial condition

$$a_{sj}(0) = a_{sj},\quad j \in \{1, 2, ..., s\}. \tag{10}$$

We multiply both sides of (9) by $a'_{si}$, summing over $i$ from 1 to $s$ and integrating with respect to time variable on $[0,t]$, we get

$$\int_0^t \|u_{s\tau}(\tau)\|^2\, ds + J(u_s(t)) \le J(u_{0s}),\quad 0 \le t \le T_{\max}. \tag{11}$$

where $T_{\max}$ is the maximal existence time of solution $u_s(t)$. We shall prove that $T_{\max} = +\infty$. By (7), (11) and the continuity of $J$, we obtain

$$J(u_s(0)) \to J(u_{0s}),\ \text{as } s \to \infty. \tag{12}$$

Thanks to $J(u_0) < d$ and the continuity of functional $J$, it follows from (12) that

$$J(u_{0s}) < d,\quad \text{for sufficiently large } m.$$

And therefore, from (11), we get

$$\int_0^t \|u_{s\tau}(\tau)\|^2\, ds + J(u_s(t)) < d,\quad 0 \le t \le T_{\max}, \tag{13}$$

for sufficiently large $s$. Next, we will study

$$u_s(t) \in \mathcal{W}_1^+, \quad t \in [0, T_{\max}), \tag{14}$$

for sufficiently large $s$. We suppose that (14) does not hold and think that there exists a smallest time $t_0$ such that $u_s(t_0) \notin \mathcal{W}_1^+$. Then, by continuity of $u_s(t_0) \in \partial\mathcal{W}_1^+$. So, we get

$$J(u_s(t_0)) = d \text{ and } I(u_s(t_0)) = 0. \tag{15}$$

Nevertheless, we see that (15) could not occur by (13) while if (15) holds then, by definition of $d$, we get

$$J(u_s(t_0)) \geq \inf_{u\in\mathcal{N}} J(u) = d,$$

which also contradicts with (13). Moreover, we have (14), i.e., $J(u_s(t)) < d$, and $I(u_s(t)) > 0$, for any $t \in [0, T_{\max})$, for sufficiently large $s$. Then, by (4), we obtain

$$\begin{aligned} d &> J(u_s(t)) \\ &\geq \left(\frac{1}{2} - \frac{1}{r}\right) \left\|P^{\frac{1}{2}} u_s(t)\right\|^2 + \frac{1}{r^2} \left\|u_s(t)\right\|_r^r, \end{aligned}$$

which gives

$$\|u_s(t)\|_r^r < r^2 d \text{ and } \left\|P^{\frac{1}{2}} u_s(t)\right\|^2 < \frac{2r}{r-2} d. \tag{16}$$

Since $u_s(x,t) \in \mathcal{W}_1^+$ for $s$ large enough, it follows from (4) that $J(u_s) \geq 0$ for $s$ large enough. So, by (13) it follows for $s$ large enough

$$\int_0^t \|u_{s\tau}(\tau)\|^2 \, ds < d. \tag{17}$$

By (16), we know that $T_{\max} = +\infty$. It follows from (16) and (17) that there exist a function $u \in H_0^m(\Omega)$ and a subsequence of $\{u_s\}_{j=1}^\infty$ still denoted by $\{u_s\}_{j=1}^\infty$ such that

$$u_s \to u \text{ weakly star in } L^\infty(0,\infty; H_0^m(\Omega)), \tag{18}$$

$$u_{st} \to u_t \text{ weakly in } L^2(0,\infty; L^2(\Omega)). \tag{19}$$

By (18), (19) and Aubin-Lions compactness theorem, we obtain

$$u_s \to u \text{ strongly in } C([0,+\infty]; L^2(\Omega)).$$

This yields that

$$|u_s|^{r-2} u_s \ln|u_s| \to |u|^{r-2} u \ln|u| \text{ a.e. } (x,t) \in \Omega \times (0,+\infty). \tag{20}$$

By (16), we have

$$\int_\Omega \left(|u_s(t)|^{r-1} \ln|u_s(t)|\right)^{\frac{2r}{2r-1}} dx \leq C_d := \left[e\,(r-1)\right]^{-\frac{2r}{2r-1}} |\Omega| + 2^{\frac{2r}{2r-1}} r^2 d. \tag{21}$$

Hence, it follows from (20) and (21) that

$$|u_s|^{r-2} u_s \ln|u_s| \to |u|^{r-2} u \ln|u| \text{ weakly star in } L^\infty(0,+\infty; L^{\frac{2r}{2r-1}}(\Omega)).$$

Then integrating (9) respect to $t$ for $0 \leq t < \infty$, we obtain

$$< u_t, w_i > + < P^{\frac{1}{2}} u, P^{\frac{1}{2}} w_i > = < |u|^{r-2} u \ln|u|, w_i > .$$

On the other hand, there exists a global weak solution $u_0 \in \mathcal{W}_1^+$ of the problem (1).

*Decay estimates*

 

Thanks to $u_0 \in \mathcal{W}_1^+$, we deduce from (4) that

$$\begin{aligned} J(u_0) &> J(u(t)) \\ &\geq \left(\frac{1}{2}-\frac{1}{r}\right)\left\|P^{\frac{1}{2}}u(t)\right\|^2+\frac{1}{r^2}\|u(t)\|_r^r. \end{aligned} \tag{22}$$

From Lemma 2, (5) and $I(u(t)) > 0$, there exists a $\lambda^* > 1$ such that $I(\lambda^* u(t)) = 0$. We get

$$\begin{aligned} d &\leq J(\lambda^* u(t)) \\ &\leq (\lambda^*)^r\left(\left(\frac{1}{2}-\frac{1}{r}\right)\left\|P^{\frac{1}{2}}u(t)\right\|^2+\frac{1}{r^2}\|u(t)\|_r^r\right). \end{aligned} \tag{23}$$

Using (22) and (23), we get

$$d \leq (\lambda^*)^r J(u_0),$$

which yields that

$$\lambda^* \geq \left(\frac{d}{J(u_0)}\right)^{\frac{1}{r}}. \tag{24}$$

By (3), we get

$$\begin{aligned} 0 &= I(\lambda^* u(t)) \\ &= (\lambda^*)^r I(u(t)) + \left[(\lambda^*)^2-(\lambda^*)^r\right]\left\|P^{\frac{1}{2}}u(t)\right\|^2-(\lambda^*)^r\ln(\lambda^*)\|u(t)\|_r^r. \end{aligned} \tag{25}$$

Using (24) and (25), we get

$$\begin{aligned} (\lambda^*)^r I(u(t)) &= \left[(\lambda^*)^r-(\lambda^*)^2\right]\left\|P^{\frac{1}{2}}u(t)\right\|^2+(\lambda^*)^r\ln(\lambda^*)\|u(t)\|_r^r \\ &\geq \left[(\lambda^*)^r-(\lambda^*)^2\right]\left\|P^{\frac{1}{2}}u(t)\right\|^2. \end{aligned} \tag{26}$$

It follows from (24), (26) and Lemma 1 that

$$\begin{aligned} I(u(t)) &\geq \left[1-\left(\frac{d}{J(u_0)}\right)^{\frac{2-r}{r}}\right]\left\|P^{\frac{1}{2}}u(t)\right\|^2 \\ &\geq C_1\|u(t)\|^2, \end{aligned} \tag{27}$$

where $C_1$ is constant. Integrating the $I(u(\tau))$ with respect to $\tau$ over $(t, T)$, we obtain

$$\begin{aligned} \int_t^T I(u(\tau))d\tau &= -\int_t^T\int_\Omega u_\tau(\tau)u(\tau)dxd\tau \\ &= \frac{1}{2}\|u(t)\|^2-\frac{1}{2}\|u(T)\|^2 \\ &\leq \frac{C_2}{2}\|u(t)\|^2, \end{aligned} \tag{28}$$

where $C_2$ is constant. From (27) and (28), we have

$$\int_t^T C_1\|u(t)\|^2 ds \leq \frac{C_2}{2}\|u(t)\|^2, \text{ for all } t \in [0, T]. \tag{29}$$

Let $T \to +\infty$ in (29), we can have

$$\int_t^\infty \|u(t)\|^2 ds \leq C_3\|u(t)\|^2,$$

where $C_3 = \frac{C_2}{2C_1}$. By Lemma 3, we have

$$\|u(t)\|^2 \leq \|u(0)\|^2 e^{1-\frac{t}{C_3}}, \ \ t \in [0, \infty).$$

□

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# Invariant Summability and Invariant Statistical Convergence of Order $\eta$ for Double Set Sequences

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**Abstract:** In this study, for double set sequences, we introduce the notions of invariant summability and invariant statistical convergence of order $\eta$ $(0 < \eta \leq 1)$ in the Wijsman sense. Also, we investigate some properties of these new notions and the relations between them.

**Keywords:** Convergence in the Wijsman convergence, Double set sequences, Invariant summability, Order $\eta$, Statistical convergence.

## 1 Introduction

Long after the notion of convergence for double sequences was introduced by Pringsheim [1], using the notions of invariant mean, statistical convergence etc., this notion was extended to new convergence notions by some authors [2, 3]. Recently, on the notions of statistical convergence of order $\alpha$ and strongly $p$-Cesàro summability of order $\alpha$ for double sequences were studied by Çolak and Altın [4] and Savaş [5].

Over the years, on the various convergence notions for set sequences have been studied by many authors. One of them, discussed in this study, is the notion of convergence in the Wijsman sense [6]-[8]. Using the notions of invariant mean, statistical convergence etc., this notion was extended to new convergence notions for double set sequences by some authors [9]-[11]. In [10], Nuray and Ulusu studied on the notions of invariant summability and invariant statistical convergence in the Wijsman sense for double set sequences. Recently, on the notions of statistical convergence of order $\alpha$ and strongly $p$-Cesàro summability of order $\alpha$ for double set sequences were studied by Ulusu and Gülle [12].

In this study, using the notions of invariant mean and order $\eta$, we studied on new convergence notions in the Wijsman sense for double set sequences.

More information on the notions of convergence for real or set sequences can be found in [14]-[17].

## 2 Definitions and notations

In this section, let us remind the basic notions that need for a better understanding of our study (see, [7], [9]-[12], [17]).

For a metric space $(Y, d)$, $\mu(y, C)$ denote the distance from $y$ to $C$ where

$$\mu(y, C) := \mu_y(C) = \inf_{c \in C} d(y, c)$$

for any $y \in Y$ and any non-empty set $C \subseteq Y$.

For a non-empty set $Y$, let a function $g : \mathbb{N} \to P_Y$ (the power set of $Y$) is defined by $g(m) = C_m \in P_Y$ for each $m \in \mathbb{N}$. Then, the sequence $\{C_m\} = \{C_1, C_2, \ldots\}$, which is the codomain elements of $g$, is called set sequences.

Throughout this study, $(Y, d)$ will be considered as a metric space and $C, C_{mn}$ will be considered as any non-empty closed subsets of $Y$.

The double set sequence $\{C_{mn}\}$ is called convergent to the set $C$ in the Wijsman sense if each $y \in Y$,

$$\lim_{m,n \to \infty} \mu_y(C_{mn}) = \mu_y(C).$$

The double set sequence $\{C_{mn}\}$ is called statistically convergent to the set $C$ in the Wijsman sense if every $\varepsilon > 0$ and each $y \in Y$,

$$\lim_{p,q \to \infty} \frac{1}{pq} \left| \left\{ (m, n) : \; m \leq p, n \leq q, \; \left| \mu_y(C_{mn}) - \mu_y(C) \right| \geq \varepsilon \right\} \right| = 0.$$

Let $\sigma$ be a mapping such that $\sigma : \mathbb{N}^+ \to \mathbb{N}^+$ (the set of positive integers). A continuous linear functional $\psi$ on $\ell_\infty$ is called an invariant mean (or a $\sigma$-mean) if it satisfies the following conditions:

 

1. $\psi(x_s) \geq 0$, when the sequence $(x_s)$ has $x_s \geq 0$ for all $s$,
2. $\psi(e) = 1$, where $e = (1, 1, 1, \ldots)$ and
3. $\psi(x_{\sigma(s)}) = \psi(x_s)$ for all $(x_s) \in \ell_\infty$.

The mappings $\sigma$ are assumed to be one to one and such that $\sigma^m(s) \neq s$ for all $m, s \in \mathbb{N}^+$, where $\sigma^m(s)$ denotes the $m$ th iterate of the mapping $\sigma$ at $s$. Thus $\psi$ extends the limit functional on $c$, in the sense that $\psi(x_s) = \lim x_s$ for all $(x_s) \in c$.

The double set sequence $\{C_{mn}\}$ is called invariant summable to the set $C$ in the Wijsman sense if each $y \in Y$,

$$\lim_{p,q\to\infty} \frac{1}{pq} \sum_{m,n=1,1}^{p,q} \mu_y(C_{\sigma^m(s)\sigma^n(t)}) = \mu_y(C)$$

uniformly in $s, t$.

The double set sequence $\{C_{mn}\}$ is called strongly invariant summable to the set $C$ in the Wijsman sense if each $y \in Y$,

$$\lim_{p,q\to\infty} \frac{1}{pq} \sum_{m,n=1,1}^{p,q} \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| = 0$$

uniformly in $s, t$.

## 3 Main results

In this section, for double set sequences, we introduce the notions of invariant summability and invariant statistical convergence of order $\eta$ $(0 < \eta \leq 1)$ in the Wijsman sense. Also, we investigate some properties of these new notions and the relations between them.

**Definition 1.** *The double set sequence $\{C_{mn}\}$ is invariant summable of order $\eta$ to the set $C$ in the Wijsman sense if each $y \in Y$,*

$$\lim_{p,q\to\infty} \frac{1}{(pq)^\eta} \sum_{m,n=1,1}^{p,q} \mu_y(C_{\sigma^m(s)\sigma^n(t)}) = \mu_y(C)$$

*uniformly in $s, t$ where $0 < \eta \leq 1$ and we denote this in $C_{mn} \overset{W_2(V_\sigma^\eta)}{\longrightarrow} C$ format.*

**Definition 2.** *The double set sequence $\{C_{mn}\}$ is strongly $r$-invariant summable of order $\eta$ to the set $C$ in the Wijsman sense if each $y \in Y$,*

$$\lim_{p,q\to\infty} \frac{1}{(pq)^\eta} \sum_{m,n=1,1}^{p,q} \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right|^r = 0$$

*uniformly in $s, t$ where $0 < \eta \leq 1$ and $0 < r < \infty$, and we denote this in $C_{mn} \overset{W_2[V_\sigma^\eta]^r}{\longrightarrow} C$ format.*

*If $r = 1$, then the double set sequence $\{C_{mn}\}$ is simply called strongly invariant summable of order $\eta$ to the set $C$ in the Wijsman sense and we denote this in $C_{mn} \overset{W_2[V_\sigma^\eta]}{\longrightarrow} C$ format.*

**Example 1.** *Let $Y = \mathbb{R}^2$ and a double set sequence $\{C_{mn}\}$ be defined as following:*

$$C_{mn} := \begin{cases} \left\{(a, b) \in \mathbb{R}^2 : (a - m)^2 + (b - n)^2 = 1\right\} & , \text{ if } m \text{ and } n \text{ are square integers}, \\ \{(0, 0)\} & , \text{ otherwise.} \end{cases}$$

*In this case, the double set sequence $\{C_{mn}\}$ is strongly invariant summable of order $\eta$ $(0 < \eta \leq 1)$ to the set $C = \{(0, 0)\}$ in the Wijsman sense.*

**Remark 1.** *For $\eta = 1$, respectively, the notions of $W_2(V_\sigma^\eta)$-summability and $W_2[V_\sigma^\eta]$-summability coincides with the notions of invariant summability and strongly invariant summability in the Wijsman sense for double set sequences in [10].*

**Theorem 1.** *If $0 < \eta \leq \vartheta \leq 1$, then*

$$C_{mn} \overset{W_2[V_\sigma^\eta]^r}{\longrightarrow} C \Rightarrow C_{mn} \overset{W_2[V_\sigma^\vartheta]^r}{\longrightarrow} C.$$

*Proof:* Let $0 < \eta \leq \vartheta \leq 1$ and suppose that $C_{mn} \overset{W_2[V_\sigma^\eta]^r}{\longrightarrow} C$. For each $y \in Y$, we have

$$\frac{1}{(mn)^\vartheta} \sum_{m,n=1,1}^{p,q} \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right|^r \leq \frac{1}{(mn)^\eta} \sum_{m,n=1,1}^{p,q} \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right|^r$$

for all $s, t$. Hence, by our assumption, we get $C_{mn} \overset{W_2[V_\sigma^\vartheta]^r}{\longrightarrow} C$. □

 

If $\vartheta = 1$ is taken in Theorem 1, then the following corollary is obtained.

**Corollary 1.** *Let $\eta \in (0, 1]$ and $0 < r < \infty$. If a double set sequence $\{C_{mn}\}$ is strongly $r$-invariant summable of order $\eta$ to a set $C$ in the Wijsman sense, then the double set sequence is strongly $r$-invariant summable to the set $C$ in the Wijsman sense, i.e.,*

$$C_{mn} \overset{W_2[V_\sigma^\eta]^r}{\longrightarrow} C \Rightarrow C_{mn} \overset{W_2[V_\sigma]^r}{\longrightarrow} C.$$

Now, we can state a theorem giving the relationship between $W_2[V_\sigma^\eta]^r$-summability and $W_2[V_\sigma^\eta]^u$-summability, where $0 < \eta \leq 1$ and $0 < r < u < \infty$.

**Theorem 2.** *Let $0 < \eta \leq 1$ and $0 < r < u < \infty$. Then,*

$$C_{mn} \overset{W_2[V_\sigma^\eta]^u}{\longrightarrow} C \Rightarrow C_{mn} \overset{W_2[V_\sigma^\eta]^r}{\longrightarrow} C.$$

*Proof:* Let $0 < \eta \leq 1$ and $0 < r < u < \infty$. Also, we suppose that $C_{mn} \overset{W_2[V_\sigma^\eta]^u}{\longrightarrow} C$. For each $y \in Y$, by Hölder inequality, we have

$$\frac{1}{(pq)^\eta} \sum_{m,n=1,1}^{p,q} \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right|^r < \frac{1}{(pq)^\eta} \sum_{m,n=1,1}^{p,q} \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right|^u$$

for all $s, t$. Hence, by our assumption, we get $C_{mn} \overset{W_2[V_\sigma^\eta]^r}{\longrightarrow} C$. □

**Definition 3.** *The double set sequence $\{C_{mn}\}$ is invariant statistically convergent of order $\eta$ to the set $C$ in the Wijsman sense if every $\varepsilon > 0$ and each $y \in Y$,*

$$\lim_{p,q\to\infty} \frac{1}{(pq)^\eta} \left| \left\{ (m,n) : \; m \leq p, n \leq q, \; \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \varepsilon \right\} \right| = 0$$

*uniformly in $s, t$ where $0 < \eta \leq 1$ and we denote this in $C_{mn} \overset{W_2(S_\sigma^\eta)}{\longrightarrow} C$ format.*

**Example 2.** *Let $Y = \mathbb{R}^2$ and a double set sequence $\{C_{mn}\}$ be defined as following:*

$$C_{mn} := \begin{cases} \left\{(a,b) \in \mathbb{R}^2 : (a)^2 + (b-1)^2 = \dfrac{1}{mn}\right\} & , \text{ if } m \text{ and } n \text{ are square integers,} \\ \{(1,0)\} & , \text{ otherwise.} \end{cases}$$

*In this case, the double set sequence $\{C_{mn}\}$ is invariant statistically convergent of order $\eta$ ($0 < \eta \leq 1$) to the set $C = \{(1, 0)\}$ in the Wijsman sense.*

**Remark 2.** *For $\eta = 1$, the notion of $W_2(S_\sigma^\eta)$-convergence coincides with the notion of invariant statistical convergence in the Wijsman sense for double set sequences in [10].*

**Theorem 3.** *If $0 < \eta \leq \vartheta \leq 1$, then*

$$C_{mn} \overset{W_2(S_\sigma^\eta)}{\longrightarrow} C \Rightarrow C_{mn} \overset{W_2(S_\sigma^\vartheta)}{\longrightarrow} C.$$

*Proof:* Let $0 < \eta \leq \vartheta \leq 1$ and suppose that $C_{mn} \overset{W_2(S_\sigma^\eta)}{\longrightarrow} C$. For every $\varepsilon > 0$ and each $y \in Y$, we have

$$\lim_{p,q\to\infty} \frac{1}{(pq)^\vartheta} \left| \left\{ (m,n) : \; m \leq p, n \leq q, \; \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \varepsilon \right\} \right|$$

$$\leq \lim_{p,q\to\infty} \frac{1}{(pq)^\eta} \left| \left\{ (m,n) : \; m \leq p, n \leq q, \; \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \varepsilon \right\} \right|.$$

for all $s, t$. Hence, by our assumption, we get $C_{mn} \overset{W_2(S_\sigma^\vartheta)}{\longrightarrow} C$. □

If $\eta = 1$ is taken in Theorem 3, then the following corollary is obtained.

**Corollary 2.** *Let $\eta \in (0, 1]$. If a double set sequence $\{C_{mn}\}$ is invariant statistically convergent of order $\eta$ to a set $C$ in the Wijsman sense, then the double set sequence is invariant statistically convergent to the set $C$ in the Wijsman sense, i.e.,*

$$C_{mn} \overset{W_2(S_\sigma^\eta)}{\longrightarrow} C \Rightarrow C_{mn} \overset{W_2(S_\sigma)}{\longrightarrow} C.$$

**Theorem 4.** *If a double set sequence $\{C_{mn}\}$ is strongly $r$-invariant summable of order $\eta$ to a set $C$ in the Wijsman sense, then the double set sequence is invariant statistically convergent of order $\vartheta$ to the set $C$ in the Wijsman sense, where $0 < \eta \leq \vartheta \leq 1$ and $0 < r < \infty$.*

*Proof:* Let $0 < \eta \leq \vartheta \leq 1$ and $0 < r < \infty$. Also, we suppose that a double set sequence $\{C_{mn}\}$ is strongly $r$-invariant summable of order $\eta$ to a set $C$ in the Wijsman sense. For every $\varepsilon > 0$ and each $y \in Y$, we have

$$\begin{aligned}
\sum_{m,n=1,1}^{p,q} \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right|^r &= \sum_{\substack{m,n=1,1 \\ \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)})-\mu_y(C)\right|\geq\varepsilon}}^{p,q} \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right|^r \\
&\quad + \sum_{\substack{m,n=1,1 \\ \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)})-\mu_y(C)\right|<\varepsilon}}^{p,q} \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right|^r \\
&\geq \sum_{\substack{m,n=1,1 \\ \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)})-\mu_y(C)\right|\geq\varepsilon}}^{p,q} \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right|^r \\
&\geq \varepsilon^r \left|\left\{(m,n):\ m \leq p, n \leq q,\ \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \varepsilon\right\}\right|
\end{aligned}$$

and so

$$\begin{aligned}
\frac{1}{(pq)^\eta} \sum_{m,n=1,1}^{p,q} \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right|^r &\geq \frac{\varepsilon^r}{(pq)^\eta} \left|\left\{(m,n):\ m \leq p, n \leq q,\ \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \varepsilon\right\}\right| \\
&\geq \frac{\varepsilon^r}{(pq)^\vartheta} \left|\left\{(m,n):\ m \leq p, n \leq q,\ \left|\mu_y(C_{\sigma^m(s)\sigma^n(t)}) - \mu_y(C)\right| \geq \varepsilon\right\}\right|.
\end{aligned}$$

Hence, by our assumption, we get the double set sequence $\{C_{mn}\}$ is invariant statistically convergent of order $\vartheta$ to the set $C$ in the Wijsman sense. □

If $\vartheta = \eta$ is taken in Theorem 4, then the following corollary is obtained.

**Corollary 3.** *Let $\eta \in (0, 1]$ and $0 < r < \infty$. If a double set sequence $\{C_{mn}\}$ is strongly $r$-invariant summable of order $\eta$ to a set $C$, then the double set sequence is invariant statistically convergent of order $\eta$ to the set $C$.*

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# Lacunary Invariant and Lacunary Invariant Statistical Equivalence of Order $\beta$ for Double Set Sequences

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**Abstract:** In this paper, for double set sequences, the concepts of asymptotical lacunary invariant equivalence and asymptotical lacunary invariant statistical equivalence of order $\beta$ $(0 < \beta \leq 1)$ in the Wijsman sense were introduced. Also, some properties of these new equivalence concepts and the relations between them were investigated.

**Keywords:** Asymptotical equivalence, Convergence in the Wijsman sense, Double lacunary sequence, Double set sequences, Invariant statistical convergence, Order $\beta$.

## 1 Introduction

Long after the concept of convergence for double sequences was introduced by Pringsheim [1], using the concepts of statistical convergence, double lacunary sequence, invariant mean etc., this concept was extended to new convergence concepts for double sequences by some authors [2]-[4]. Recently, for double sequences, on two new convergence concepts called double almost statistical and double almost lacunary statistical convergence of order $\alpha$ were studied by Savaş [5, 6]. Moreover, for double sequences, the concept of asymptotical equivalence was introduced by Patterson [7].

Over the years, on the various convergence concepts for set sequences have been studied by many authors. One of them, discussed in this study, is the concept of convergence in the Wijsman sense [8]-[10]. Using the concepts of statistical convergence, double lacunary sequence, invariant mean etc., this concept was extended to new convergence concepts for double set sequences by some authors [11]-[14]. In [13], Nuray and Ulusu studied on the concepts of lacunary invariant summability and lacunary invariant statistical convergence in the Wijsman sense for double set sequences. Furthermore, for double set sequences, the concepts of asymptotical equivalence in the Wijsman sense were introduced by Nuray et al. [15] and then these concepts were studied by some authors [16].

In this paper, using the concepts of invariant mean and order $\beta$, we studied on new asymptotical equivalence concepts in the Wijsman sense for double set sequences.

More information on the concepts of convergence or asymptotical equivalence for real or set sequences can be found in [17]-[25].

## 2 Definitions and notations

In this section, let us recall the basic concepts necessary for a better understanding of our study (see, [3, 9], [11, 12], [14]-[16], [20, 21] ).

For a metric space $(Y, \rho)$, $d(y, C)$ denote the distance from $y$ to $C$ where

$$d(y, C) := d_y(C) = \inf_{c \in C} \rho(y, c)$$

for any $y \in Y$ and any non-empty set $C \subseteq Y$.

For a non-empty set $Y$, let a function $g : \mathbb{N} \to 2^Y$ (the power set of $Y$) is defined by $g(i) = C_i \in 2^Y$ for each $i \in \mathbb{N}$. Then the sequence $\{C_i\} = \{C_1, C_2, \ldots\}$, which is the codomain elements of $g$, is called set sequences.

Throughout this study, $(Y, \rho)$ will be considered as a metric space and $C, C_{ij}, D_{ij}$ will be considered as any non empty closed subsets of $Y$.

The double set sequence $\{C_{ij}\}$ is called convergent to the set $C$ in the Wijsman sense if each $y \in Y$,

$$\lim_{i,j \to \infty} d_y(C_{ij}) = d_y(C).$$

The double set sequence $\{C_{ij}\}$ is called statistical convergent to the set $C$ in the Wijsman sense if every $\varepsilon > 0$ and each $y \in Y$,

$$\lim_{m,n \to \infty} \frac{1}{mn} \left| \left\{ (i,j) : i \leq m, j \leq n, \ \left| d_y(C_{ij}) - d_y(C) \right| \geq \varepsilon \right\} \right| = 0.$$

A double sequence $\theta_2 = \{(j_u, k_v)\}$ is called a double lacunary sequence if there exist increasing sequences $(j_u)$ and $(k_v)$ of the integers such that

$$j_0 = 0,\ h_u = j_u - j_{u-1} \to \infty \ \text{ and } \ k_0 = 0,\ \bar{h}_v = k_v - k_{v-1} \to \infty \ \text{ as } \ u, v \to \infty.$$

In general, the following notations are used for any double lacunary sequence:

$$h_{uv} = h_u \bar{h}_v,\ \ I_{uv} = \big\{(i,j) : j_{u-1} < i \le j_u \ \text{ and } \ k_{v-1} < j \le k_v\big\}.$$

Throughout this study, $\theta_2 = \{(j_u, k_v)\}$ will be considered as a double lacunary sequence.

The double set sequence $\{C_{ij}\}$ is called lacunary statistically convergent to the set $C$ in the Wijsman sense if every $\varepsilon > 0$ and each $y \in Y$,

$$\lim_{u,v\to\infty} \frac{1}{h_{uv}} \left| \Big\{(i,j) \in I_{uv} : \big|d_y(C_{ij}) - d_y(C)\big| \ge \varepsilon \Big\} \right| = 0.$$

The term $d_y\Big(\dfrac{C_{ij}}{D_{ij}}\Big)$ is defined as follows:

$$d_y\Big(\frac{C_{ij}}{D_{ij}}\Big) = \begin{cases} \dfrac{d(y, C_{ij})}{d(y, D_{ij})} & , \ \ y \notin C_{ij} \cup D_{ij} \\ \lambda & , \ \ y \in C_{ij} \cup D_{ij}. \end{cases}$$

The double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are called asymptotically equivalent in the Wijsman sense if each $y \in Y$,

$$\lim_{i,j\to\infty} d_y\Big(\frac{C_{ij}}{D_{ij}}\Big) = 1$$

and denoted by $C_{ij} \overset{W}{\sim} D_{ij}$.

Let $\sigma$ be a mapping such that $\sigma : \mathbb{N}^+ \to \mathbb{N}^+$ (the set of positive integers). A continuous linear functional $\psi$ on $\ell_\infty$ is called an invariant mean (or a $\sigma$-mean) if it satisfies the following conditions:

1. $\psi(x_s) \ge 0$, when the sequence $(x_s)$ has $x_s \ge 0$ for all $s$,
2. $\psi(e) = 1$, where $e = (1, 1, 1, \ldots)$ and
3. $\psi(x_{\sigma(s)}) = \psi(x_s)$ for all $(x_s) \in \ell_\infty$.

The mappings $\sigma$ are assumed to be one to one and such that $\sigma^i(s) \ne s$ for all $i, s \in \mathbb{N}^+$, where $\sigma^i(s)$ denotes the $i$ th iterate of the mapping $\sigma$ at $s$. Thus $\psi$ extends the limit functional on $c$, in the sense that $\psi(x_s) = \lim x_s$ for all $(x_s) \in c$.

The double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are called asymptotically lacunary invariant equivalent to multiple $\lambda$ in the Wijsman sense if each $y \in Y$,

$$\lim_{u,v\to\infty} \frac{1}{h_{uv}} \sum_{(i,j)\in I_{uv}} d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}}\Big) = \lambda.$$

The double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ is called asymptotically strong lacunary invariant equivalent to multiple $\lambda$ in the Wijsman sense if each $y \in Y$,

$$\lim_{u,v\to\infty} \frac{1}{h_{uv}} \sum_{(i,j)\in I_{uv}} \left| d_y\big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}}\big) - \lambda \right| = 0.$$

## 3 Main results

In this section, for double set sequences, the concepts of asymptotical lacunary invariant equivalence and asymptotical lacunary invariant statistical equivalence of order $\beta$ $(0 < \beta \le 1)$ in the Wijsman sense were introduced. Also, some properties of these new equivalence concepts and the relations between them were investigated.

**Definition 1.** *The double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are asymptotically lacunary invariant equivalent to multiple $\lambda$ of order $\beta$ in the Wijsman sense if each $y \in Y$,*

$$\lim_{u,v\to\infty} \frac{1}{h_{uv}^\beta} \sum_{(i,j)\in I_{uv}} d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{C_{\sigma^i(s)\sigma^j(t)}}\Big) = \lambda$$

*uniformly in $s, t$ where $0 < \beta \le 1$ and we denote this in $C_{ij} \overset{W_2^\lambda(N_{\sigma\theta}^\beta)}{\longrightarrow} D_{ij}$ format, and simply called asymptotically lacunary invariant equivalent of order $\beta$ in the Wijsman sense if $\lambda = 1$.*

**Definition 2.** *The double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are asymptotically strong lacunary q-invariant equivalent to multiple $\lambda$ of order $\beta$ in the Wijsman sense if each $y \in Y$,*

$$\lim_{u,v\to\infty} \frac{1}{h_{uv}^{\beta}} \sum_{(i,j)\in I_{uv}} \left| d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}}\Big) - \lambda \right|^q = 0$$

*uniformly in $s,t$ where $0<\beta\leq 1$ and $0<q<\infty$. We denote this in $C_{ij} \overset{W_2^{\lambda}[N_{\sigma\theta}^{\beta}]^q}{\longrightarrow} D_{ij}$ format and simply called asymptotically strong lacunary q-invariant equivalent of order $\beta$ in the Wijsman sense if $\lambda = 1$.*

*If $q=1$, then the double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are simply called asymptotically strong lacunary invariant equivalent to multiple $\lambda$ of order $\beta$ and we denote this in $C_{ij} \overset{W_2^{\lambda}[N_{\sigma\theta}^{\beta}]}{\longrightarrow} D_{ij}$ format.*

**Example 1.** *Let $Y=\mathbb{R}^2$ and double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ be defined as following:*

$$C_{ij} := \begin{cases} \Big\{(a,b)\in\mathbb{R}^2 : a^2+(b+1)^2 = \frac{1}{ij}\Big\} & ; \text{ if } (i,j)\in I_{uv},\ i \text{ and } j \text{ are square integers}, \\ \{(2,0)\} & ; \text{ otherwise.} \end{cases}$$

*and*

$$D_{ij} := \begin{cases} \Big\{(a,b)\in\mathbb{R}^2 : a^2+(b-1)^2 = \frac{1}{ij}\Big\} & ; \text{ if } (i,j)\in I_{uv},\ i \text{ and } j \text{ are square integers}, \\ \{(2,0)\} & ; \text{ otherwise.} \end{cases}$$

*In this case, the double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are asymptotically strong lacunary invariant equivalent of order $\beta$ ($0<\beta\leq 1$) in the Wijsman sense.*

**Remark 1.** *For $\beta = 1$, respectively, the concepts of $W_2^{\lambda}(N_{\sigma\theta}^{\beta})$-equivalence and $W_2^{\lambda}[N_{\sigma\theta}^{\beta}]^q$-equivalence coincide with the concepts of asymptotically lacunary invariant equivalence and asymptotically strong lacunary q-invariant equivalence in the Wijsman sense for double set sequences in [16].*

**Theorem 1.** *If $0<\beta\leq\gamma\leq 1$, then*

$$C_{ij} \overset{W_2^{\lambda}[N_{\sigma\theta}^{\beta}]^q}{\longrightarrow} D_{ij} \Rightarrow C_{ij} \overset{W_2^{\lambda}[N_{\sigma\theta}^{\gamma}]^q}{\longrightarrow} D_{ij}$$

*for every double lacunary sequence $\theta_2 = \{(j_u,k_v)\}$.*

*Proof:* Let $0<\beta\leq\gamma\leq 1$ and suppose that $C_{ij} \overset{W_2^{\lambda}[N_{\sigma\theta}^{\beta}]^q}{\longrightarrow} D_{ij}$. For each $y\in Y$, we have

$$\frac{1}{h_{uv}^{\gamma}} \sum_{(i,j)\in I_{uv}} \left| d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}}\Big) - \lambda \right|^q \leq \frac{1}{h_{uv}^{\beta}} \sum_{(i,j)\in I_{uv}} \left| d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}}\Big) - \lambda \right|^q$$

for all $s,t$. Hence, by our assumption, we get $C_{ij} \overset{W_2^{\lambda}[N_{\sigma\theta}^{\gamma}]^q}{\longrightarrow} D_{ij}$. □

If $\gamma = 1$ is taken in Theorem 1, then following corollary is obtained.

**Corollary 1.** *Let $\beta\in(0,1]$ and $0<q<\infty$. If double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are asymptotically strong lacunary q-invariant equivalent to multiple $\lambda$ of order $\beta$ in the Wijsman sense, then the double set sequences are asymptotically strong lacunary q-invariant equivalent to multiple $\lambda$ in the Wijsman sense, i.e.,*

$$C_{ij} \overset{W_2^{\lambda}[N_{\sigma\theta}^{\beta}]^q}{\longrightarrow} D_{ij} \Rightarrow C_{ij} \overset{W_2^{\lambda}[N_{\sigma\theta}]^q}{\longrightarrow} D_{ij}.$$

Now, we can state a theorem giving the relationship between $W_2^{\lambda}[N_{\sigma\theta}^{\beta}]^q$-equivalence and $W_2^{\lambda}[N_{\sigma\theta}^{\beta}]^p$-equivalence, where $0<\beta\leq 1$ and $0<q<p<\infty$.

**Theorem 2.** *Let $0<\beta\leq 1$ and $0<q<p<\infty$. Then,*

$$C_{ij} \overset{W_2^{\lambda}[N_{\sigma\theta}^{\beta}]^p}{\longrightarrow} D_{ij} \Rightarrow C_{ij} \overset{W_2^{\lambda}[N_{\sigma\theta}^{\beta}]^q}{\longrightarrow} D_{ij}$$

*for every double lacunary sequence $\theta_2 = \{(j_u,k_v)\}$.*

*Proof:* Let $0 < \beta \leq 1$ and $0 < q < p < \infty$. Also, we suppose that $C_{ij} \overset{W_2^\lambda[N_{\sigma\theta}^\beta]^p}{\longrightarrow} D_{ij}$. For each $y \in Y$, by Hölder inequality, we have

$$\frac{1}{h_{uv}^\beta} \sum_{(i,j)\in I_{uv}} \left| d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}}\Big) - \lambda \right|^q < \frac{1}{h_{uv}^\beta} \sum_{(i,j)\in I_{uv}} \left| d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}}\Big) - \lambda \right|^p$$

for all $s, t$. Hence, by our assumption, we get $C_{ij} \overset{W_2^\lambda[N_{\sigma\theta}^\beta]^q}{\longrightarrow} D_{ij}$. □

**Definition 3.** *The double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are asymptotically lacunary invariant statistical equivalent to multiple $\lambda$ of order $\beta$ in the Wijsman sense if every $\varepsilon > 0$ and each $y \in Y$,*

$$\lim_{u,v\to\infty} \frac{1}{h_{uv}^\beta} \left| \left\{ (i,j) \in I_{uv} : \left| d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}}\Big) - \lambda \right| \geq \varepsilon \right\} \right| = 0$$

*uniformly in $s, t$ where $0 < \beta \leq 1$ and we denote this in $C_{ij} \overset{W_2^\lambda(S_{\sigma\theta}^\beta)}{\longrightarrow} D_{ij}$ format, and simply called asymptotically lacunary invariant statistical equivalent of order $\beta$ in the Wijsman sense if $\lambda = 1$ .*

**Example 2.** *Let $Y = \mathbb{R}^2$ and double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ be defined as following:*

$$C_{ij} := \begin{cases} \Big\{(a,b) \in \mathbb{R}^2 : a^2 + b^2 = -ij\, b\Big\} & ; \text{ if } (i,j) \in I_{uv},\ i \text{ and } j \text{ are square integers} \\ \{(1,-2)\} & ; \text{ otherwise.} \end{cases}$$

*and*

$$D_{ij} := \begin{cases} \Big\{(a,b) \in \mathbb{R}^2 : a^2 + b^2 = ij\, b\Big\} & ; \text{ if } (i,j) \in I_{uv},\ i \text{ and } j \text{ are square integers} \\ \{(1,-2)\} & ; \text{ otherwise.} \end{cases}$$

*In this case, the double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are asymptotically lacunary invariant statistical equivalent of order $\beta$ ($0 < \beta \leq 1$) in the Wijsman sense.*

**Remark 2.** *For $\beta = 1$, the concept of $W_2^\lambda(S_{\sigma\theta}^\beta)$-equivalence coincides with the concept of asymptotically lacunary invariant statistical equivalence in the Wijsman sense for double set sequences in [16].*

**Theorem 3.** *If $0 < \beta \leq \gamma \leq 1$, then*

$$C_{ij} \overset{W_2^\lambda(S_{\sigma\theta}^\beta)}{\longrightarrow} D_{ij} \Rightarrow C_{ij} \overset{W_2^\lambda(S_{\sigma\theta}^\gamma)}{\longrightarrow} D_{ij}$$

*for every double lacunary sequence $\theta_2 = \{(j_u, k_v)\}$.*

*Proof:* Let $0 < \beta \leq \gamma \leq 1$ and suppose that $C_{ij} \overset{W_2^\lambda(S_{\sigma\theta}^\beta)}{\longrightarrow} D_{ij}$. For every $\varepsilon > 0$ and each $y \in Y$, we have

$$\frac{1}{h_{uv}^\gamma} \left| \left\{ (i,j) \in I_{uv} : \left| d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}}\Big) - \lambda \right| \geq \varepsilon \right\} \right| \leq \frac{1}{h_{uv}^\beta} \left| \left\{ (i,j) \in I_{uv} : \left| d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^i(s)\sigma^j(t)}}\Big) - \lambda \right| \geq \varepsilon \right\} \right|.$$

for all $s, t$. Hence, by our assumption, we get $C_{ij} \overset{W_2^\lambda(S_{\sigma\theta}^\vartheta)}{\longrightarrow} D_{ij}$. □

If $\gamma = 1$ is taken in Theorem 3, then the following corollary is obtained.

**Corollary 2.** *Let $\beta \in (0, 1]$. If double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are asymptotically lacunary invariant statistical equivalent to multiple $\lambda$ of order $\beta$ in the Wijsman sense, then the double set sequences are asymptotically lacunary invariant statistical equivalent to multiple $\lambda$ in the Wijsman sense, i.e.,*

$$C_{ij} \overset{W_2^\lambda(S_{\sigma\theta}^\beta)}{\longrightarrow} D_{ij} \Rightarrow C_{ij} \overset{W_2^\lambda(S_{\sigma\theta})}{\longrightarrow} D_{ij}.$$

**Theorem 4.** *If double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are asymptotically strong lacunary q-invariant equivalent to multiple $\lambda$ of order $\beta$ in the Wijsman sense, then the double set sequences are asymptotically lacunary invariant statistical equivalent to multiple $\lambda$ of order $\gamma$ in the Wijsman sense, where $0 < \beta \leq \gamma \leq 1$ and $0 < q < \infty$.*

*Proof:* Let $0 < \beta \le \gamma \le 1$ and $0 < q < \infty$. Also, we suppose that the double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are asymptotically strong lacunary $q$-invariant equivalent to multiple $\lambda$ of order $\beta$ in the Wijsman sense. For every $\varepsilon > 0$ and each $y \in Y$, we have

$$\begin{aligned}
\sum_{(i,j)\in I_{uv}} \left| d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^m(s)\sigma^n(t)}}\Big) - \lambda \right|^q &= \sum_{\substack{(i,j)\in I_{uv} \\ \left|d_y\left(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^m(s)\sigma^n(t)}}\right)-\lambda\right|\ge\varepsilon}} \left| d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^m(s)\sigma^n(t)}}\Big) - \lambda \right|^q \\
&\quad + \sum_{\substack{(i,j)\in I_{uv} \\ \left|d_y\left(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^m(s)\sigma^n(t)}}\right)-\lambda\right|<\varepsilon}} \left| d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^m(s)\sigma^n(t)}}\Big) - \lambda \right|^q \\
&\ge \sum_{\substack{(i,j)\in I_{uv} \\ \left|d_y\left(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^m(s)\sigma^n(t)}}\right)-\lambda\right|\ge\varepsilon}} \left| d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^m(s)\sigma^n(t)}}\Big) - \lambda \right|^q \\
&\ge \varepsilon^q \left| \left\{ (i,j) \in I_{uv} : \left| d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^m(s)\sigma^n(t)}}\Big) - \lambda \right| \ge \varepsilon \right\} \right|
\end{aligned}$$

and so

$$\begin{aligned}
\frac{1}{h_{uv}^{\beta}} \sum_{(i,j)\in I_{uv}} \left| d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^m(s)\sigma^n(t)}}\Big) - \lambda \right|^q &\ge \frac{\varepsilon^q}{h_{uv}^{\beta}} \left| \left\{ (i,j) \in I_{uv} : \left| d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^m(s)\sigma^n(t)}}\Big) - \lambda \right| \ge \varepsilon \right\} \right| \\
&\ge \frac{\varepsilon^q}{h_{uv}^{\gamma}} \left| \left\{ (i,j) \in I_{uv} : \left| d_y\Big(\frac{C_{\sigma^i(s)\sigma^j(t)}}{D_{\sigma^m(s)\sigma^n(t)}}\Big) - \lambda \right| \ge \varepsilon \right\} \right|.
\end{aligned}$$

for all $s, t$. Hence, by our assumption, we get that the double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are asymptotically lacunary invariant statistical equivalent to multiple $\lambda$ of order $\gamma$ in the Wijsman sense. □

If $\gamma = \beta$ is taken in Theorem 4, then the following corollary is obtained.

**Corollary 3.** *Let $\beta \in (0, 1]$ and $0 < q < \infty$. If double set sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are asymptotically strong lacunary q-invariant equivalent to multiple $\lambda$ of order $\beta$ in the Wijsman sense, then the double set sequences are asymptotically lacunary invariant statistical equivalent to multiple $\lambda$ of order $\beta$ in the Wijsman sense.*

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# Some Types of Boundedness for the Fuzzy Soft Sets

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**Abstract:** General bornological spaces play a key role in recent research of convergence structures on hyperspaces, in optimization theory and in the study of topologies on function spaces. In order to generalize this structure, in the present study, we attemp to define and investigate the concept of boundedness for the fuzzy soft sets. Hence we deal with the softification of the bornological spaces with the help of the parametrization tool. Moreover, we examine several basic and categorical properties of the proposed concepts.

**Keywords:** Boundedness, Bounded fuzzy soft map, Category, Fuzzy soft set.

## 1 Introduction and motivation

In our work two lattices $L$ and $M$, will play the fundamental role. The first one is a frame, that is a complete lattice $L = (L, \leq, \wedge, \vee)$, satisfying the infinite distributivity law

$$\alpha \wedge (\vee_{i\in I}\beta_i) = \vee_{i\in I}(\alpha \wedge \beta_i), \forall \alpha \in L, \{\beta_i\}_{i\in I} \subset L.$$

The top and the bottom elements of $L$ are denoted by $1_L$ and $0_L$, respectively. By $M$ we denote the complete completely distributive lattice $M = (M, \leq, \wedge, \vee)$ whose the bottom and the top elements are denoted by $0_M$ and $1_M$, respectively. For a complete lattice $M$ and $\alpha, \beta \in M$, the wedge-below relation $\preceq$ is defined on $M$ as follows: $\beta \preceq \alpha \Leftrightarrow$ if $K \subseteq M$ and $\alpha \leq \vee K$ then $\exists \gamma \in K, \beta \leq \gamma$.
As shown in [8] a lattice $M$ is completely distributive if and only if the wedge-below relation has the following property,
$\alpha = \vee\{\beta \in M \mid \beta \preceq \alpha\}$, for each $\alpha \in M$. For more details about the lattices, see [3, 8].

An element $\alpha$ in $M$ is said to be coprime if $\alpha \leq \beta \vee \gamma$ implies that $\alpha \leq \beta$ or $\alpha \leq \gamma$. The set of all nonzero coprime elements of $M$ is denoted by $c(M)$. We denote $M^o = \{\alpha \in M \mid \alpha \preceq 1_M\}$.
Throughout this work, $X$ refers to a nonempty initial universe and $E$ denotes an arbitrary nonempty set viewed on the sets of parameters.

### *1.1 L-fuzzy soft sets*

In this subsection, we recall some basic notions and notations of the theory of $L$-fuzzy soft sets defined by Maji et al.[6] . We also remind the operations on $L$-fuzzy soft sets.

**Definition 1** [6] A pair $(f, E)$ is called an $L$-fuzzy soft set over $X$ if $f$ is a mapping from $E$ into the set of all $L$-fuzzy subsets of $X$; i.e., $f : E \to L^X$. This means that $f_e := f(e) : X \to L$ is an $L$-fuzzy set on $X$ for each $e \in E$. In what follows an $L$-fuzzy soft set $(f, E)$ over $X$ is denoted by a triple $(f, E, X)$. Sometimes the mapping $f : E \to L^X$ is referred to an $L$-fuzzy soft structure over the pair $(E, X)$.

If one considers $L = I = [0, 1]$ and $A \subseteq E$, then $(f, A, X)$ is a said to be a fuzzy soft set on $X$. As it is seen from the former definition, a fuzzy soft set is a parameterized collection of fuzzy sets.

**Definition 2** [7] Let $(f_1, E, X)$ and $(f_2, E, X)$ be two $L$-fuzzy soft structures such that for any $e \in E$, $f_1(e) \leq f_2(e)$. Then $f_1$ is said to be coarser than $f_2$ and denoted by $f_1 \leq f_2$.

**Definition 3** [6] Let $(f, E, X)$ and $(g, E, X)$ be two $L$-fuzzy soft sets, then

(1) $(f, E, X)$ is said to be a subset of $(g, E, X)$ denoted by $(f, E, X) \sqsubseteq (g, E, X)$ if $f(e) \leq g(e)$, for each $e \in E$. In this case $(f, E, X) = (g, E, X)$ if and only if f $(f, E, X)$ is subset of $(g, E, X)$ and vice-versa.

(2) The union $(k, E, X) = (f, E, X) \sqcup (g, E, X)$ is defined by $k(e) = f(e) \vee g(e)$ for all $e \in E$.

(3) The intersection $(h, E, X) = (f, E, X) \sqcap (g, E, X)$ is defined by $h(e) = f(e) \wedge g(e)$ for all $e \in E$.

(4) The complement of the $L$-fuzzy soft set $(f, E, X)$ is denoted by $(f, E, X)^c = (f^c, E, X)$, where $f^c : E \to L^X$ is a mapping given by $f^c(e) = (f(e))'$ for all $e \in E$.

(5) $(f, E, X)$ is said to be the null $L$-fuzzy soft set and denoted by $\Phi$ iff $f(e) = 0_X$ for each $e \in E$.

(6) $(f, E, X)$ is said to be the universal $L$-fuzzy soft set and denoted by $\mathcal{X}$ iff $f(e) = 1_X$ for each $e \in E$.

**Theorem 1.** [7] Let $FS(X, E)$ denotes the family of all $L$-fuzzy soft structures over $(E, X)$ which equipped with the partial order $\leq$ . Then $(FS(X, E), \leq)$ is a complete lattice.

 

**Definition 4** [5] Let $(f, E_1, X)$ and $(g, E_2, Y)$ be two $L$-fuzzy soft sets and $(\varphi, \psi) : (f, E_1, X) \to (g, E_2, Y)$ be a soft mapping. Then
(1) the image of $(f, E_1, X)$ under $(\varphi, \psi)$ is denoted by $(\varphi, \psi)(f, E_1, X) = (\varphi(f), \psi(E_1), Y)$ and defined by $\varphi(f)_k(y) = \vee_{\varphi(x)=y} \vee_{\psi(a)=k} f_e(x)$ for all $k \in E_2, y \in Y$
(2) the pre-image of $(g, E_2, Y)$ under $(\varphi, \psi)$ is denoted by $(\varphi, \psi)^{-1}(g, E_2, Y) = (\varphi^{\leftarrow}(g), \psi^{-1}(E_2), X)$ and it is defined by follows: $\varphi^{\leftarrow}(g)_{e_1}(x) = g_{\psi(e_1)}(\varphi(x))$, for all $e_1 \in E_1$ and for all $x \in X$.

### *1.2 Bornological structures in the context of fuzzy sets*

The concept of bornology (or so called abstract boundedness) was introduced by S.T.Hu [4] to make "being bounded for a set" meaningful in general spaces. This definition opened a way of modeling macroscopic phenomena in general topological spaces. In addition, it was improved in several directions. Later Abel and Sostak [1] and Sostak and Uljane [10, 11] presented and studied the concept of bornology in the context of fuzzy sets as follows.

**Definition 5** [1] An $L$-bornology on a set $X$ is a family $\mathcal{B} \subseteq L^X$ which satisfies the following axioms:
(1LB) $\vee\{B \mid B \in \mathcal{B}\} = 1_X$.
(2LB) If $B \in \mathcal{B}$ and $A \le B$, then $A \in \mathcal{B}$.
(3LB) If $B_1, B_2 \in \mathcal{B}$, then $B_1 \vee B_2 \in \mathcal{B}$.
The pair $(X, \mathcal{B})$ is called an $L$-bornological space and $L$-sets which belongs to $\mathcal{B}$ are called bounded in this space. An $L$-bornology $\mathcal{B}$ is called strict if it satisfies the following stronger version of the first axiom:
(1LB*) $\chi_{\{x\}} \in \mathcal{B}$ for all $x \in X$.
Given two $L$-bornological spaces $(X_1, \mathcal{B}^1)$ and $(X_2, \mathcal{B}^2)$, a function $\varphi : X_1 \to X_2$ is called bounded if $\varphi^{\to}(B) \in \mathcal{B}^2$, for every $B \in \mathcal{B}^1$.

By considering the fuzzy analogue of a classical bornological structure when both the structure itself is fuzzy, and it acts on families of $L$-sets, Sostak and Uljane [11] defined the concept of an $LM$-valued bornology. This fuzzy-fuzzy approach of bornology was described in a somewhat different way. In this approach, Sostak and Uljane [11] deal with a many-valued set, that is a set endowed with some many-valued equality instead of an ordinary set. Since this definition depends on the $L$-valued equality, we consider the following modified version of the original definition. In case when the relation $R$ is the ordinary relation $=$; i.e., $R(x, y) = 1$ if and only if $x = y$, then the original definition coincides with the following specific version.

**Definition 6** Given a set $X$, a mapping $\mathcal{B} : L^X \to M$ is said to be an $LM$-valued bornology on $X$ if it satisfies the following axioms:
(1LMB) $\forall \alpha \in M^o, \exists \mathcal{U} \subseteq L^X$ s.t. $\vee\mathcal{U} = 1_{L^X}$ and $\mathcal{B}(U) \ge \alpha, \forall U \in \mathcal{U}$.
(2LMB) If $A \le B$, then $\mathcal{B}(A) \ge \mathcal{B}(B)$, for all $A, B \in L^X$.
(3LMB) $\mathcal{B}(A \vee B) \ge \mathcal{B}(A) \wedge \mathcal{B}(B)$, for all $A, B \in L^X$.
The pair $(X, \mathcal{B})$ is called an $LM$-valued bornological space and the value $\mathcal{B}(A)$ is interpreted as the degree of boundedness of an $L$-fuzzy set $A$ in this space.
One can prefer to consider the following stronger version of the first axiom:
(1*LMB) $\vee\{A \in L^X \mid \mathcal{B}(A) = 1_M\} = 1_{L^X}$.
The mapping $\mathcal{B} : L^X \to M$ which satisfies the axioms (1*LMB), (2LMB) and (3LMB) is said to be a strong $LM$-valued bornology on $X$.
Given two $LM$-valued bornological spaces $(X_1, \mathcal{B}^1)$ and $(X_2, \mathcal{B}^2)$, a function $\varphi : X_1 \to X_2$ is called bounded if $\mathcal{B}^1(A) \le \mathcal{B}^2(\varphi(A))$ whenever $A \in L^X$.

**Remark 1.** *In case when $L = M = 2$, then we return to the original definition of a bornology given by Hu [4].*

We refer to the paper of Sostak and Uljane [11] for the generalized (in fact, original) version of the $LM$-valued bornology which is defined on an $L$-valued set $(X, R)$.

## 2 Boundedness in the fuzzy soft universe

Çetkin [2] defined the notion of "boundedness" for the soft sets in a different direction from Reddy and Jalil [9], without using soft metric structure. In this section, we present the notion of bornology in the context of lattice valued fuzzy soft sets as the extensions of the fuzzy bornologies. Hence, we describe the notion of "boundedness" for the $L$-fuzzy soft sets by using the similar way of pictured in [2].

### *2.1 Soft L-bornological spaces*

In this subsection, we present the parameterized extension of an $L$-bornological space, which we call as soft $L$-bornological space. As a result, we identify being bounded for an $L$-fuzzy soft set in such spaces. Besides we demonstrate some structural properties.

**Definition 7** A mapping $\mathcal{B} : E \to 2^{L^X}$ (where, $\mathcal{B}(e) := \mathcal{B}_e \in 2^{L^X}$, for all $e \in E$) is said to be a soft $L$-bornology on a set $X$ with respect to the parameter set E, if the mappings $\mathcal{B}(e) : L^X \to 2$ is an $L$-bornology on $X$, for each $e \in E$. In this case, a soft $L$-bornology stands for a parameterized family of $L$-bornologies.
Then in this case a soft $L$-bornology on $X$ with respect to the parameter set E, is denoted by $\mathcal{B}(X, E)$ and the triple $(X, \mathcal{B}, E)$ denotes the soft $L$-bornological space.

**Example 1** (1) Let $E = \{*\}$ and let $(\mathcal{B}, \{*\})$ be a soft set on $L^X$; i.e., $\mathcal{B} : \{*\} \to 2^{L^X}$. If in this case, $\mathcal{B}(*)$ is an $L$-bornology on $X$, then $\mathcal{B}$ is a soft $L$-bornology on $X$ w.r.t $\{*\}$.
(2) Let $E = \{0, 1\}$ and let $(\mathcal{B}, \{0, 1\})$ be a soft set on $L^X$; i.e., $\mathcal{B} : \{0, 1\} \to 2^{L^X}$. If $\mathcal{B}(0)$ and $\mathcal{B}(1)$ are $L$-bornologies on $X$, then $\mathcal{B}$ is a soft $L$-bornology on $X$ w.r.t $\{0, 1\}$.
(3) Let $E = [0, 1] = I$ and let $(\mathcal{B}, I)$ be a soft set on $L^X$; i.e., $\mathcal{B} : I \to 2^{L^X}$. If $\mathcal{B}(\alpha)$ is an $L$-bornology on $X$ for all $\alpha \in I$, then $\mathcal{B}$ is a soft $L$-bornology on $X$ w.r.t $I$.

**Remark 2.** *It is clear that a soft $L$-bornology on a set $X$ is just a parameterized family of ideals in the lattice $L^X$.*

**Definition 8** Let $(X, \mathcal{B}, E)$ be a soft $L$-bornological space and $(f, E, X)$ be an $L$-fuzzy soft set. Then $(f, E, X)$ is said to be a bounded $L$-fuzzy soft set if $f(e) \in \mathcal{B}(e)$, for all $e \in E$.

It is obviously seen that boundedness for fuzzy soft sets is hereditary and closed under finite unions.

**Definition 9** Let $(X, \mathcal{B}, E)$ be a soft $L$-bornological space and $\mathcal{D} : E \to 2^{L^X}$ be a mapping such that $\mathcal{D}(e) \subseteq \mathcal{B}(e)$, for each $e \in E$. Then $\mathcal{D}(X, E)$ is said to be a soft $L$-bornology base of $\mathcal{B}(X, E)$, if $\mathcal{D}(e)$ is a $L$-bornology base for $\mathcal{B}(e)$; i.e., each elements of $\mathcal{B}(e)$ is a subset of an element of $\mathcal{D}(e)$.

**Definition 10** Let $(X, \mathcal{B}^1, E_1)$ and $(Y, \mathcal{B}^2, E_2)$ be two soft $L$-bornological spaces. Then the soft mapping $(\varphi, \psi) : (X, \mathcal{B}^1, E_1) \to (Y, \mathcal{B}^2, E_2)$ is called soft $L$-bounded if the mappings $\varphi_e : (X, \mathcal{B}^1(e)) \to (Y, \mathcal{B}^2(\psi(e)))$ are $L$-bounded for all $e \in E_1$. In other words,

$(\varphi, \psi) : (X, \mathcal{B}^1, E_1) \to (Y, \mathcal{B}^2, E_2)$ is a soft $L$-bounded map iff $\varphi^{\Rightarrow}(\mathcal{B}^1(e)) \leq \mathcal{B}^2(\psi(e))$, for each $e \in E_1$. That is, $(\varphi, \psi) : (X, \mathcal{B}^1, E_1) \to (Y, \mathcal{B}^2, E_2)$ is a soft $L$-bounded map if and only if $\varphi^{\to}(B) \in \mathcal{B}^2_{\psi(e)}$, whenever $B \in \mathcal{B}^1_e$, for all $e \in E_1$.

**Theorem 2.** *If the soft mappings $(\varphi_1, \psi_1) : (X, \mathcal{B}^1, E_1) \to (Y, \mathcal{B}^2, E_2)$ and $(\varphi_2, \psi_2) : (Y, \mathcal{B}^2, E_2) \to (Z, \mathcal{B}^3, E_3)$ are soft L-bounded, then their composition $(\varphi_2, \psi_2) \circ (\varphi_1, \psi_1) : (X, \mathcal{B}^1, E_1) \to (Z, \mathcal{B}^3, E_3)$ is a soft L-bounded map, too.*

*Proof:* The proof is easily obtained by using the former definition. □

In addition, since the identity function $id_X : (X, \mathcal{B}(e)) \to (X, \mathcal{B}(e))$ is $L$-bounded between $L$-bornological spaces for each $e \in E$, then the identity soft mapping $(id_X, id_E) : (X, \mathcal{B}, E) \to (X, \mathcal{B}, E)$ is soft $L$-bounded. Hence soft $L$-bornological spaces and the soft $L$-bounded mappings between them form a category which will be denoted $\mathbf{SL - BOR}$ and called the category of soft $L$-bornological spaces.

In case $L = 2$ is a two-point lattice, the concept of a soft 2-bornological space is equivalent to the category of soft bornological spces $\mathbf{SBOR}$. Hence $\mathbf{S2 - BOR}$ is actually a category whose objects are soft bornological spaces and the morphisms are soft bounded mappings.

Let $\mathcal{B}(X, E, L)$ be the family of all soft $L$-bornologies on $X$ with respect to the parameter set $E$. Let us define an order $\leq$ on $\mathcal{B}(X, E, L)$ by setting;

$\mathcal{B}^1 \leq \mathcal{B}^2 :\Leftrightarrow \mathcal{B}^2(e) \subseteq \mathcal{B}^1(e)$, for all $e \in E$ and $\mathcal{B}^1, \mathcal{B}^2 \in \mathcal{B}(X, E, L)$.

In this case, $\mathcal{B}^2$ is said to be coarser (or stronger) than $\mathcal{B}^1$.

**Theorem 3.** *The pair $(\mathcal{B}(X, E, L), \leq)$ is a partially ordered set. Additionally, $(\mathcal{B}(X, E, L), \leq)$ is a complete infinitely distributive lattice.*

*Proof:* By the soft $L$-bornology axioms, it is easily seen that the bottom element $\mathcal{B}_0$ of $\mathcal{B}(X, E, L)$ is identified by $\mathcal{B}_0(e) = L^X$ for each $e \in E$. Now let us describe its top element $\mathcal{B}_1$. Given a soft set $(F, E, X)$ and a fixed parameter $e \in E$, define a mapping $\lambda : F(e) \to c(L)$. Let $J_F := \{\vee_{x \in F(e)} x^{\lambda(x)} \mid \lambda \in (c(L))^{F(e)}\}$. Then $\mathcal{B}_1(e) = \{J_F \mid F \in S(X, E), |F(e)| < \aleph_0\}$ is the strongest $L$-bornology on $X$ for each $e \in E$. Hence the mapping $\mathcal{B}_1$ is the strongest soft $L$-bornology on $X$ with respect to $E$. Besides the meets and the joins are described similarly, by taking into consideration the parametrization tool in the $L$-bornological case. □

**Theorem 4.** *Let $X$ be a set and $(Y, \mathcal{B}^2, E_2)$ be a soft L-bornological space. Then the weakest soft L-bornology $B : E \to 2^{L^X}$ on $(X, E)$, which makes the soft mapping $(\varphi, \psi) : (X, \mathcal{B}, E) \to (Y, \mathcal{B}^2, E_2)$ bounded, is identified by follows:*

*$\mathcal{B}_e = \{A \in L^X \mid \exists B \in \mathcal{B}^2_{\psi(e)}$ such that $A \leq \varphi^{-1}(B)\}$ for any $e \in E$.*

*Proof:* Since being bounded for a soft mapping $(\varphi, \psi) : (X, \mathcal{B}, E) \to (Y, \mathcal{B}^2, E_2)$ is described over the parameters, that is since it depends on the boundedness of the mappings of $\{\varphi_e : (X, \mathcal{B}_e) \to (Y, \mathcal{B}^2_{\psi(e)})\}$, for all $e \in E$, it is sufficient to prove that the subfamily $\mathcal{B}_e \subseteq L^X$ is the unique initial structure of the given source $\{\varphi_e : X \to (Y, \mathcal{B}^2_{\psi(e)})\}$, for any $e \in E$. It is also provided by Abel and Sostak [1]. □

If one generalizes the above claim to the family of mappings, then the following is obtained.

**Theorem 5.** *For any given source $\{(\varphi, \psi)_i : (X, E) \to (Y_i, \mathcal{B}^i, E_i) \mid i \in \Gamma\}$, there exists a unique initial lift $\{(\varphi, \psi)_i : (X, \mathcal{B}, E) \to (Y_i, \mathcal{B}^i, E_i) \mid i \in \Gamma\}$ in the category of $\mathbf{SL - BOR}$.*

**Theorem 6.** *Let $Y$ be a set, $(X, \mathcal{B}^1, E_1)$ be a soft L-bornological space and the soft mapping $(\varphi, \psi) : (X, \mathcal{B}^1, E_1) \to Y$ be surjective. Then the mapping $\mathcal{B} : E_2 \to 2^{L^Y}$ which is described below is a soft L-bornology on $(Y, E_2)$ which makes the soft mapping $(\varphi, \psi) : (X, \mathcal{B}^1, E_1) \to (Y, \mathcal{B}, E_2)$ bounded.*

*$\mathcal{B}_{\psi(e)} = \{\varphi^{\to}(B) \in L^Y \mid B \in \mathcal{B}^1_e\}$, for each $e \in E_1$.*

*Proof:* Since the functions $\varphi$ and $\psi$ are both surjective, then consider the mapping $\varphi_e : (X, \mathcal{B}^1_e) \to Y$ for a fixed parameter $e \in E_1$. Then it is easy to verify that the family $\mathcal{B}_{\psi(e)}$ described above is an $L$-bornology on $Y$ which makes the function $\varphi_e : (X, \mathcal{B}^1_e) \to (Y, \mathcal{B}_{\psi(e)})$ is bounded. By the arbitrariness of the parameter, this witnesses the proof. □

**Corollary 1.** *Products exist in the category $\mathbf{SL - BOR}$.*

### *2.2 LM-valued soft bornological spaces*

In this subsection, we present the parameterized extension of the $LM$-valued bornology described in Definition 6. Hence we provide the definition of an $LM$-valued soft bornological space and identify the parameterized degree of being bounded for an $L$-fuzzy soft set in such spaces. Besides, we observe some fundamental properties of this extended concept with the help of the parameters.

**Definition 11** An $LM$-valued soft bornology on a set $X$ with respect to the parameters of $E$, is a mapping $\mathcal{B} : E \to M^{L^X}$ such that for all $e \in E$, $\mathcal{B}(e) \in M^{L^X}$; i.e., $\mathcal{B}(e) : L^X \to M$ is an $LM$-valued bornology on $X$. This means that if $\mathcal{B} : E \to M^{L^X}$ is an $LM$-valued soft bornology, then $\mathcal{B}(e) := \mathcal{B}_e : L^X \to M$ is an $LM$-valued bornology in the sense of Definition 6, for each $e \in E$ and vice versa. Hence an $LM$-valued soft bornology on a set $X$, is actually a parameterized family of $LM$-valued bornologies on $X$.

An $LM$-valued soft bornology on a set $X$ with respect to the parameters of $E$, is denoted by $\mathcal{B}(X, E)$. In this case, the triple $(X, \mathcal{B}, E)$ is called an $LM$-valued soft bornological space.

**Example 2** (1) Let $(X, \mathcal{B}, E)$ be a soft bornological space, that is $\mathcal{B}$ is a bornology of soft sets on $X$. If for each $e \in E$, we define $\mathcal{B}(e) := \chi_{\mathcal{B}(e)} : 2^X \to 2$, then we can consider $(X, \mathcal{B}, E)$ as an $LM$-valued soft bornological space.

(2) Let $(X, \mathcal{B}, E)$ be a soft $L$-bornological space. If for each $e \in E$, we define $\mathcal{B}(e) := \chi_{\mathcal{B}(e)} : L^X \to 2$, then we can consider $(X, \mathcal{B}, E)$ as an $LM$-valued soft bornological space.

**Definition 12** Let $(X, \mathcal{B}, E)$ be an $LM$-valued soft bornological space and $(f, E, X)$ be an $L$-fuzzy soft set. Then the value $\mathcal{B}_e(f(e))$ is interpreted as the parameterized degree of boundeness of the $L$-fuzzy soft set $(f, E, X)$ with respect to the parameter $e \in E$.

**Definition 13** Let $\mathcal{L}$ be a subset of $L^X$ that is closed under finite unions, then the mapping $\mathcal{D} : E \to M^{\mathcal{L}}$ is said to be an $LM$-valued soft bornology base if for all $e \in E$, the followings are satisfied.

(1) $\forall \alpha \in M^o, \exists \mathcal{U} \subseteq L^X$ s.t. $\vee \mathcal{U} = 1_{L^X}$ and $\mathcal{D}_e(U) \geq \alpha, \forall U \in \mathcal{U}$.

(2) $\mathcal{D}_e(A \vee B) = \mathcal{D}_e(A) \wedge \mathcal{D}_e(B)$, for each $A, B \in \mathcal{L}$.

**Theorem 7.** *Let $\mathcal{D} : E \to M^{\mathcal{L}}$ be an $LM$-valued soft bornology base on $X$. Then the mapping $\mathcal{B}^* := \langle \mathcal{D} \rangle : E \to M^{L^X}$ defined by $\mathcal{B}^*_e(A) = \vee\{\mathcal{D}_e(B) \mid B \in \mathcal{L}, A \leq B\}$ is an $LM$-valued soft bornology on $X$.*

*Proof:* In order to prove the claim, it is sufficient to show the third axiom of the definition, since the others are easy to see.

$\mathcal{B}^*_e(A_1 \vee A_2) = \vee\{\mathcal{D}_e(B) \mid B \geq A_1 \vee A_2\}$
$= \vee\{\mathcal{D}_e(B_1 \vee B_2 \mid B_i \geq A_i, i = 1, 2)\}$
$= \vee\{\mathcal{D}_e(B_1) \wedge \mathcal{D}_e(B_2) \mid B_i \geq A_i, i = 1, 2\}$
$\geq \vee\{\mathcal{D}_e(B_1) \mid B_1 \geq A_1\} \wedge \vee\{\mathcal{D}_e(B_2) \mid B_2 \geq A_2\} = \mathcal{B}^*_e(A_1) \wedge \mathcal{B}^*_e(A_2)$, for each $e \in E$.

Since for each $e \in E$, $\mathcal{B}^*_e$ is an $LM$-valued bornology on $X$, then $\mathcal{B}^*$ is an $LM$-valued soft bornology on $X$. □

**Definition 14** A fuzzy soft mapping $(\varphi, \psi) : (X, \mathcal{B}^1, E_1) \to (Y, \mathcal{B}^2, E_2)$ is called bounded between $LM$-valued soft bornological spaces if the fuzzy functions $\varphi_e : (X, \mathcal{B}^1(e)) \to (Y, \mathcal{B}^2(\psi(e)))$ are bounded for all $e \in E_1$; i.e., $\mathcal{B}^2(\psi(e)) \circ \varphi^{\to} \geq \mathcal{B}^1(e)$ for all $e \in E_1$.

In other words,
$(\varphi, \psi) : (X, \mathcal{B}^1, E_1) \to (Y, \mathcal{B}^2, E_2)$ is bounded if and only if $\mathcal{B}^2_{\psi(e)}(\varphi^{\to}(A)) \geq \mathcal{B}^1_e(A)$ for each $A \in L^X$ and for each $e \in E_1$.

**Theorem 8.** *Composition of two bounded fuzzy soft mappings is bounded, too.*

*Proof:* Since the boundedness of fuzzy soft mappings described by the boundedness of some fuzzy mappings based on the parameters, the proof is easy to verify. □

Since the identity fuzzy mapping $id_X : (X, \mathcal{B}(e)) \to (X, \mathcal{B}(e))$ is bounded for any $e \in E$, then the identical fuzzy soft mapping $(id_X, id_E) : (X, \mathcal{B}, E) \to (X, \mathcal{B}, E)$ is bounded. In conclude one may infer that the following.

**Theorem 9.** *$LM$-valued soft bornological spaces and bounded fuzzy soft mappings between them form a category which is denoted by* $\mathbf{SBOR}(\mathbf{L}, \mathbf{M})$.

**Definition 15** Let $\mathfrak{B}(L, M, X)$ be the family of all $LM$-valued soft bornologies on $X$ with respect to the parameters of $E$. Define a partial order " $\leq$ " by setting for $\mathcal{B}^1, \mathcal{B}^2 \in \mathfrak{B}(L, M, X) : \mathcal{B}^1 \leq \mathcal{B}^2 :\Leftrightarrow \mathcal{B}^1_e(A) \geq \mathcal{B}^2_e(A)$ for all $e \in E$ and $A \in L^X$.

In this case, we say that $\mathcal{B}^1$ is coarser (or stronger) than $\mathcal{B}^2$.

**Theorem 10.** *The partially ordered set $(\mathfrak{B}(L, M, X), \preceq)$ is a complete lattice.*

*Proof:* In order to obtain the bottom element, we constitute the mapping $\mathcal{B}^{\perp} : E \to M^{L^X}$ as $\mathcal{B}^{\perp}_e(A) = 1_M$ for each $e \in E$. Then it is obvious that $\mathcal{B}^{\perp}$ is the coarsest element of $\mathfrak{B}(L, M, X)$. To obtain the top element, for a given soft set $(F, E, X)$ and a fixed parameter $e \in E$, first define a fuzzy set $f^{\alpha}(x) = \alpha$ if $x \in F(e)$, otherwise $f^{\alpha}(x) = 0_L$, where $\alpha \in L$. Now define the mapping $\mathcal{B}^{\top} : E \to M^{L^X}$ as

$$\mathcal{B}^{\top}_e(A) = \begin{cases} 1_M, & \text{if } \exists F \in S(X, E), |F(e)| < \aleph_0, \exists \alpha \text{ such that } A \leq f^{\alpha}, \\ 0_M, & \text{otherwise} \end{cases}$$

Then $\mathcal{B}^{\top}$ is the finest $LM$-valued soft bornology in $\mathfrak{B}(L, M, X)$. Further given a family of $\{\mathcal{B}^i : E \to M^{L^X} \mid i \in \Gamma\}$ of $LM$-valued soft bornologies on $X$, we define the join $\mathcal{B}^* : E \to M^{L^X}$ by setting $\mathcal{B}^*_e(A) = \wedge_{i \in \Gamma} \mathcal{B}^i_e(A)$ for all $A \in {}^X$. Hence it is a join semi-lattice. The existence of the meets in the family of $\mathfrak{B}(L, M, X)$ can be demonstrated in a similar way of $LM$-valued bornology with the observations of the parameters. □

**Theorem 11.** *Let $\mathcal{B} : E \to M^{L^X}$ be an $LM$-valued soft bornology on $X$ and $\alpha \in M$, then the mapping $\mathcal{B}^{\alpha} : E \to 2^{L^X}$ defined by for each $e \in E$, $\mathcal{B}^{\alpha}_e = \{A \in L^X \mid \mathcal{B}_e(A) \geq \alpha\}$ is a soft $L$-bornology on $X$.*

*Proof:* Let $\mathcal{B} : E \to M^{L^X}$ be an $LM$-valued soft bornology on $X$. Then $\mathcal{B}_e : L^X \to M$ is an $LM$-valued bornology in the sense of Definition 6. Hence, each level sets $\mathcal{B}^{\alpha}_e = \{A \in L^X \mid \mathcal{B}_e(A) \geq \alpha\}$ is an $L$-bornology in the sense of Definition 5, for any parameters of $E$. This makes the mapping $\mathcal{B}^{\alpha} : E \to 2^{L^X}$ a soft $L$-bornology as claimed. □

Further, for a fixed parameter $e \in E$, the family of $\alpha$-levels $\{\mathcal{B}_e^\alpha \mid \alpha \in M\}$ of $L$-brnology is lower semi-continuous, in the following sense

$$\mathcal{B}_e^\alpha = \cap\{\mathcal{B}_e^\beta \mid \beta \preceq \alpha,\ \beta \in M\}, \forall \alpha \in M.$$

It is evident that $\mathcal{B}_e^{0_M} = 1_{L^X}$ as the intersection of empty set. Since $M$ is a completely distributive lattice, an $LM$-valued soft bornology $\mathcal{B}$ can be characterized by its lower semi-continuous decomposition into parameterized level $L$-bornologies as $\{\mathcal{B}_e^\alpha = \vee_{\beta \preceq \alpha} \mathcal{B}_e^\beta \mid \alpha \in M\}$, for any $e \in E$.

Conversely, one can construct an $LM$-valued soft bornology from a given family of soft $L$-bornologies as follows: Let $\{\mathcal{C}^\alpha \mid \alpha \in c(M)\}$ be a family of soft $L$-bornologies such that $\alpha \leq \beta$ implies $\mathcal{C}_e^\beta \subseteq \mathcal{C}_e^\alpha$. Then the mapping $\mathcal{B} : E \to M^{L^X}$ which is defined by $\mathcal{B}_e(A) = \vee\{\alpha \in c(M) \mid A \in \mathcal{C}_e^\alpha\}$ for all $e \in E$, is an $LM$-valued soft bornology on $X$. In addition, $\mathcal{B}_e^\alpha = \cap\{\mathcal{C}_e^\beta \mid \beta \in c(M),\ \beta \preceq \alpha\}$, for any $e \in E$.

**Theorem 12.** *Let $(\varphi, \psi) : (X, E) \to (Y, \mathcal{B}^2, E_2)$ be a fuzzy soft mapping and let $\mathcal{L} := \{A = \varphi^{-1}(B) \mid B \in L^Y\}$. Then $\mathcal{D} : E \to M^{\mathcal{L}}$ which is described by for any $e \in E$, $\mathcal{D}_e(A) = \mathcal{B}^2_{\psi(e)}(B)$, is an $LM$-valued soft bornology on $X$. Also the soft mapping $(\varphi, \psi)$ is soft bounded according to the induced $LM$-valued soft bornology.*

*Proof:* It is easy to verify, therefore omitted. □

By concerning the above initial bornology construction, we may generalize it to the family of mappings and obtain the following.

**Theorem 13.** *For any given source $\{(\varphi, \psi)_i : (X, E) \to (Y_i, \mathcal{B}^i, E_i) \mid i \in \Gamma\}$, there exists a unique initial lift $\{(\varphi, \psi)_i : (X, \mathcal{B}, E) \to (Y_i, \mathcal{B}^i, E_i) \mid i \in \Gamma\}$ in the category $\mathbf{SBOR}(\mathbf{L}, \mathbf{M})$ of $LM$-valued soft bornological spaces.*

**Theorem 14.** *Let $(\varphi, \psi) : (X, \mathcal{B}^1, E_1) \to (Y, E_2)$ be a surjective fuzzy soft mapping. Then the mapping $\mathcal{B} : E_2 \to M^{L^X}$ which is described by for any $e \in E_1$, $\mathcal{B}_{\psi(e)}(B) = \mathcal{B}^1_e(\varphi^{-1}(B))$, is an $LM$-valued soft bornology on $Y$, which makes the soft mapping bounded.*

*Proof:* Since it is easy to verify, therefore omitted. □

**Corollary 2.** *Products exist in the category* $\mathbf{SBOR}(\mathbf{L}, \mathbf{M})$.

When we compare the definitions proposed in this section, we observe that the category $SL - BOR$ of soft $L$-bornological spaces is exactly a subcategory of the $\mathbf{SBOR}(\mathbf{L}, \mathbf{M})$ of $LM$-valued soft bornological spaces.

## 3 Conclusion

In this study, we defined the notions of soft $L$-bornological spaces and $LM$-valued soft bornological spaces as the extensions of the fuzzy bornological spaces to the more general cases by using the parameters. Therefore, we provided the word of "boundedness" meaningful for the fuzzy soft sets in the presented spaces. We considered fundamental descriptions and their relations in categorical point of view. In conclude, we hope that the results presented in this research will open a new perspective for applied sciences. For further research, we plan to apply soft boundedness in the selection principles theory.

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# A note on $f$-$CLS$-modules

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**Abstract:** Recall that a submodule $N$ of $M$ is called *fully invariant*, if $f(N) \subseteq N$ for all $f \in End(M_R)$. In this paper, we call a submodule $N$ is *$f$-closed*, if $N$ is fully invariant such that $M/N$ is nonsingular. The fundamental properties of $f$-closed submodules are investigated. Our focus is to develop the class of $f$-$CLS$-modules in which every $f$-closed submodule is a direct summand. We obtain characterizations between the generalizations of extending modules and $f$-$CLS$-modules.

**Keywords:** $CLS$-module, Extending module, Fully invariant submodule.

## 1 Introduction

Throughout the paper, all rings are associative with unity and modules are unital right modules. We use $R$ and $M$ to denote such a ring and a module, respectively. Recall that a module $M$ is said to be *extending* [3], if every complement submodule is a direct summand of $M$, or equivalently, every submodule of $M$ is essential in a direct summand of $M$. Several authors have studied the assorted generalization of extending modules with respect to different sets of submodules. A submodule $X$ is called *fully invariant* [4], provided that $g(X) \subseteq X$ for all $g \in End(M_R)$. There are many examples of fully invariant submodules in various algebraic constructions. Authors in [1] introduced a generalization of extending modules by using fully invariant submodules. A module $M$ is called *$FI$-extending* [1], if every fully invariant submodule of $M$ is essential in a direct summand of $M$. Module theoretical properties of $FI$-extending modules were analyzed in [1]. Observe from [7] that a submodule $X$ of $M$ is called *z-closed* when $M/X$ is nonsingular. These submodules are named as complement in [5]. A module $M$ is called a $CLS$-module [7], if every z-closed submodule of $M$ is a direct summand of $M$. Notice that extending modules fulfill $CLS$-module condition.

In this paper, we introduce the notions of $f$-closed submodule and $f$-$CLS$-module. We say a submodule $X$ of $M$ is *$f$-closed*, if $X$ is fully invariant in $M$ and $M/X$ is nonsingular. The properties of $f$-closed submodules are explored. Moreover, we call a module $M$ is an *$f$-$CLS$-module*, if every $f$-closed submodule is a direct summand. The concept of $f$-$CLS$-modules generalizes that of $CLS$-module by asking that only every $f$-closed submodule is a direct summand rather than every $z$-closed submodule. This new class contains not only $CLS$-modules but also $FI$-extending modules. We provide connections between $f$-$CLS$-modules and related notions. Some structural properties such as direct sums and direct summands are discussed. In contrast to $CLS$-modules, the former class is closed under finite direct sums. Moreover, a decomposition result with respect to second singular submodule is acquired. Finally, we give a characterization for the $f$-$CLS$-modules in terms of lifting homomorphisms from $f$-closed submodule to the module. Examples are given to illustrate our results.

For the notations $L \leq M$, $L \leq_e M$, $L \leq_c M$, $L \leq_d M$, $L \trianglelefteq M$, $Z(M)$, $Z_2(M)$ and $End(M_R)$, we mean that $L$ is a right $R$-submodule of $M$, $L$ is an essential submodule of $M$, $L$ is a complement submodule of $M$, $L$ is a direct summand of $M$, $L$ is a fully invariant submodule of $M$, the singular submodule of $M$, the second singular submodule of $M$, and the endomorphism ring of $M$, respectively. Recall that a ring is called *Abelian*, if every idempotent elements are central. For unknown notation and terminology, we refer to [3, 5, 8].

## 2 The class of $f$-$CLS$-modules

We start this section to determine the basic properties of $f$-closed submodules. For the properties of fully invariant submodules, we refer to [4] in the following result.

**Lemma 1.** *[4] Let $M$ be a module.*

*$(i)$ Assume $\{X_i \mid i \in I\}$ is the family of fully invariant submodules of $M$. Then $\bigcap_{i\in I} X_i$ and $\sum_{i\in I} X_i$ are fully invariant submodule of $M$.*

*$(ii)$ Let $X_1 \leq X_2 \leq M$ such that $X_1 \trianglelefteq X_2$ and $X_2 \trianglelefteq M$. Then $X_1 \trianglelefteq M$.*

*$(iii)$ Assume $M = \bigoplus_{i\in I} M_i$ and $X \trianglelefteq M$. Then $X = \bigoplus_{i\in I}(X \cap M_i)$, where $X \cap M_i \trianglelefteq M_i$ for each $i \in I$.*

**Definition 1.** *A submodule $N$ of $M$ is $f$-closed (denoted by, $N \leq_f M$) provided that $N$ is a fully invariant submodule of $M$ and $M/N$ is nonsingular.*

**Lemma 2.** $(i)$ *Any intersection of $f$-closed submodule of $M$ is an $f$-closed submodule of $M$.*

$(ii)$ *Let $A_1, A_2 \leq M$ such that $A_1 \leq A_2$. If $A_1 \leq_f A_2$ and $A_2 \leq_f M$, then $A_1 \leq_f M$.*

*Proof:* $(i)$ Let $X_1 \leq_f M$ and $X_2 \leq_f M$. Then $X_1 \trianglelefteq M$ and $X_2 \trianglelefteq M$ such that $Z(M/X_1) = 0$ and $Z(M/X_2) = 0$. Notice from Lemma 1, $X_1 \cap X_2 \trianglelefteq M$. Define the homomorphism $\theta : M \to M/(X_1 \cap X_2)$, by $\theta(m) = (m + X_1, m + X_2)$. Then $M/(X_1 \cap X_2) \cong \theta(M) \leq (M/X_1) \oplus (M/X_2)$. Since $Z(M/X_1) = 0$ and $Z(M/X_2) = 0$, $Z(\theta(M)) = 0$. It follows that $Z(M/(X_1 \cap X_2)) = 0$. Therefore $X_1 \cap X_2 \leq_f M$.

$(ii)$ Suppose $A_1 \leq_f A_2$ and $A_2 \leq_f M$. It follows from Lemma 1 that $A_1 \trianglelefteq M$. Since $(M/A_1)/(A_2/A_1) \cong M/A_2$, it can be checked that $Z(M/A_1) = 0$. Hence $A_1 \leq_f M$.

□

**Lemma 3.** *Every $f$-closed submodule of $M$ is a complement in $M$.*

*Proof:* Let $X \leq_f M$. Then $X \trianglelefteq M$ and $Z(M/X) = 0$. Assume that there exists $T \leq M$ such that $X \leq_e T \leq_c M$. Then $T/X$ is singular, and hence $T/X \subseteq Z(M/X)$. Since $Z(M/X) = 0$, $T = X$. Thus $X$ has no proper essential extension, $X$ is a complement in $M$.

□

The next example explains that the converse of Lemma 3 need not to be true, in general.

**Example 1.** *Let $F$ be a field and $V_F$ be a vector space over the field $F$ with $dim(V_F) \geq 2$. Consider $R$ as an $R$-module such that*

$$R = \begin{bmatrix} F & V \\ 0 & F \end{bmatrix} = \left\{ \begin{bmatrix} a & v \\ 0 & a \end{bmatrix} : a \in F, v \in V \right\}.$$

*Let $W = \begin{bmatrix} 0 & Fv \\ 0 & 0 \end{bmatrix}$ be a submodule of $R_R$. Then $W$ is a complement in $R_R$, but $W$ is not an $f$-closed submodule in $R_R$.*

**Definition 2.** *A module $M$ is $f$-$CLS$-module, if every $f$-closed submodule of $M$ is a direct summand of $M$.*

In the following result, we provide that the class of $f$-$CLS$-modules contains both $CLS$-modules and $FI$-extending modules.

**Proposition 1.** *Consider the following assertions for a module $M$:*

$(1)$ *$M$ is an extending module.*

$(2)$ *$M$ is a $CLS$-module.*

$(3)$ *$M$ is an $FI$-extending module.*

$(4)$ *$M$ is an $f$-$CLS$-module.*

*Then $(1) \Rightarrow (2) \Rightarrow (4)$ and $(1) \Rightarrow (3) \Rightarrow (4)$, but these arrows are not reversible, in general.*

*Proof:* $(1) \Rightarrow (2)$ and $(1) \Rightarrow (3)$. These implications are clear from [7, Corollary 5] and [2, Proposition 3.7], respectively.

$(2) \Rightarrow (4)$. It is straightforward from definitions.

$(3) \Rightarrow (4)$. Let $V \leq_f M$. Then $V \trianglelefteq M$ such that $Z(M/V) = 0$. Thus $V \leq_e T \leq_d M$ for some submodule $T$ of $M$. Hence $T/V$ is singular, so $T/V \leq Z(M/V) = 0$. Then $V = T$. Consequently, $M$ is an $f$-$CLS$-module.

$(2) \not\Rightarrow (1)$ and $(3) \not\Rightarrow (1)$. Let $M_{\mathbb{Z}} = (\mathbb{Z}/\mathbb{Z}p) \oplus (\mathbb{Z}/\mathbb{Z}p^3)$ for any prime $p$. Then $M_{\mathbb{Z}}$ is a $CLS$-module which is not extending by [7, Example 6]. On the other hand, it is $FI$-extending module by [6, p.1814] and [2, Proposition 3.7].

$(4) \not\Rightarrow (2)$. Let $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$ be the upper triangular matrix ring over $\mathbb{Z}$. It is well known that $R_R$ is $FI$-extending, so it is a $f$-$CLS$-module by the above implication. However, $R_R$ is not extending. Since $Z(R_R) = 0$, $R_R$ is not $CLS$-module by [8, Corollary 5.60].

$(4) \not\Rightarrow (3)$. Let $R = \begin{bmatrix} F & V \\ 0 & F \end{bmatrix} = \left\{ \begin{bmatrix} a & v \\ 0 & a \end{bmatrix} : a \in F, v \in V \right\}$, where$F$ is a field and $V_F$ is a vector space over the field $F$ with $dim(V_F) = 2$. Note that $R$ is a commutative indecomposable ring. Hence $R_R$ is not $FI$-extending. It can be seen that $R_R$ is the only $f$-closed submodule of $R_R$. Therefore it is an $f$-$CLS$-module.

□

**Proposition 2.** $(i)$ *If $M$ is nonsingular, then $M$ is an $f$-$CLS$-module if and only if $M$ is an $FI$-extending module.*

$(ii)$ *If $M$ is a multiplication (resp., duo) module, then $M$ is an $f$-$CLS$-module if and only if $M$ is a $CLS$-module.*

*Proof:* $(i)$ Let $M$ is an $f$-$CLS$-module and $X \trianglelefteq M$. Then there exists a submodule $T$ of $M$ such that $X \leq_e T \leq_c M$. Since $M$ is nonsingular, it follows from [8, Lemma 5.58] and [2, Proposition 2.4] that $T \leq_f M$. Thus $T$ is a direct summand of $M$ which yields that $M$ is $FI$-extending.

$(ii)$ It follows from the fact that every submodule of a multiplication (resp., duo) module is fully invariant.

□

The next result explains when the aforementioned property is inherited by submodules.

**Proposition 3.** *If $M$ be an $f$-$CLS$-module, then every $f$-closed submodule of $M$ is an $f$-$CLS$-module.*

*Proof:* Let $A \leq_f M$ and $Y \leq_f A$. Hence $Y \leq_f M$ by Lemma 2$(ii)$. Therefore $Y$ is a direct summand of $M$. Thus $M = Y \oplus Y'$ for some submodule $Y'$ of $M$. It follows from Lemma 1 that $A = A \cap (Y \oplus Y') = Y \oplus (A \cap Y')$. Then $Y$ is a direct summand of $A$. Thus $A$ is a $f$-$CLS$-module.

□

Observe from [7] that $CLS$-modules is not closed under direct sums. Contrary to $CLS$-modules, $f$-$CLS$-modules enjoy with the direct sums property.

**Theorem 1.** *Let $M = \bigoplus_{i=1}^{k} M_i$ for some $M_i \leq M$. If $M_i$ is an $f$-$CLS$-module for all $1 \leq i \leq k$, then $M$ is an $f$-$CLS$-module.*

*Proof:* The proof follows from the induction argument on $k$. Thus, it is enough to prove the result for the case $k = 2$. Let $M = M_1 \oplus M_2$ and $Y \leq_f M$. Thus $Y$ is a fully invariant submodule in $M$ and $M/Y$ is nonsingular. By Lemma 1, $Y = (Y \cap M_1) \oplus (Y \cap M_2)$ such that $Y \cap M_1 \trianglelefteq M_1$ and $Y \cap M_2 \trianglelefteq M_2$. Observe that $M_1/(Y \cap M_1) \cong (M_1 + Y)/Y$ which is nonsingular. Thus $Y \cap M_1 \leq_f M_1$. Thereby $Y \cap M_1$ is a direct summand of $M_1$. Hence $M_1 = (Y \cap M_1) \oplus T_1$ for some submodule $T_1$ of $M_1$. Similarly, $Y \cap M_2$ is also a direct summand of $M_2$. Thus $M_2 = (Y \cap M_2) \oplus T_2$ for some submodule $T_2$ of $M_2$. Therefore $M = M_1 \oplus M_2 = (Y \cap M_1) \oplus (Y \cap M_2) \oplus T_1 \oplus T_2 = Y \oplus T$, where $T = T_1 \oplus T_2$. Hence $M$ is an $f$-$CLS$-module.

□

We explore when the direct summand of an $f$-$CLS$-module fulfills the $f$-$CLS$-module condition in the subsequent result.

**Proposition 4.** *Let $M = M_1 \oplus M_2$ be an $f$-$CLS$-module for some $M_1, M_2 \leq M$. If $M_1 \trianglelefteq M$, then $M_1$ and $M_2$ are $f$-$CLS$-modules.*

*Proof:* Let $M = M_1 \oplus M_2$ be an $f$-$CLS$-module and $M_1 \trianglelefteq M$. Let $X_1 \leq_f M_1$. Then $X_1 \trianglelefteq M_1$ and $M_1/X_1$ is nonsingular. Hence $X_1 \trianglelefteq M$ by Lemma 1, and $M/X_1$ is nonsingular by [5, Proposition 1.22]. Thus $X_1 \leq_f M$, so $X_1$ is a direct summand of $M$. Then $M = X_1 \oplus X_2$ for some submodules $X_1, X_2$ of $M$. It follows that $M_1 = M_1 \cap (X_1 \oplus X_2) = X_1 \oplus (M_1 \cap X_2)$, so $X_1$ is a direct summand of $M_1$. Thence $M_1$ is an $f$-$CLS$-module. Now, let $X_2 \leq_f M_2$. Further $M_1 \oplus X_2$ is a fully invariant in $M$ by [2, Lemma 4.11]. Observe that $M/(M_1 \oplus X_2) \cong M_2/X_2$ is nonsingular. Hence $M_1 \oplus X_2 \leq_f M$. Thus $M_1 \oplus X_2$ is a direct summand of $M$, so $X_2$ is a direct summand of $M_2$. Hence $M_2$ is a $f$-$CLS$-module.

□

**Corollary 1.** *Let $M$ be a module with an Abelian endomorphism ring. If $M$ is an $f$-$CLS$- module, then every direct summand of $M$ is an $f$-$CLS$-module.*

*Proof:* Observe that every direct summand is fully invariant when the module has an Abelian endomorphism ring. Thus the proof follows from Proposition 4.

□

**Corollary 2.** *Suppose $M = \bigoplus_{i=1}^{k} M_i$, where $M_i \trianglelefteq M$ for all $1 \leq i \leq k$. Then $M$ is an $f$-$CLS$-module if and only if $M_i$ is an $f$-$CLS$-module.*

*Proof:* It is a consequence of Theorem 1 and Proposition 4.

□

Now we acquire the decomposition result with respect to the second singular submodule for the class of $f$-$CLS$-modules.

**Proposition 5.** *Let $M$ be an $f$-$CLS$-module and $K$ an $f$-closed submodule of $M$. Then $M = Z_2(M) \oplus T \oplus Y$ such that $K$ and $Y$ are $f$-$CLS$-modules, where $K = Z_2(M) \oplus T$.*

*Proof:* Let $M$ be an $f$-$CLS$-module and $K \leq_f M$. Then $M = K \oplus K'$ for some submodule $K'$ of $M$. Since $K \trianglelefteq M$, $K$ and $K'$ are $f$-$CLS$-modules by Proposition 4. Recall that $Z_2(M) \subseteq K$, as $M/K$ is nonsingular. Note that $Z_2(M) \trianglelefteq K$ and $Z(K/Z_2(M)) = 0$, so $Z_2(M) \leq_f K$. It follows that $K = Z_2(M) \oplus T$ for some some submodule $T$ of $K$. Therefore $M = Z_2(M) \oplus T \oplus K'$. Hence $K' = Y$ is the desired direct summand.

□

Finally, we obtain a characterization of $f$-$CLS$-modules with respect to lifting homomorphism from $f$-closed submodules to the module.

**Theorem 2.** *$M$ is an $f$-$CLS$-module if and only if for each $f$-closed submodule $N$ of $M$ and $R$-homomorphism $\varphi : N \to X$, there exists $\theta : M \to X$ such that $\theta|_N = \varphi$, for any $R$-modules $X$ and $M$.*

*Proof:* Let $M$ be an $f$-$CLS$-module and $N \leq_f M$. Then $N$ is a direct summand of $M$. Let $f : N \to X$ be a homomorphism. Define $g : M \to X$ by $g = f\pi$, where $\pi : M \to N$ is a projection map. Hence $g|_N = f$. Conversely, assume $M$ fulfills the former property. Let $K \leq_f M$. Hence $\iota : K \to K$ identity map can be extended to $g : M \to K$. Therefore $M = K \oplus \ker g$. Consequently $M$ is an $f$-$CLS$-module.

□

## 3 Conclusions

In this study, we investigate the class of modules whose $f$-closed submodules are direct summands. Some structural properties are obtained. As a further work, some generalizations of extending modules can be explored, by using the class of $f$-closed submodules.

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# A Study on Matrix Sequence of Generalized Third-Order Jacobsthal Numbers

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**Abstract:** In this paper, we introduce and investigate the generalized third-order Jacobsthal matrix sequence and we deal with, in detail, three special cases of this sequence which we call them third-order Jacobsthal, third-order Jacobsthal-Lucas and modified third-order Jacobsthal matrix sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

**Keywords:** Third-order Jacobsthal numbers, Third-order Jacobsthal sequence, Third-order Jacobsthal matrix sequence, Third-order Jacobsthal-Lucas matrix sequence.

## 1 Introduction and preliminaries

A generalized third order Jacobsthal sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = V_{n-1} + V_{n-2} + 2V_{n-3} \tag{1}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{1}{2}V_{-(n-1)} - \frac{1}{2}V_{-(n-2)} + \frac{1}{2}V_{-(n-3)}$$

for $n = 1, 2, 3, ....$ Therefore, recurrence (1) holds for all integer $n$.

Binet formula of generalized third order Jacobsthal numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1 = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0,\ b_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0,\ b_3 = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0. \tag{2}$$

Here, $\alpha, \beta$ and $\gamma$ are the roots of the cubic equation $x^3 - x^2 - x - 2 = 0$. Moreover

$$\begin{aligned} \alpha &= 2 \\ \beta &= \frac{-1+i\sqrt{3}}{2} \\ \gamma &= \frac{-1-i\sqrt{3}}{2}. \end{aligned}$$

Note that

$$\begin{aligned} \alpha+\beta+\gamma &= 1, \\ \alpha\beta+\alpha\gamma+\beta\gamma &= -1, \\ \alpha\beta\gamma &= 2. \end{aligned}$$

The first few generalized third order Jacobsthal numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized third order Jacobsthal numbers

| $n$ | $V_n$ | $V_{-n}$ |
|---|---|---|
| 0 | $V_0$ | |
| 1 | $V_1$ | $\frac{1}{2}V_2-\frac{1}{2}V_1-\frac{1}{2}V_0$ |
| 2 | $V_2$ | $-\frac{1}{4}V_2+\frac{3}{4}V_1-\frac{1}{4}V_0$ |
| 3 | $V_2+V_1+2V_0$ | $-\frac{1}{8}V_2-\frac{1}{8}V_1+\frac{7}{8}V_0$ |
| 4 | $2V_2+3V_1+2V_0$ | $\frac{7}{16}V_2-\frac{9}{16}V_1-\frac{9}{16}V_0$ |
| 5 | $5V_2+4V_1+4V_0$ | $-\frac{9}{32}V_2+\frac{23}{32}V_1-\frac{9}{32}V_0$ |
| 6 | $9V_2+9V_1+10V_0$ | $-\frac{9}{64}V_2-\frac{9}{64}V_1+\frac{55}{64}V_0$ |
| 7 | $18V_2+19V_1+18V_0$ | $\frac{55}{128}V_2-\frac{73}{128}V_1-\frac{73}{128}V_0$ |
| 8 | $37V_2+36V_1+36V_0$ | $-\frac{73}{256}V_2+\frac{183}{256}V_1-\frac{73}{256}V_0$ |

Now we present three special case of the sequence $\{V_n\}$. Third-order Jacobsthal sequence$\{J_n\}_{n\geq 0}$, third-order Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$ and modified third-order Jacobsthal sequence $\{K_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$J_{n+3} = J_{n+2}+J_{n+1}+2J_n, \quad J_0=0, J_1=1, J_2=1, \tag{3}$$

$$j_{n+3} = j_{n+2}+j_{n+1}+2j_n, \quad j_0=2, j_1=1, j_2=5, \tag{4}$$

$$K_{n+3} = K_{n+2}+K_{n+1}+2K_n, \quad K_0=3, K_1=1, K_2=3. \tag{5}$$

The sequences $\{J_n\}_{n\geq 0}$ and $\{j_n\}_{n\geq 0}$ are defined in [5] and $\{K_n\}_{n\geq 0}$ is given in [2]. The sequences $\{J_n\}_{n\geq 0}$, $\{j_n\}_{n\geq 0}$ and $\{K_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} J_{-n} &= -\frac{1}{2}J_{-(n-1)}-\frac{1}{2}J_{-(n-2)}+\frac{1}{2}J_{-(n-3)}, \\ j_{-n} &= -\frac{1}{2}j_{-(n-1)}-\frac{1}{2}j_{-(n-2)}+\frac{1}{2}j_{-(n-3)}, \\ K_{-n} &= -\frac{1}{2}K_{-(n-1)}-\frac{1}{2}K_{-(n-2)}+\frac{1}{2}K_{-(n-3)} \end{aligned}$$

for $n=1,2,3,...$ respectively. For more information on generalized third-order Jacobsthal numbers, see [7].

In the rest of the paper, for easy writing, we drop the superscripts and write $J_n, j_n$ and $K_n$ for $J_n^{(3)}, j_n^{(3)}$ and $K_n^{(3)}$ respectively. Note that $J_n$ is the sequence A077947 in [8] associated with the expansion of $1/(1-x-x^2-2x^3)$, $j_n$ is the sequence A226308 in [8] and $K_n$ is the sequence A186575 in [8] associated with the expansion of $(1+2x+6x^2)/(1-x-x^2-2x^3)$ in powers of $x$.

Next, we present the first few values of the third-order Jacobsthal, third-order Jacobsthal-Lucas and modified third-order Jacobsthal numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $J_n$ | 0 | 1 | 1 | 2 | 5 | 9 | 18 | 37 | 73 | 146 | 293 | 585 | 1170 | 2341 |
| $J_{-n}$ | | 0 | $\frac{1}{2}$ | $-\frac{1}{4}$ | $-\frac{1}{8}$ | $\frac{7}{16}$ | $-\frac{9}{32}$ | $-\frac{9}{64}$ | $\frac{55}{128}$ | $-\frac{73}{256}$ | $-\frac{73}{512}$ | $\frac{439}{1024}$ | $-\frac{585}{2048}$ | $-\frac{585}{4096}$ |
| $j_n$ | 2 | 1 | 5 | 10 | 17 | 37 | 74 | 145 | 293 | 586 | 1169 | 2341 | 4682 | 9361 |
| $j_{-n}$ | | 1 | $-1$ | 1 | $\frac{1}{2}$ | $-\frac{5}{4}$ | $\frac{7}{8}$ | $\frac{7}{16}$ | $-\frac{41}{32}$ | $\frac{55}{64}$ | $\frac{55}{128}$ | $-\frac{329}{256}$ | $\frac{439}{512}$ | $\frac{439}{1024}$ |
| $K_n$ | 3 | 1 | 3 | 10 | 15 | 31 | 66 | 127 | 255 | 514 | 1023 | 2047 | 4098 | 8191 |
| $K_{-n}$ | | $-\frac{1}{2}$ | $-\frac{3}{4}$ | $\frac{17}{8}$ | $-\frac{15}{16}$ | $-\frac{31}{32}$ | $\frac{129}{64}$ | $-\frac{127}{128}$ | $-\frac{255}{256}$ | $\frac{1025}{512}$ | $-\frac{1023}{1024}$ | $-\frac{2047}{2048}$ | $\frac{8193}{4096}$ | $-\frac{8191}{8192}$ |

For all integers $n$, third-order Jacobsthal, Jacobsthal-Lucas and modified Jacobsthal numbers can be expressed using Binet's formulas as

$$\begin{aligned} J_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \\ j_n &= \frac{(2\alpha^2-\alpha+2)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{(2\beta^2-\beta+2)\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{(2\gamma^2-\gamma+2)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}, \\ K_n &= \alpha^n+\beta^n+\gamma^n, \end{aligned}$$

respectively.

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} V_n x^n$ of the sequence $V_n$.

**Lemma 1.** *Suppose that $f_{V_n}(x)=\sum\limits_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized third-order Jacobsthal sequence $\{V_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} V_n x^n$ is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2}{1-x-x^2-2x^3}. \tag{6}$$

The previous Lemma gives the following results as particular examples.

**Corollary 1.** *Generated functions of third-order Jacobsthal, Jacobsthal-Lucas and modified Jacobsthal numbers are*

$$\begin{aligned}\sum_{n=0}^{\infty} J_n x^n &= \frac{x}{1-x-x^2-2x^3},\\ \sum_{n=0}^{\infty} j_n x^n &= \frac{2-x+2x^2}{1-x-x^2-2x^3},\\ \sum_{n=0}^{\infty} K_n x^n &= \frac{3-2x-x^2}{1-x-x^2-2x^3},\end{aligned}$$

*respectively.*

## 2 The matrix sequences of third-order Jacobsthal and third-order Jacobsthal-Lucas numbers

Recently, there have been so many studies of the sequences of numbers in the literature that concern about subsequences of the Horadam (Generalized Fibonacci) numbers and generalized Tribonacci numbers such as Fibonacci, Lucas, Pell and Jacobsthal numbers; third-order Pell, third-order Pell-Lucas, Padovan, Perrin, Padovan-Perrin, third-order Jacobsthal, third order Jacobsthal and third order Jacobsthal-Lucas numbers. The sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. On the other hand, the matrix sequences have taken so much interest for different type of numbers. We present some works on matrix sequences of the numbers in the following Table 2.

Table 2. A few special study on the matrix sequences of the numbers.

| Name of sequence | work on the matrix sequences of the numbers |
|---|---|
| Generalized Fibonacci | [3, 4, 6, 14–18, 21] |
| Generalized Tribonacci | [1, 10–12, 19, 20] |
| Generalized Tetranacci | [9] |

In this section we define generalized third-order Jacobsthal matrix sequence and investigate their properties.

**Definition 2.** *For any integer $n \geq 0$, the third-order Jacobsthal matrix $(\mathcal{V}_n)$ and third-order Jacobsthal-Lucas matrix $(\mathcal{M}_n)$ are defined by*

$$\mathcal{V}_n = \mathcal{V}_{n-1} + \mathcal{V}_{n-2} + 2\mathcal{V}_{n-3} \tag{7}$$

*with initial conditions*

$$\begin{aligned}\mathcal{V}_0 &= \begin{pmatrix} V_1 & V_2-V_1 & 2V_0 \\ V_0 & V_1-V_0 & V_2-V_1-V_0 \\ \frac{1}{2}(V_2-V_1-V_0) & \frac{1}{2}(3V_0+V_1-V_2) & \frac{1}{2}(3V_1-V_0-V_2) \end{pmatrix},\\ \mathcal{V}_1 &= \begin{pmatrix} V_2 & 2V_0+V_1 & 2V_1 \\ V_1 & V_2-V_1 & 2V_0 \\ V_0 & V_1-V_0 & V_2-V_1-V_0 \end{pmatrix},\\ \mathcal{V}_2 &= \begin{pmatrix} 2V_0+V_1+V_2 & 2V_1+V_2 & 2V_2 \\ V_2 & 2V_0+V_1 & 2V_1 \\ V_1 & V_2-V_1 & 2V_0 \end{pmatrix}.\end{aligned}$$

*The sequence $\{\mathcal{V}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining*

$$\mathcal{V}_{-n} = -\frac{1}{2}\mathcal{V}_{-(n-1)} - \frac{1}{2}\mathcal{V}_{-(n-2)} + \frac{1}{2}\mathcal{V}_{-(n-3)}$$

*for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (7) holds for all integers $n$.*

Three special cases of generalized third-order Jacobsthal matrix sequence (take $V_n = J_n, V_n = j_n, V_n = K_n$, respectively) can be defined as follows.

**Definition 3.** *For any integer $n \geq 0$, the third-order Jacobsthal matrix $(\mathcal{J}_n)$ and third-order Jacobsthal-Lucas matrix $(\mathcal{M}_n)$ and modified third-order Jacobsthal matrix $(\mathcal{K}_n)$ are defined by*

$$\begin{aligned}\mathcal{J}_n &= \mathcal{J}_{n-1} + \mathcal{J}_{n-2} + 2\mathcal{J}_{n-3},\\ \mathcal{M}_n &= \mathcal{M}_{n-1} + \mathcal{M}_{n-2} + 2\mathcal{M}_{n-3},\\ \mathcal{K}_n &= \mathcal{K}_{n-1} + \mathcal{K}_{n-2} + 2\mathcal{K}_{n-3},\end{aligned}$$

*respectively, with initial conditions*

$$\begin{aligned}
\mathcal{J}_0 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{J}_1 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathcal{J}_2 = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}, \\
\mathcal{M}_0 &= \begin{pmatrix} 1 & 4 & 4 \\ 2 & -1 & 2 \\ 1 & 1 & -2 \end{pmatrix}, \mathcal{M}_1 = \begin{pmatrix} 5 & 5 & 2 \\ 1 & 4 & 4 \\ 2 & -1 & 2 \end{pmatrix}, \mathcal{M}_2 = \begin{pmatrix} 10 & 7 & 10 \\ 5 & 5 & 2 \\ 1 & 4 & 4 \end{pmatrix}, \\
\mathcal{K}_0 &= \begin{pmatrix} 1 & 2 & 6 \\ 3 & -2 & -1 \\ -\frac{1}{2} & \frac{7}{2} & -\frac{3}{2} \end{pmatrix}, \mathcal{K}_1 = \begin{pmatrix} 3 & 7 & 2 \\ 1 & 2 & 6 \\ 3 & -2 & -1 \end{pmatrix}, \mathcal{K}_2 = \begin{pmatrix} 10 & 5 & 6 \\ 3 & 7 & 2 \\ 1 & 2 & 6 \end{pmatrix}
\end{aligned}$$

The sequences $\{\mathcal{J}_n\}_{n\geq 0}$, $\{\mathcal{M}_n\}_{n\geq 0}$ and $\{\mathcal{K}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
\mathcal{J}_{-n} &= -\frac{1}{2}\mathcal{J}_{-(n-1)} - \frac{1}{2}\mathcal{J}_{-(n-2)} + \frac{1}{2}\mathcal{J}_{-(n-3)}, \\
\mathcal{M}_{-n} &= -\frac{1}{2}\mathcal{M}_{-(n-1)} - \frac{1}{2}\mathcal{M}_{-(n-2)} + \frac{1}{2}\mathcal{M}_{-(n-3)}, \\
\mathcal{K}_{-n} &= -\frac{1}{2}\mathcal{K}_{-(n-1)} - \frac{1}{2}\mathcal{K}_{-(n-2)} + \frac{1}{2}\mathcal{K}_{-(n-3)},
\end{aligned}$$

for $n = 1, 2, 3, ...$ respectively.

The following theorem gives the $n$th general terms of the generalized third-order Jacobsthal matrix sequence.

**Theorem 4.** *For any integer $n$, we have the following formulas of the matrix sequences:*

$$\mathcal{V}_n = \begin{pmatrix} V_{n+1} & V_n + 2V_{n-1} & 2V_n \\ V_n & V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2} + 2V_{n-3} & 2V_{n-2} \end{pmatrix} \tag{8}$$

Proof. Suppose that $n \geq 0$. We prove (8) by strong mathematical induction on $n$. If $n = 0$ then, since $V_{-1} = \frac{1}{2}V_2 - \frac{1}{2}V_1 - \frac{1}{2}V_0$, $V_{-2} = -\frac{1}{4}V_2 + \frac{3}{4}V_1 - \frac{1}{4}V_0$, $V_{-3} = -\frac{1}{8}V_2 - \frac{1}{8}V_1 + \frac{7}{8}V_0$, we have

$$\begin{aligned}
\mathcal{V}_0 &= \begin{pmatrix} V_1 & V_0 + 2V_{-1} & 2V_0 \\ V_0 & V_{-1} + 2V_{-2} & 2V_{-1} \\ V_{-1} & V_{-2} + 2V_{-3} & 2V_{-2} \end{pmatrix} \\
&= \begin{pmatrix} V_1 & V_2 - V_1 & 2V_0 \\ V_0 & V_1 - V_0 & V_2 - V_1 - V_0 \\ \frac{1}{2}(V_2 - V_1 - V_0) & \frac{1}{2}(3V_0 + V_1 - V_2) & \frac{1}{2}(3V_1 - V_0 - V_2) \end{pmatrix}
\end{aligned}$$

which is true. Assume that the equality holds for $n \leq k$. For $n = k + 1$, we have

$$\begin{aligned}
\mathcal{V}_{k+1} &= \mathcal{V}_{k+1-1} + \mathcal{V}_{k+1-2} + 2\mathcal{V}_{k+1-3} \\
&= \mathcal{V}_k + \mathcal{V}_{k-1} + 2\mathcal{V}_{k-2} \\
&= \begin{pmatrix} V_{k+1} & V_k + 2V_{k-1} & 2V_k \\ V_k & V_{k-1} + 2V_{k-2} & 2V_{k-1} \\ V_{k-1} & V_{k-2} + 2V_{k-3} & 2V_{k-2} \end{pmatrix} + \begin{pmatrix} V_{k-1+1} & V_{k-1} + 2V_{k-1-1} & 2V_{k-1} \\ V_{k-1} & V_{k-1-1} + 2V_{k-1-2} & 2V_{k-1-1} \\ V_{k-1-1} & V_{k-1-2} + 2V_{k-1-3} & 2V_{k-1-2} \end{pmatrix} \\
&\quad +2\begin{pmatrix} V_{k-2+1} & V_{k-2} + 2V_{k-2-1} & 2V_{k-2} \\ V_{k-2} & V_{k-2-1} + 2V_{k-2-2} & 2V_{k-2-1} \\ V_{k-2-1} & V_{k-2-2} + 2V_{k-2-3} & 2V_{k-2-2} \end{pmatrix} \\
&= \begin{pmatrix} V_{k+1} + V_k + 2V_{k-1} & V_k + 3V_{k-1} + 4V_{k-2} + 4V_{k-3} & 2V_k + 2V_{k-1} + 4V_{k-2} \\ V_k + V_{k-1} + 2V_{k-2} & V_{k-1} + 3V_{k-2} + 4V_{k-3} + 4V_{k-4} & 2V_{k-1} + 2V_{k-2} + 4V_{k-3} \\ V_{k-1} + V_{k-2} + 2V_{k-3} & V_{k-2} + 3V_{k-3} + 4V_{k-4} + 4V_{k-5} & 2V_{k-2} + 2V_{k-3} + 4V_{k-4} \end{pmatrix} \\
&= \begin{pmatrix} V_{k+2} & V_{k+1} + 2V_k & 2V_{k+1} \\ V_{k+1} & V_k + 2V_{k-1} & 2V_k \\ V_k & V_{k-1} + 2V_{k-2} & 2V_{k-1} \end{pmatrix} \\
&= \begin{pmatrix} V_{k+1+1} & V_{k+1} + 2V_{k+1-1} & 2V_{k+1} \\ V_{k+1} & V_{k+1-1} + 2V_{k+1-2} & 2V_{k+1-1} \\ V_{k+1-1} & V_{k+1-2} + 2V_{k+1-3} & 2V_{k+1-2} \end{pmatrix}.
\end{aligned}$$

Thus, by strong induction on $k + 1$, this proves (8).

For the case $n \leq 0$, similarly, (8) can be proved by strong mathematical induction on $n$. □

 

The following theorem gives the $n$th general terms of the third-order Jacobsthal, third-order Jacobsthal-Lucas and modified third-order Jacobsthal matrix sequences.

**Corollary 2.** *For any integer $n$, we have the following formulas of the matrix sequences:*

$$\begin{aligned} \mathcal{J}_n &= \begin{pmatrix} J_{n+1} & J_n + 2J_{n-1} & 2J_n \\ J_n & J_{n-1} + 2J_{n-2} & 2J_{n-1} \\ J_{n-1} & J_{n-2} + 2J_{n-3} & 2J_{n-2} \end{pmatrix}, \\ \mathcal{M}_n &= \begin{pmatrix} j_{n+1} & j_n + 2j_{n-1} & 2j_n \\ j_n & j_{n-1} + 2j_{n-2} & 2j_{n-1} \\ j_{n-1} & j_{n-2} + 2j_{n-3} & 2j_{n-2} \end{pmatrix}, \\ \mathcal{K}_n &= \begin{pmatrix} K_{n+1} & K_n + 2K_{n-1} & 2K_n \\ K_n & K_{n-1} + 2K_{n-2} & 2K_{n-1} \\ K_{n-1} & K_{n-2} + 2K_{n-3} & 2K_{n-2} \end{pmatrix}. \end{aligned}$$

We now give the Binet's formula for the generalized third-order Jacobsthal matrix sequence.

**Theorem 5.** *For every integer $n$, the Binet's formula of the generalized third-order Jacobsthal matrix sequence are given by*

$$\mathcal{V}_n = A\alpha^n + B\beta^n + C\gamma^n$$

*where*

$$A = \frac{\alpha\mathcal{V}_2 + \alpha(\alpha-1)\mathcal{V}_1 + 2\mathcal{V}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, B = \frac{\beta\mathcal{V}_2 + \beta(\beta-1)\mathcal{V}_1 + 2\mathcal{V}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, C = \frac{\gamma\mathcal{V}_2 + \gamma(\gamma-1)\mathcal{V}_1 + 2\mathcal{V}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}$$

Proof. We need to prove the theorem only for $n \geq 0$. By the assumption, the characteristic equation of (7) is $x^3 - x^2 - x - 2 = 0$ and the roots of it are $\alpha, \beta$ and $\gamma$. So it's general solution is given by

$$\mathcal{V}_n = A\alpha^n + B\beta^n + C\gamma^n.$$

Using initial condition which is given in Definition 2, and also applying lineer algebra operations, we obtain the matrices $A, B, C$ as desired. This gives the formula for $\mathcal{V}_n$. □

The following theorem gives the Binet's formulas of the third-order Jacobsthal, third-order Jacobsthal-Lucas and modified third-order Jacobsthal matrix sequences.

**Corollary 3.** *For every integer $n$, the Binet formulas of the third-order Jacobsthal and third-order Jacobsthal-Lucas matrix sequences are given by*

$$\begin{aligned} \mathcal{J}_n &= A_1\alpha^n + B_1\beta^n + C_1\gamma^n, \\ \mathcal{M}_n &= A_2\alpha^n + B_2\beta^n + C_2\gamma^n, \\ \mathcal{K}_n &= A_3\alpha^n + B_3\beta^n + C_3\gamma^n, \end{aligned}$$

*where*

$$\begin{aligned} A_1 &= \frac{\alpha\mathcal{J}_2 + \alpha(\alpha-1)\mathcal{J}_1 + 2\mathcal{J}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, B_1 = \frac{\beta\mathcal{J}_2 + \beta(\beta-1)\mathcal{J}_1 + 2\mathcal{J}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, C_1 = \frac{\gamma\mathcal{J}_2 + \gamma(\gamma-1)\mathcal{J}_1 + 2\mathcal{J}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}, \\ A_2 &= \frac{\alpha\mathcal{M}_2 + \alpha(\alpha-1)\mathcal{M}_1 + 2\mathcal{M}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, B_2 = \frac{\beta\mathcal{M}_2 + \beta(\beta-1)\mathcal{M}_1 + 2\mathcal{M}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, C_2 = \frac{\gamma\mathcal{M}_2 + \gamma(\gamma-1)\mathcal{M}_1 + 2\mathcal{M}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}, \\ A_3 &= \frac{\alpha\mathcal{K}_2 + \alpha(\alpha-1)\mathcal{K}_1 + 2\mathcal{K}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}, B_3 = \frac{\beta\mathcal{K}_2 + \beta(\beta-1)\mathcal{K}_1 + 2\mathcal{K}_0}{\beta(\beta-\gamma)(\beta-\alpha)}, C_3 = \frac{\gamma\mathcal{K}_2 + \gamma(\gamma-1)\mathcal{K}_1 + 2\mathcal{K}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}. \end{aligned}$$

Now we will obtain these functions in terms of generalized third-order Jacobsthal matrix sequence as a consequence of Theorems 4 and 5. To do this, we will give the formulas for these numbers by means of the related matrix sequences. In fact, in the proof of next corollary, we will just compare the linear combination of the 2nd row and 1st column entries of the matrices.

**Corollary 4.** *For every integers $n$, the Binet's formulas for the generalized third-order Jacobsthal numbers is given as*

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

*where*

$$b_1 = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0,\ b_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0,\ b_3 = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0.$$

*Proof.* From Theorem 5, we have

$$\begin{aligned}
\mathcal{V}_n &= A\alpha^n + B\beta^n + C\gamma^n \\
&= \frac{\alpha\mathcal{V}_2 + \alpha(\alpha-1)\mathcal{V}_1 + 2\mathcal{V}_0}{\alpha(\alpha-\gamma)(\alpha-\beta)}\alpha^n + \frac{\beta\mathcal{V}_2 + \beta(\beta-1)\mathcal{V}_1 + 2\mathcal{V}_0}{\beta(\beta-\gamma)(\beta-\alpha)}\beta^n \\
&\quad + \frac{\gamma\mathcal{V}_2 + \gamma(\gamma-1)\mathcal{V}_1 + 2\mathcal{V}_0}{\gamma(\gamma-\beta)(\gamma-\alpha)}\gamma^n \\
&= \frac{\alpha^{n-1}}{(\alpha-\gamma)(\alpha-\beta)}\begin{pmatrix} & \cdot & \cdot & \cdot \\ \alpha V_2 + \alpha(\alpha-1)V_1 + 2V_0 & & \cdot & \cdot \\ & \cdot & \cdot & \cdot \end{pmatrix} \\
&\quad + \frac{\beta^{n-1}}{(\beta-\gamma)(\beta-\alpha)}\begin{pmatrix} & \cdot & \cdot & \cdot \\ \beta V_2 + \beta(\beta-1)V_1 + 2V_0 & & \cdot & \cdot \\ & \cdot & \cdot & \cdot \end{pmatrix} \\
&\quad + \frac{\gamma^{n-1}}{(\gamma-\beta)(\gamma-\alpha)}\begin{pmatrix} & \cdot & \cdot & \cdot \\ \gamma V_2 + \gamma(\gamma-1)V_1 + 2V_0 & & \cdot & \cdot \\ & \cdot & \cdot & \cdot \end{pmatrix}.
\end{aligned}$$

(we only write the 2nd row and 1st column entries of the matrices). By Theorem 4, we know that

$$\mathcal{V}_n = \begin{pmatrix} V_{n+1} & V_n + 2V_{n-1} & 2V_n \\ V_n & V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2} + 2V_{n-3} & 2V_{n-2} \end{pmatrix}.$$

Now, if we compare the 2nd row and 1st column entries with the matrices in the above two equations, then we obtain

$$\begin{aligned}
V_n &= \frac{\alpha^{n-1}}{(\alpha-\gamma)(\alpha-\beta)}(\alpha V_2 + \alpha(\alpha-1)V_1 + 2V_0) \\
&\quad + \frac{\beta^{n-1}}{(\beta-\gamma)(\beta-\alpha)}(\beta V_2 + \beta(\beta-1)V_1 + 2V_0) \\
&\quad + \frac{\gamma^{n-1}}{(\gamma-\beta)(\gamma-\alpha)}(\gamma V_2 + \gamma(\gamma-1)V_1 + 2V_0) \\
&= \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}
\end{aligned}$$

where

$$b_1 = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0,\ b_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0,\ b_3 = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0.$$

Note that

$$\begin{aligned}
\alpha V_2 + \alpha(\alpha-1)V_1 + 2V_0 &= \alpha(V_2 + (\alpha-1)V_1 + \frac{2}{\alpha}V_0) \\
&= \alpha(V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0) = \alpha b_1, \\
\beta V_2 + \beta(\beta-1)V_1 + 2V_0 &= \beta(V_2 + (\beta-1)V_1 + \frac{2}{\beta}V_0) \\
&= \beta(V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0) = \beta b_2, \\
\gamma V_2 + \gamma(\gamma-1)V_1 + 2V_0 &= \gamma(V_2 + (\gamma-1)V_1 + \frac{2}{\gamma}V_0) \\
&= \gamma(V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0) = \gamma b_3.
\end{aligned}$$

□

Now, we present summation formulas for the generalized third-order Jacobsthal matrix sequence.

**Theorem 6.** *For all integers $m, j$ we have*

$$\sum_{k=0}^{n-1}\mathcal{V}_{mk+j} = \frac{\mathcal{V}_{mn+m+j} + 2^m\mathcal{V}_{mn-m+j} + (1-K_m)\mathcal{V}_{mn+j} - \mathcal{V}_{m+j} - 2^m\mathcal{V}_{j-m} + (K_m-1)\mathcal{V}_j}{K_m + 2^m(1-K_{-m}) - 1} \tag{9}$$

*Proof. Note that*

$$\begin{aligned}
\sum_{i=0}^{n-1}\mathcal{V}_{mi+j} &= \sum_{i=0}^{n-1}(A\alpha^{mi+j} + B\beta^{mi+j} + C\gamma^{mi+j}) \\
&= A\alpha^j\left(\frac{\alpha^{mn}-1}{\alpha^m-1}\right) + B\beta^j\left(\frac{\beta^{mn}-1}{\beta^m-1}\right) + C\gamma^j\left(\frac{\gamma^{mn}-1}{\gamma^m-1}\right)
\end{aligned}$$

Simplifying and rearranging the last equalities in the last two expression imply (9) as required. □

As in Corollary 4, in the proof of next Corollary, we just compare the linear combination of the 2nd row and 1st column entries of the relevant matrices to obtain summation formula for the generalized third-order Jacobsthal sequence..

**Corollary 5.** *For all integers $m, j$ we have*

$$\sum_{k=0}^{n-1} V_{mk+j} = \frac{V_{mn+m+j} + 2^m V_{mn-m+j} + (1-K_m)V_{mn+j} - V_{m+j} - 2^m V_{j-m} + (K_m - 1)V_j}{K_m + 2^m(1-K_{-m}) - 1}.$$

We now give generating functions of $\mathcal{V}_n$ .

**Theorem 7.** *The generating function for the generalized third-order Jacobsthal matrix sequences is given as*

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{V}_n x^n &= \frac{\mathcal{V}_0 + (\mathcal{V}_1 - \mathcal{V}_0)x + (\mathcal{V}_2 - \mathcal{V}_1 - \mathcal{V}_0)x^2}{1 - x - x^2 - 2x^3} \\ &= \frac{1}{1 - x - x^2 - 2x^3} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \end{aligned}$$

*where*

$$\begin{aligned} a_{11} &= V_1 + (-V_1 + V_2)x + 2V_0 x^2 \\ a_{21} &= V_0 + (-V_0 + V_1)x + (-V_0 - V_1 + V_2)x^2 \\ a_{31} &= \frac{1}{2}(V_2 - V_1 - V_0) + \frac{1}{2}(3V_0 + V_1 - V_2)x + \frac{1}{2}(-V_2 + 3V_1 - V_0)x^2 \end{aligned}$$

$$\begin{aligned} a_{12} &= V_2 - V_1 + (2V_0 + 2V_1 - V_2)x + (-2V_0 + 2V_1)x^2 \\ a_{22} &= V_1 - V_0 + (V_0 - 2V_1 + V_2)x + (3V_0 + V_1 - V_2)x^2 \\ a_{32} &= \frac{1}{2}(3V_2 - 5V_1 - V_0)x^2 + \frac{1}{2}(V_2 + V_1 - 5V_0)x + \frac{1}{2}(3V_0 + V_1 - V_2) \end{aligned}$$

$$\begin{aligned} a_{13} &= (-2V_0 - 2V_1 + 2V_2)x^2 + (-2V_0 + 2V_1)x + 2V_0 \\ a_{23} &= V_2 - V_1 - V_0 + (3V_0 + V_1 - V_2)x + (-V_0 + 3V_1 - V_2)x^2 \\ a_{33} &= \frac{1}{2}(3V_2 - 5V_1 - V_0)x + \frac{1}{2}(7V_0 - V_1 - V_2)x^2 + \frac{1}{2}(-V_2 + 3V_1 - V_0) \end{aligned}$$

*Proof.* Suppose that $g(x) = \sum_{n=0}^{\infty} \mathcal{V}_n x^n$ is the generating function for the sequence $\{\mathcal{V}_n\}_{n\geq 0}$. Using the definition of the matrix sequence of generalized third-order Jacobsthal numbers (2), and substracting $x\sum_{n=0}^{\infty} \mathcal{V}_n x^n$, $x^2\sum_{n=0}^{\infty} \mathcal{V}_n x^n$ and $2x^3\sum_{n=0}^{\infty} \mathcal{V}_n x^n$ from $\sum_{n=0}^{\infty} \mathcal{V}_n x^n$ we obtain

$$\begin{aligned} (1 - x - x^2 - 2x^3)\sum_{n=0}^{\infty} \mathcal{V}_n x^n &= \sum_{n=0}^{\infty} \mathcal{V}_n x^n - x\sum_{n=0}^{\infty} \mathcal{V}_n x^n - x^2\sum_{n=0}^{\infty} \mathcal{V}_n x^n - 2x^3\sum_{n=0}^{\infty} \mathcal{V}_n x^n \\ &= \sum_{n=0}^{\infty} \mathcal{V}_n x^n - \sum_{n=0}^{\infty} \mathcal{V}_n x^{n+1} - \sum_{n=0}^{\infty} \mathcal{V}_n x^{n+2} - 2\sum_{n=0}^{\infty} \mathcal{V}_n x^{n+3} \\ &= \sum_{n=0}^{\infty} \mathcal{V}_n x^n - \sum_{n=1}^{\infty} \mathcal{V}_{n-1} x^n - \sum_{n=2}^{\infty} \mathcal{V}_{n-2} x^n - 2\sum_{n=3}^{\infty} \mathcal{V}_{n-3} x^n \\ &= (\mathcal{V}_0 + \mathcal{V}_1 x + \mathcal{V}_2 x^2) - (\mathcal{V}_0 x + \mathcal{V}_1 x^2) - \mathcal{V}_0 x^2 \\ &\quad + \sum_{n=3}^{\infty} (\mathcal{V}_n - \mathcal{V}_{n-1} - \mathcal{V}_{n-2} - 2\mathcal{V}_{n-3}) x^n \\ &= \mathcal{V}_0 + \mathcal{V}_1 x + \mathcal{V}_2 x^2 - \mathcal{V}_0 x - \mathcal{V}_1 x^2 - \mathcal{V}_0 x^2 \\ &= \mathcal{V}_0 + (\mathcal{V}_1 - \mathcal{V}_0)x + (\mathcal{V}_2 - \mathcal{V}_1 - \mathcal{V}_0)x^2. \end{aligned}$$

Rearranging above equation, we obtain

$$\sum_{n=0}^{\infty} \mathcal{V}_n x^n = \frac{\mathcal{V}_0 + (\mathcal{V}_1 - \mathcal{V}_0)x + (\mathcal{V}_2 - \mathcal{V}_1 - \mathcal{V}_0)x^2}{1 - x - x^2 - 2x^3}$$

which equals the $\sum_{n=0}^{\infty} \mathcal{V}_n x^n$ in the Theorem. This completes the proof. □

The following corollary gives the generating functions of the third-order Jacobsthal, third-order Jacobsthal-Lucas and modified third-order Jacobsthal matrix sequences.

**Corollary 6.** *The generating functions for the third-order Jacobsthal, third-order Jacobsthal-Lucas and modified third-order Jacobsthal matrix sequences are given as*

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{J}_n x^n &= \frac{1}{1-x-x^2-2x^3}\begin{pmatrix} 1 & 2x^2+x & 2x \\ x & 1-x & 2x^2 \\ x^2 & x-x^2 & -x^2-x+1 \end{pmatrix}, \\
\sum_{n=0}^{\infty}\mathcal{M}_n x^n &= \frac{1}{1-x-x^2-2x^3}\begin{pmatrix} 4x^2+4x+1 & -2x^2+x+4 & 4x^2-2x+4 \\ 2x^2-x+2 & 2x^2+5x-1 & -4x^2+2x+2 \\ -2x^2+x+1 & 4x^2-2x+1 & 4x^2+4x-2 \end{pmatrix}, \\
\sum_{n=0}^{\infty}\mathcal{K}_n x^n &= \frac{1}{1-x-x^2-2x^3}\begin{pmatrix} 6x^2+2x+1 & -4x^2+5x+2 & -2x^2-4x+6 \\ -x^2-2x+3 & 7x^2+4x-2 & -3x^2+7x-1 \\ -\frac{3}{2}x^2+\frac{7}{2}x-\frac{1}{2} & \frac{1}{2}x^2-\frac{11}{2}x+\frac{7}{2} & \frac{17}{2}x^2+\frac{1}{2}x-\frac{3}{2} \end{pmatrix},
\end{aligned}$$

The well known generating function for generalized third-order Jacobsthal numbers is as in (6). However, we will obtain these functions in terms of generalized third-order Jacobsthal matrix sequences as a consequence of Theorem 7. To do this, we will again compare the the 2nd row and 1st column entries with the matrices in Theorem 7. Thus we have the following corollary.

**Corollary 7.** *The generating function for the generalized third-order Jacobsthal sequence $\{V_n\}$ is given as*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2}{1-x-x^2-2x^3}.$$

Using Theorem 4 and Corollary 2, we see that

$$\begin{aligned}
\mathcal{V}_{-1} &= \begin{pmatrix} V_0 & V_1-V_0 & V_2-V_1-V_0 \\ \frac{1}{2}(V_2-V_1-V_0) & \frac{1}{2}(3V_0+V_1-V_2) & \frac{1}{2}(3V_1-V_0-V_2) \\ \frac{1}{4}(3V_1-V_0-V_2) & \frac{1}{4}(3V_2-5V_1-V_0) & \frac{1}{4}(7V_0-V_1-V_2) \end{pmatrix}, \\
\mathcal{V}_{-2} &= \begin{pmatrix} \frac{1}{2}(V_2-V_1-V_0) & \frac{1}{2}(3V_0+V_1-V_2) & \frac{1}{2}(3V_1-V_0-V_2) \\ \frac{1}{4}(3V_1-V_0-V_2) & \frac{1}{4}(3V_2-5V_1-V_0) & \frac{1}{4}(7V_0-V_1-V_2) \\ \frac{1}{8}(7V_0-V_1-V_2) & \frac{1}{8}(7V_1-9V_0-V_2) & \frac{1}{8}(7V_2-9V_1-9V_0) \end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{J}_{-1} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \ \mathcal{J}_{-2} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \end{pmatrix}, \\
\mathcal{M}_{-1} &= \begin{pmatrix} 2 & -1 & 2 \\ 1 & 1 & -2 \\ -1 & 2 & 2 \end{pmatrix}, \ \mathcal{M}_{-2} = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 2 \\ 1 & -2 & 1 \end{pmatrix}, \\
\mathcal{K}_{-1} &= \begin{pmatrix} 3 & -2 & -1 \\ -\frac{1}{2} & \frac{7}{2} & -\frac{3}{2} \\ -\frac{3}{4} & \frac{1}{4} & \frac{17}{4} \end{pmatrix}, \ \mathcal{K}_{-2} = \begin{pmatrix} -\frac{1}{2} & \frac{7}{2} & -\frac{3}{2} \\ -\frac{3}{4} & \frac{1}{4} & \frac{17}{4} \\ \frac{17}{8} & -\frac{23}{8} & -\frac{15}{8} \end{pmatrix}.
\end{aligned}$$

We now give generating functions of the generalized third-order Jacobsthal matrix sequence $\mathcal{V}_n$ for negative indices.

**Theorem 8.** *For negative indices, the generating function for the generalized third-order Jacobsthal matrix sequence is given as*

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n &= \frac{2\mathcal{V}_0+(\mathcal{V}_2-\mathcal{V}_1)x+\mathcal{V}_1x^2}{2+x+x^2-x^3} \\
&= \frac{1}{2+x+x^2-x^3}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}
\end{aligned}$$

*where*

$$\begin{aligned}
b_{11} &= 2V_1+(2V_0+V_1)x+V_2x^2 \\
b_{21} &= 2V_0+(-V_1+V_2)x+V_1x^2 \\
b_{31} &= V_2-V_1-V_0+(-V_0+V_1)x+V_0x^2
\end{aligned}$$

 

*and*

$$\begin{aligned} b_{12} &= 2V_2 - 2V_1 + x\left(V_1 - 2V_0 + V_2\right) + x^2\left(2V_0 + V_1\right) \\ b_{22} &= 2V_1 - 2V_0 + \left(2V_0 + 2V_1 - V_2\right)x + \left(-V_1 + V_2\right)x^2 \\ b_{32} &= -V_2 + V_1 + 3V_0 + \left(V_0 - 2V_1 + V_2\right)x + \left(-V_0 + V_1\right)x^2 \end{aligned}$$

*and*

$$\begin{aligned} b_{13} &= 4V_0 + \left(-2V_1 + 2V_2\right)x + 2V_1x^2 \\ b_{23} &= 2V_2 - 2V_1 - 2V_0 + \left(-2V_0 + 2V_1\right)x + 2V_0x^2 \\ b_{33} &= -V_2 + 3V_1 - V_0 + \left(3V_0 + V_1 - V_2\right)x + \left(-V_0 - V_1 + V_2\right)x^2 \end{aligned}$$

*Proof.* Then, using Definition 2, and substracting $-\frac{1}{2}x\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n$, $-\frac{1}{2}x^2\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n$ and $\frac{1}{2}x^3\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n$ from $\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n$ we obtain

$$\begin{aligned} (1+\frac{1}{2}x+\frac{1}{2}x^2-\frac{1}{2}x^3)\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n &= \sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n + \frac{1}{2}x\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n + \frac{1}{2}x^2\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n - \frac{1}{2}x^3\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n \\ &= \sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n + \frac{1}{2}\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^{n+1} + \frac{1}{2}\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^{n+2} - \frac{1}{2}\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^{n+3} \\ &= \sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n + \frac{1}{2}\sum_{n=1}^{\infty}\mathcal{V}_{-n+1}x^n + \frac{1}{2}\sum_{n=2}^{\infty}\mathcal{V}_{-n+2}x^n - \frac{1}{2}\sum_{n=3}^{\infty}\mathcal{V}_{-n+3}x^n \\ &= (\mathcal{V}_0+\mathcal{V}_{-1}x+\mathcal{V}_{-2}x^2) + \frac{1}{2}(\mathcal{V}_0x+\mathcal{V}_{-1}x^2) + \frac{1}{2}\mathcal{V}_0x^2 \\ &\quad + \sum_{n=3}^{\infty}(\mathcal{V}_{-n} + \frac{1}{2}\mathcal{V}_{-n+1} + \frac{1}{2}\mathcal{V}_{-n+2} - \frac{1}{2}\mathcal{V}_{-n+3})x^n \\ &= (\mathcal{V}_0+\mathcal{V}_{-1}x+\mathcal{V}_{-2}x^2) + \frac{1}{2}(\mathcal{V}_0x+\mathcal{V}_{-1}x^2) + \frac{1}{2}\mathcal{V}_0x^2 \\ &= \mathcal{V}_0 + \frac{1}{2}(\mathcal{V}_2-\mathcal{V}_1)x + \frac{1}{2}\mathcal{V}_1x^2 \end{aligned}$$

where

$$\begin{aligned} \mathcal{V}_{-1} &= \frac{1}{t}(\mathcal{V}_2 - r\mathcal{V}_1 - s\mathcal{V}_0), \\ \mathcal{V}_{-2} &= \frac{1}{t^2}(-s\mathcal{V}_2 + t\mathcal{V}_1 + s^2\mathcal{V}_0 + rs\mathcal{V}_1 - rt\mathcal{V}_0). \end{aligned}$$

Rearranging above equation, we get

$$\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n = \frac{2\mathcal{V}_0 + (\mathcal{V}_2-\mathcal{V}_1)x + \mathcal{V}_1x^2}{2+x+x^2-x^3}$$

which equals the $\sum_{n=0}^{\infty}\mathcal{V}_{-n}x^n$ in the Theorem. □

The following corollary gives the generating functions of the third-order Jacobsthal, third-order Jacobsthal-Lucas and modified third-order Jacobsthal matrix sequences with negative indices .

**Corollary 8.** *The generating functions for the third-order Jacobsthal, third-order Jacobsthal-Lucas and modified third-order Jacobsthal matrix sequences with negative indices are given as*

$$\begin{aligned} \sum_{n=0}^{\infty}\mathcal{J}_{-n}x^n &= \frac{1}{2+x+x^2-x^3}\begin{pmatrix} x^2+x+2 & x^2+2x & 2x^2 \\ x^2 & x+2 & 2x \\ x & x^2-x & 2 \end{pmatrix}, \\ \sum_{n=0}^{\infty}\mathcal{M}_{-n}x^n &= \frac{1}{2+x+x^2-x^3}\begin{pmatrix} 5x^2+5x+2 & 5x^2+2x+8 & 2x^2+8x+8 \\ x^2+4x+4 & 4x^2+x-2 & 4x^2-2x+4 \\ 2x^2-x+2 & -x^2+5x+2 & 2x^2+2x-4 \end{pmatrix}, \\ \sum_{n=0}^{\infty}\mathcal{K}_{-n}x^n &= \frac{1}{2+x+x^2-x^3}\begin{pmatrix} 3x^2+7x+2 & 7x^2-2x+4 & 2x^2+4x+12 \\ x^2+2x+6 & 2x^2+5x-4 & 6x^2-4x-2 \\ 3x^2-2x-1 & -2x^2+4x+7 & -x^2+7x-3 \end{pmatrix}, \end{aligned}$$

*respectively.*

Now, we will obtain generating functions for generalized third-order Jacobsthal numbers in terms of generalized third-order Jacobsthal matrix sequences with negative indices as a consequence of Theorem 8. To do this, we will again compare the the 2nd row and 1st column entries with the matrices in Theorem 8. Thus we have the following corollary.

**Corollary 9.** *The generating functions for the generalized third-order Jacobsthal sequence $\{V_{-n}\}_{n\geq 0}$ is given as*

$$\sum_{n=0}^{\infty} V_{-n}x^n = \frac{2V_0 + (-V_1 + V_2)\,x + V_1x^2}{2 + x + x^2 - x^3}.$$

The previous corollary gives the following results as particular examples.

**Corollary 10.** *Generated functions of third-order Jacobsthal, third-order Jacobsthal-Lucas and modified third-order Jacobsthal numbers with negative indices are*

$$\begin{aligned}
\sum_{n=0}^{\infty} J_{-n}x^n &= \frac{2J_0 + (-J_1 + J_2)\,x + J_1x^2}{2 + x + x^2 - x^3} = \frac{x^2}{2 + x + x^2 - x^3}, \\
\sum_{n=0}^{\infty} j_{-n}x^n &= \frac{2j_0 + (-j_1 + j_2)\,x + j_1x^2}{2 + x + x^2 - x^3} = \frac{4 + 4x + x^2}{2 + x + x^2 - x^3}, \\
\sum_{n=0}^{\infty} K_{-n}x^n &= \frac{2K_0 + (-K_1 + K_2)\,x + K_1x^2}{2 + x + x^2 - x^3} = \frac{6 + 2x + x^2}{2 + x + x^2 - x^3},
\end{aligned}$$

*respectively.*

## 3 Some identities

In this section, we assume that $m$ and $n$ are arbitrary integers, unless otherwise mentioned. In this section, we obtain some identities of generalized third-order Jacobsthal and third-order Jacobsthal, third-order Jacobsthal-Lucas and modified third-order Jacobsthal numbers. First, we can give a few basic relations between $\{V_n\}$ and $\{J_n\}$.

**Lemma 9.** *The following equalities are true:*

*(a)*$8V_n = (7V_0 - V_1 - V_2)J_{n+4} + (-9V_0 + 7V_1 - V_2)J_{n+3} + (-9V_0 - 9V_1 + 7V_2)J_{n+2}.$
*(b)*$4V_n = (-V_0 + 3V_1 - V_2)J_{n+3} + (-V_0 - 5V_1 + 3V_2)J_{n+2} + (7V_0 - V_1 - V_2)J_{n+1}.$
*(c)*$2V_n = (-V_0 - V_1 + V_2)J_{n+2} + (3V_0 + V_1 - V_2)J_{n+1} + (-V_0 + 3V_1 - V_2)J_n.$
*(d)*$V_n = V_0J_{n+1} + (V_1 - V_0)J_n + (V_2 - V_1 - V_0)J_{n-1}.$
*(e)*$V_n = V_1J_n + (V_2 - V_1)J_{n-1} + 2V_0J_{n-2}.$

Proof. Note that all the identities hold for all integers $n$. We prove (a). Writing

$$V_n = a \times J_{n+4} + b \times J_{n+3} + c \times J_{n+2}$$

and solving the system of equations

$$\begin{aligned}
V_0 &= a \times J_4 + b \times J_3 + c \times J_2 \\
V_1 &= a \times J_5 + b \times J_4 + c \times J_3 \\
V_2 &= a \times J_6 + b \times J_5 + c \times J_4
\end{aligned}$$

we find that $a = \frac{1}{8}(7V_0 - V_1 - V_2), b = \frac{1}{8}(-9V_0 + 7V_1 - V_2), c = \frac{1}{8}(7V_2 - 9V_1 - 9V_0)$. The other equalities can be proved similarly. □

Note that all the identities in the above lemma can be proved by induction as well.

Next, we present a few basic relations between $\{J_n\}$ and $\{V_n\}$.

**Lemma 10.** *The following equalities are true:*

*(a)*$2(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)J_n = (-V_1^2 - V_1V_2 - 2V_0V_1 + V_2^2)V_{n+4} + (3V_1^2 + V_1V_2 + 2V_0V_1 - V_2^2 - 2V_0V_2)V_{n+3} + (4V_0^2 + 4V_0V_1 + 2V_0V_2 + V_1^2 - V_1V_2 - V_2^2)V_{n+2}.$
*(b)*$(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)J_n = (V_1^2 - V_0V_2)V_{n+3} + (2V_0^2 + V_0V_1 + V_0V_2 - V_1V_2)V_{n+2} + (-V_1^2 - V_1V_2 - 2V_0V_1 + V_2^2)V_{n+1}.$
*(c)*$(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)J_n = (2V_0^2 + V_0V_1 + V_1^2 - V_2V_1)V_{n+2} + (V_2^2 - 2V_0V_1 - V_0V_2 - V_1V_2)V_{n+1} + 2(V_1^2 - V_0V_2)V_n.$
*(d)*$(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)J_n = (2V_0^2 - V_0V_1 - V_0V_2 + V_1^2 - 2V_1V_2 + V_2^2)V_{n+1} + (2V_0^2 + V_0V_1 - 2V_2V_0 + 3V_1^2 - V_2V_1)V_n + 2(2V_0^2 + V_0V_1 + V_1^2 - V_2V_1)V_{n-1}.$
*(e)*$(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)J_n = (4V_0^2 - 3V_0V_2 + 4V_1^2 - 3V_1V_2 + V_2^2)V_n + (6V_0^2 + V_0V_1 - V_0V_2 + 3V_1^2 - 4V_1V_2 + V_2^2)V_{n-1} + 2(2V_0^2 - V_0V_1 - V_0V_2 + V_1^2 - 2V_1V_2 + V_2^2)V_{n-2}.$

Now, we give a few basic relations between $\{V_n\}$ and $\{j_n\}$.

**Lemma 11.** *The following equalities are true:*

(a) $96V_n = (-27V_0 + 5V_1 + 5V_2)j_{n+4} + (37V_0 - 27V_1 + 5V_2)j_{n+3} + (37V_0 + 37V_1 - 27V_2)j_{n+2}$.
(b) $48V_n = (5V_0 - 11V_1 + 5V_2)j_{n+3} + (5V_0 + 21V_1 - 11V_2)j_{n+2} + (-27V_0 + 5V_1 + 5V_2)j_{n+1}$.
(c) $24V_n = (5V_0 + 5V_1 - 3V_2)j_{n+2} + (-11V_0 - 3V_1 + 5V_2)j_{n+1} + (5V_0 - 11V_1 + 5V_2)j_n$.
(d) $12V_n = (-3V_0 + V_1 + V_2)j_{n+1} + (5V_0 - 3V_1 + V_2)j_n + (5V_0 + 5V_1 - 3V_2)j_{n-1}$.
(e) $6V_n = (V_0 - V_1 + V_2)j_n + (V_0 + 3V_1 - V_2)j_{n-1} + (-3V_0 + V_1 + V_2)j_{n-2}$.

Next, we present a few basic relations between $\{j_n\}$ and $\{V_n\}$.

**Lemma 12.** *The following equalities are true:*

(a) $(V_0 + V_1 + V_2)(4V_0^2 + 3V_1^2 + V_2^2 - 2V_0V_2 - 3V_1V_2)j_n = (2V_0^2 + 3V_1^2 - V_2^2 + 3V_0V_1 - V_0V_2)V_{n+4} + (2V_0^2 - 3V_1^2 + 2V_2^2 - 3V_0V_1 + 2V_0V_2 - 3V_1V_2)V_{n+3} + 2(-2V_0^2 - 2V_0V_2 - 3V_1V_0 + V_2^2)V_{n+2}$.
(b) $(V_0 + V_1 + V_2)(4V_0^2 + 3V_1^2 + V_2^2 - 2V_0V_2 - 3V_1V_2)j_n = (4V_0^2 + V_2^2 + V_0V_2 - 3V_1V_2)V_{n+3} + (-2V_0^2 - 3V_0V_1 - 5V_0V_2 + 3V_1^2 + V_2^2)V_{n+2} + 2(2V_0^2 + 3V_1^2 - V_2^2 + 3V_0V_1 - V_0V_2)V_{n+1}$.
(c) $(V_0 + V_1 + V_2)(4V_0^2 + 3V_1^2 + V_2^2 - 2V_0V_2 - 3V_1V_2)j_n = (2V_0^2 + 3V_1^2 + 2V_2^2 - 3V_0V_1 - 4V_0V_2 - 3V_1V_2)V_{n+2} + (8V_0^2 + 6V_1^2 - V_2^2 + 6V_0V_1 - V_0V_2 - 3V_1V_2)V_{n+1} + 2(4V_0^2 + V_2^2 + V_0V_2 - 3V_1V_2)V_n$.
(d) $(V_0 + V_1 + V_2)(4V_0^2 + 3V_1^2 + V_2^2 - 2V_0V_2 - 3V_1V_2)j_n = (10V_0^2 + 9V_1^2 + V_2^2 + 3V_0V_1 - 5V_0V_2 - 6V_1V_2)V_{n+1} + (10V_0^2 + 3V_1^2 + 4V_2^2 - 3V_0V_1 - 2V_0V_2 - 9V_1V_2)V_n + 2(2V_0^2 + 3V_1^2 + 2V_2^2 - 3V_0V_1 - 4V_0V_2 - 3V_1V_2)V_{n-1}$.
(e) $(V_0 + V_1 + V_2)(4V_0^2 + 3V_1^2 + V_2^2 - 2V_0V_2 - 3V_1V_2)j_n = (20V_0^2 + 12V_1^2 + 5V_2^2 - 7V_0V_2 - 15V_1V_2)V_n + (14V_0^2 + 15V_1^2 + 5V_2^2 - 3V_0V_1 - 13V_0V_2 - 12V_1V_2)V_{n-1} + 2(10V_0^2 + 9V_1^2 + V_2^2 + 3V_0V_1 - 5V_0V_2 - 6V_1V_2)V_{n-2}$.

Now, we give a few basic relations between $\{V_n\}$ and $\{K_n\}$.

**Lemma 13.** *The following equalities are true:*

(a) $588V_n = (-85V_0 - 57V_1 + 55V_2)K_{n+4} + (195V_0 + 27V_1 - 57V_2)K_{n+3} + (-29V_0 + 195V_1 - 85V_2)K_{n+2}$.
(b) $294V_n = (55V_0 - 15V_1 - V_2)K_{n+3} + (-57V_0 + 69V_1 - 15V_2)K_{n+2} + (-85V_0 - 57V_1 + 55V_2)K_{n+1}$.
(c) $147V_n = (-V_0 + 27V_1 - 8V_2)K_{n+2} + (-15V_0 - 36V_1 + 27V_2)K_{n+1} + (55V_0 - 15V_1 - V_2)K_n$.
(d) $147V_n = (-16V_0 - 9V_1 + 19V_2)K_{n+1} + (54V_0 + 12V_1 - 9V_2)K_n + 2(-V_0 + 27V_1 - 8V_2)K_{n-1}$.
(e) $147V_n = (38V_0 + 3V_1 + 10V_2)K_n + (-18V_0 + 45V_1 + 3V_2)K_{n-1} + 2(-16V_0 - 9V_1 + 19V_2)K_{n-2}$.

Next, we present a few basic relations between $\{K_n\}$ and $\{V_n\}$.

**Lemma 14.** *The following equalities are true:*

(a) $4(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)K_n = (-4V_0^2 + 4V_0V_1 - 14V_0V_2 + 15V_1^2 + 5V_1V_2 - 3V_2^2)V_{n+4} + (28V_0^2 + 12V_0V_1 + 22V_0V_2 - 9V_1^2 - 15V_1V_2 + V_2^2)V_{n+3} + (-12V_0^2 - 40V_0V_1 - 2V_0V_2 - 21V_1^2 - 11V_1V_2 + 17V_2^2)V_{n+2}$.
(b) $2(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)K_n = (12V_0^2 + 8V_0V_1 + 4V_0V_2 + 3V_1^2 - 5V_1V_2 - V_2^2)V_{n+3} + (-8V_0^2 - 18V_0V_1 - 8V_0V_2 - 3V_1^2 - 3V_1V_2 + 7V_2^2)V_{n+2} + (-4V_0^2 + 4V_0V_1 - 14V_0V_2 + 15V_1^2 + 5V_1V_2 - 3V_2^2)V_{n+1}$.
(c) $(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)K_n = (2V_0^2 - 2V_0V_2 - 5V_1V_0 + 3V_2^2 - 4V_1V_2)V_{n+2} + (4V_0^2 + 6V_0V_1 - 5V_0V_2 + 9V_1^2 - 2V_2^2)V_{n+1} + (12V_0^2 + 8V_0V_1 + 4V_0V_2 + 3V_1^2 - 5V_1V_2 - V_2^2)V_n$.
(d) $(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)K_n = (6V_0^2 + V_0V_1 - 7V_0V_2 + 9V_1^2 - 4V_1V_2 + V_2^2)V_{n+1} + (14V_0^2 + 3V_0V_1 + 2V_0V_2 + 3V_1^2 - 9V_1V_2 + 2V_2^2)V_n + 2(2V_0^2 - 2V_0V_2 - 5V_1V_0 + 3V_2^2 - 4V_1V_2)V_{n-1}$.
(e) $(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)K_n = (20V_0^2 + 4V_0V_1 - 5V_0V_2 + 12V_1^2 - 13V_1V_2 + 3V_2^2)V_n + (10V_0^2 - 9V_0V_1 - 11V_0V_2 + 9V_1^2 - 12V_1V_2 + 7V_2^2)V_{n-1} + 2(6V_0^2 + V_0V_1 - 7V_0V_2 + 9V_1^2 - 4V_1V_2 + V_2^2)V_{n-2}$.

## 4 Relation between generalized third-order Jacobsthal matrix sequences and its special cases

In this section, we assume that $m$ and $n$ are arbitrary integers, unless otherwise mentioned.

The following theorem shows that there always exist interrelation between generalized third-order Jacobsthal and third-order Jacobsthal matrix sequences.

**Theorem 15.** *For the matrix sequences $\{\mathcal{V}_n\}$ and $\{\mathcal{J}_n\}$, we have the following identities.*

(a) $8\mathcal{V}_n = (7V_0 - V_1 - V_2)\mathcal{J}_{n+4} + (-9V_0 + 7V_1 - V_2)\mathcal{J}_{n+3} + (-9V_0 - 9V_1 + 7V_2)\mathcal{J}_{n+2}$.
(b) $4\mathcal{V}_n = (-V_0 + 3V_1 - V_2)\mathcal{J}_{n+3} + (-V_0 - 5V_1 + 3V_2)\mathcal{J}_{n+2} + (7V_0 - V_1 - V_2)\mathcal{J}_{n+1}$.
(c) $2\mathcal{V}_n = (-V_0 - V_1 + V_2)\mathcal{J}_{n+2} + (3V_0 + V_1 - V_2)\mathcal{J}_{n+1} + (-V_0 + 3V_1 - V_2)\mathcal{J}_n$.
(d) $\mathcal{V}_n = V_0\mathcal{J}_{n+1} + (V_1 - V_0)\mathcal{J}_n + (V_2 - V_1 - V_0)\mathcal{J}_{n-1}$.
(e) $\mathcal{V}_n = V_1\mathcal{J}_n + (V_2 - V_1)\mathcal{J}_{n-1} + +2V_0\mathcal{J}_{n-2}$.
(f) $2(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)\mathcal{J}_n = (-V_1^2 - V_1V_2 - 2V_0V_1 + V_2^2)\mathcal{V}_{n+4} + (3V_1^2 + V_1V_2 + 2V_0V_1 - V_2^2 - 2V_0V_2)\mathcal{V}_{n+3} + (4V_0^2 + 4V_0V_1 + 2V_0V_2 + V_1^2 - V_1V_2 - V_2^2)\mathcal{V}_{n+2}$.
(g) $(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)\mathcal{J}_n = (V_1^2 - V_0V_2)\mathcal{V}_{n+3} + (2V_0^2 + V_0V_1 + V_0V_2 - V_1V_2)\mathcal{V}_{n+2} + (-V_1^2 - V_1V_2 - 2V_0V_1 + V_2^2)\mathcal{V}_{n+1}$.
(h) $(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)\mathcal{J}_n = (2V_0^2 + V_0V_1 + V_1^2 - V_2V_1)\mathcal{V}_{n+2} + (V_2^2 - 2V_0V_1 - V_0V_2 - V_1V_2)\mathcal{V}_{n+1} + 2(V_1^2 - V_0V_2)\mathcal{V}_n$.

(i) $(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)\mathcal{J}_n = (2V_0^2 - V_0V_1 - V_0V_2 + V_1^2 - 2V_1V_2 + V_2^2)\mathcal{V}_{n+1} + (2V_0^2 + V_0V_1 - 2V_2V_0 + 3V_1^2 - V_2V_1)\mathcal{V}_n + 2(2V_0^2 + V_0V_1 + V_1^2 - V_2V_1)\mathcal{V}_{n-1}$.
(j) $(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)\mathcal{J}_n = (4V_0^2 - 3V_0V_2 + 4V_1^2 - 3V_1V_2 + V_2^2)\mathcal{V}_n + (6V_0^2 + V_0V_1 - V_0V_2 + 3V_1^2 - 4V_1V_2 + V_2^2)\mathcal{V}_{n-1} + 2(2V_0^2 - V_0V_1 - V_0V_2 + V_1^2 - 2V_1V_2 + V_2^2)\mathcal{V}_{n-2}$.

*Proof.* From Lemmas 9 and 10, (a)-(j) follow. □

The following theorem shows that there always exist interrelation between generalized third-order Jacobsthal and third-order Jacobsthal-Lucas matrix sequences.

**Theorem 16.** *For the matrix sequences $\{\mathcal{V}_n\}$ and $\{\mathcal{M}_n\}$, we have the following identities.*

(a) $96\mathcal{V}_n = (-27V_0 + 5V_1 + 5V_2)\mathcal{M}_{n+4} + (37V_0 - 27V_1 + 5V_2)\mathcal{M}_{n+3} + (37V_0 + 37V_1 - 27V_2)\mathcal{M}_{n+2}$.
(b) $48\mathcal{V}_n = (5V_0 - 11V_1 + 5V_2)\mathcal{M}_{n+3} + (5V_0 + 21V_1 - 11V_2)\mathcal{M}_{n+2} + (-27V_0 + 5V_1 + 5V_2)\mathcal{M}_{n+1}$.
(c) $24\mathcal{V}_n = (5V_0 + 5V_1 - 3V_2)\mathcal{M}_{n+2} + (-11V_0 - 3V_1 + 5V_2)\mathcal{M}_{n+1} + (5V_0 - 11V_1 + 5V_2)\mathcal{M}_n$.
(d) $12\mathcal{V}_n = (-3V_0 + V_1 + V_2)\mathcal{M}_{n+1} + (5V_0 - 3V_1 + V_2)\mathcal{M}_n + (5V_0 + 5V_1 - 3V_2)\mathcal{M}_{n-1}$.
(e) $6\mathcal{V}_n = (V_0 - V_1 + V_2)\mathcal{M}_n + (V_0 + 3V_1 - V_2)\mathcal{M}_{n-1} + (-3V_0 + V_1 + V_2)\mathcal{M}_{n-2}$.
(f) $(V_0 + V_1 + V_2)(4V_0^2 + 3V_1^2 + V_2^2 - 2V_0V_2 - 3V_1V_2)\mathcal{M}_n = (2V_0^2 + 3V_1^2 - V_2^2 + 3V_0V_1 - V_0V_2)\mathcal{V}_{n+4} + (2V_0^2 - 3V_1^2 + 2V_2^2 - 3V_0V_1 + 2V_0V_2 - 3V_1V_2)\mathcal{V}_{n+3} + 2(-2V_0^2 - 2V_0V_2 - 3V_1V_0 + V_2^2)\mathcal{V}_{n+2}$.
(g) $(V_0 + V_1 + V_2)(4V_0^2 + 3V_1^2 + V_2^2 - 2V_0V_2 - 3V_1V_2)\mathcal{M}_n = (4V_0^2 + V_2^2 + V_0V_2 - 3V_1V_2)\mathcal{V}_{n+3} + (-2V_0^2 - 3V_0V_1 - 5V_0V_2 + 3V_1^2 + V_2^2)\mathcal{V}_{n+2} + 2(2V_0^2 + 3V_1^2 - V_2^2 + 3V_0V_1 - V_0V_2)\mathcal{V}_{n+1}$.
(h) $(V_0 + V_1 + V_2)(4V_0^2 + 3V_1^2 + V_2^2 - 2V_0V_2 - 3V_1V_2)\mathcal{M}_n = (2V_0^2 + 3V_1^2 + 2V_2^2 - 3V_0V_1 - 4V_0V_2 - 3V_1V_2)\mathcal{V}_{n+2} + (8V_0^2 + 6V_1^2 - V_2^2 + 6V_0V_1 - V_0V_2 - 3V_1V_2)\mathcal{V}_{n+1} + 2(4V_0^2 + V_2^2 + V_0V_2 - 3V_1V_2)\mathcal{V}_n$.
(i) $(V_0 + V_1 + V_2)(4V_0^2 + 3V_1^2 + V_2^2 - 2V_0V_2 - 3V_1V_2)\mathcal{M}_n = (10V_0^2 + 9V_1^2 + V_2^2 + 3V_0V_1 - 5V_0V_2 - 6V_1V_2)\mathcal{V}_{n+1} + (10V_0^2 + 3V_1^2 + 4V_2^2 - 3V_0V_1 - 2V_0V_2 - 9V_1V_2)\mathcal{V}_n + 2(2V_0^2 + 3V_1^2 + 2V_2^2 - 3V_0V_1 - 4V_0V_2 - 3V_1V_2)\mathcal{V}_{n-1}$.
(j) $(V_0 + V_1 + V_2)(4V_0^2 + 3V_1^2 + V_2^2 - 2V_0V_2 - 3V_1V_2)\mathcal{M}_n = (20V_0^2 + 12V_1^2 + 5V_2^2 - 7V_0V_2 - 15V_1V_2)\mathcal{V}_n + (14V_0^2 + 15V_1^2 + 5V_2^2 - 3V_0V_1 - 13V_0V_2 - 12V_1V_2)\mathcal{V}_{n-1} + 2(10V_0^2 + 9V_1^2 + V_2^2 + 3V_0V_1 - 5V_0V_2 - 6V_1V_2)\mathcal{V}_{n-2}$.

*Proof.* From Lemmas 11 and 12, (a)-(j) follow. □

The following theorem shows that there always exist interrelation between generalized third-order Jacobsthal and modified third-order Jacobsthal matrix sequences.

**Theorem 17.** *For the matrix sequences $\{\mathcal{V}_n\}$ and $\{\mathcal{K}_n\}$ we have the following identities.*

(a) $588\mathcal{V}_n = (-85V_0 - 57V_1 + 55V_2)\mathcal{K}_{n+4} + (195V_0 + 27V_1 - 57V_2)\mathcal{K}_{n+3} + (-29V_0 + 195V_1 - 85V_2)\mathcal{K}_{n+2}$.
(b) $294\mathcal{V}_n = (55V_0 - 15V_1 - V_2)\mathcal{K}_{n+3} + (-57V_0 + 69V_1 - 15V_2)\mathcal{K}_{n+2} + (-85V_0 - 57V_1 + 55V_2)\mathcal{K}_{n+1}$.
(c) $147\mathcal{V}_n = (-V_0 + 27V_1 - 8V_2)\mathcal{K}_{n+2} + (-15V_0 - 36V_1 + 27V_2)\mathcal{K}_{n+1} + (55V_0 - 15V_1 - V_2)\mathcal{K}_n$.
(d) $147\mathcal{V}_n = (-16V_0 - 9V_1 + 19V_2)\mathcal{K}_{n+1} + (54V_0 + 12V_1 - 9V_2)\mathcal{K}_n + 2(-V_0 + 27V_1 - 8V_2)\mathcal{K}_{n-1}$.
(e) $147\mathcal{V}_n = (38V_0 + 3V_1 + 10V_2)\mathcal{K}_n + (-18V_0 + 45V_1 + 3V_2)\mathcal{K}_{n-1} + 2(-16V_0 - 9V_1 + 19V_2)\mathcal{K}_{n-2}$.
(f) $4(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)\mathcal{K}_n = (-4V_0^2 + 4V_0V_1 - 14V_0V_2 + 15V_1^2 + 5V_1V_2 - 3V_2^2)\mathcal{V}_{n+4} + (28V_0^2 + 12V_0V_1 + 22V_0V_2 - 9V_1^2 - 15V_1V_2 + V_2^2)\mathcal{V}_{n+3} + (-12V_0^2 - 40V_0V_1 - 2V_0V_2 - 21V_1^2 - 11V_1V_2 + 17V_2^2)\mathcal{V}_{n+2}$.
(g) $2(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)\mathcal{K}_n = (12V_0^2 + 8V_0V_1 + 4V_0V_2 + 3V_1^2 - 5V_1V_2 - V_2^2)\mathcal{V}_{n+3} + (-8V_0^2 - 18V_0V_1 - 8V_0V_2 - 3V_1^2 - 3V_1V_2 + 7V_2^2)\mathcal{V}_{n+2} + (-4V_0^2 + 4V_0V_1 - 14V_0V_2 + 15V_1^2 + 5V_1V_2 - 3V_2^2)\mathcal{V}_{n+1}$.
(h) $(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)\mathcal{K}_n = (2V_0^2 - 2V_0V_2 - 5V_1V_0 + 3V_2^2 - 4V_1V_2)\mathcal{V}_{n+2} + (4V_0^2 + 6V_0V_1 - 5V_0V_2 + 9V_1^2 - 2V_2^2)\mathcal{V}_{n+1} + (12V_0^2 + 8V_0V_1 + 4V_0V_2 + 3V_1^2 - 5V_1V_2 - V_2^2)\mathcal{V}_n$.
(i) $(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)\mathcal{K}_n = (6V_0^2 + V_0V_1 - 7V_0V_2 + 9V_1^2 - 4V_1V_2 + V_2^2)\mathcal{V}_{n+1} + (14V_0^2 + 3V_0V_1 + 2V_0V_2 + 3V_1^2 - 9V_1V_2 + 2V_2^2)\mathcal{V}_n + 2(2V_0^2 - 2V_0V_2 - 5V_1V_0 + 3V_2^2 - 4V_1V_2)\mathcal{V}_{n-1}$.
(j) $(4V_0^3 + 4V_0^2V_1 + 2V_0^2V_2 + 3V_0V_1^2 - 5V_0V_1V_2 - V_0V_2^2 + 3V_1^3 - 2V_1V_2^2 + V_2^3)\mathcal{K}_n = (20V_0^2 + 4V_0V_1 - 5V_0V_2 + 12V_1^2 - 13V_1V_2 + 3V_2^2)\mathcal{V}_n + (10V_0^2 - 9V_0V_1 - 11V_0V_2 + 9V_1^2 - 12V_1V_2 + 7V_2^2)\mathcal{V}_{n-1} + 2(6V_0^2 + V_0V_1 - 7V_0V_2 + 9V_1^2 - 4V_1V_2 + V_2^2)\mathcal{V}_{n-2}$.

*Proof.* From Lemmas 13 and 14, (a)-(j) follow. □

To prove the following Lemma 19 (c) we need the next lemma.

**Lemma 18.** *Let $A, B, C$ as in Theorem 5 and $A_1, B_1, C_1; A_2, B_2, C_2; A_3, B_3, C_3$ as in Corollary 3. Then the following relations hold:*

$$\begin{aligned} A_1^2 &= A_1,\ B_1^2 = B_1,\ C_1^2 = C_1, \\ AB &= BA = AC = CA = CB = BC = (0)\,, \\ A_1B_1 &= B_1A_1 = A_1C_1 = C_1A_1 = C_1B_1 = B_1C_1 = (0)\,, \\ A_2B_2 &= B_2A_2 = A_2C_2 = C_2A_2 = C_2B_2 = B_2C_2 = (0)\,, \\ A_3B_3 &= B_3A_3 = A_3C_3 = C_3A_3 = C_3B_3 = B_3C_3 = (0)\,. \end{aligned}$$

*Proof.* Using $\alpha + \beta + \gamma = 1$, $\alpha\beta + \alpha\gamma + \beta\gamma = -1$ and $\alpha\beta\gamma = 2$, required equalities can be established by matrix calculations. □

**Lemma 19.** *For all integers $m$ and $n$, we have the following identities.*

(a) $\mathcal{J}_0\mathcal{V}_n = \mathcal{V}_n\mathcal{J}_0 = \mathcal{V}_n$.
(b) $\mathcal{V}_0\mathcal{J}_n = \mathcal{J}_n\mathcal{V}_0 = \mathcal{V}_n$.
(c) $\mathcal{J}_m\mathcal{J}_n = \mathcal{J}_n\mathcal{J}_m = \mathcal{J}_{m+n}$

*(d)*$\mathcal{J}_m\mathcal{V}_n = \mathcal{V}_n\mathcal{J}_m = \mathcal{V}_{m+n}$.
*(e)*$\mathcal{J}_m\mathcal{M}_n = \mathcal{M}_n\mathcal{J}_m = \mathcal{M}_{m+n}$.
*(f)*$\mathcal{J}_m\mathcal{K}_n = \mathcal{K}_n\mathcal{J}_m = \mathcal{K}_{m+n}$.
*(g)*$\mathcal{V}_0\mathcal{V}_n = \mathcal{V}_n\mathcal{V}_0$.
*(h)*$\mathcal{V}_n\mathcal{V}_m = \mathcal{V}_m\mathcal{V}_n = \mathcal{V}_0\mathcal{V}_{m+n}$.
*(i)*$\mathcal{J}_{-n} = (\mathcal{J}_n)^{-1}$.
*(j)*$\mathcal{V}_{-n} = (\mathcal{V}_0)^{1-n}(\mathcal{V}_{-1})^n$

*Proof.* Identities can be established easily.

(a)Since $\mathcal{J}_0$ is the identity matrix, (a) follows.
(b)It can be seen by using Lemma 9.
(c)(c) is given in [10]. We supply the proof for completeness. Using Lemma 18 we obtain

$$\begin{aligned}
\mathcal{J}_m\mathcal{J}_n &= (A_1\alpha^m + B_1\beta^m + C_1\gamma^m)(A_1\alpha^n + B_1\beta^n + C_1\gamma^n) \\
&= A_1^2\alpha^{m+n} + B_1^2\beta^{m+n} + C_1^2\gamma^{m+n} + A_1B_1\alpha^m\beta^n + B_1A_1\alpha^n\beta^m \\
&\quad +A_1C_1\alpha^m\gamma^n + C_1A_1\alpha^n\gamma^m + B_1C_1\beta^m\gamma^n + C_1B_1\beta^n\gamma^m \\
&= A_1\alpha^{m+n} + B_1\beta^{m+n} + C_1\gamma^{m+n} \\
&= \mathcal{J}_{m+n}.
\end{aligned}$$

(d)From (b), we have

$$\mathcal{J}_m\mathcal{V}_n = \mathcal{J}_m\mathcal{J}_n j_0.$$

Now from (c) and again from (b), we obtain $\mathcal{J}_m\mathcal{V}_n = \mathcal{J}_{m+n}\mathcal{V}_0 = \mathcal{V}_{m+n}$.
It can be shown similarly that $\mathcal{V}_n\mathcal{J}_m = \mathcal{V}_{m+n}$.
(e)Take $\mathcal{V}_n = \mathcal{M}_n$ in (d).
(f)Take $\mathcal{V}_n = \mathcal{K}_n$ in (d).
(g)After matrix multiplication, just compare the row and column entries of the matrices.
(h)Using (d) and (g) and (b) we get

$$\mathcal{V}_0\mathcal{V}_{m+n} = \mathcal{V}_0\mathcal{V}_n\mathcal{J}_m = \mathcal{V}_n\mathcal{V}_0\mathcal{J}_m = \mathcal{V}_n\mathcal{V}_m.$$

Again, using (d) and (g) and (b), we obtain

$$\mathcal{V}_0\mathcal{V}_{m+n} = \mathcal{V}_0\mathcal{V}_m\mathcal{J}_n = \mathcal{V}_m\mathcal{V}_0\mathcal{J}_n = \mathcal{V}_m\mathcal{V}_n.$$

This completes the proof of (h).
(i)Suppose first that $n \geq 0$. We prove by mathematical induction. If $n = 0$ then we have

$$\mathcal{J}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = (\mathcal{J}_0)^{-1}$$

which is true and

$$\mathcal{J}_{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = (\mathcal{J}_1)^{-1}$$

which is true. Assume that the equality holds for $n \leq k$. For $n = k+1$, by using (c), we obtain

$$\begin{aligned}
(\mathcal{J}_{k+1})^{-1} &= (\mathcal{J}_k\mathcal{J}_1)^{-1} = (\mathcal{J}_1)^{-1}(\mathcal{J}_k)^{-1} = \mathcal{J}_{-1}\mathcal{J}_{-k} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} J_{-k+1} & J_{-k} + 2J_{-k-1} & 2J_{-k} \\ J_{-k} & J_{-k-1} + 2J_{-k-2} & 2J_{-k-1} \\ J_{-k-1} & J_{-k-2} + 2J_{-k-3} & 2J_{-k-2} \end{pmatrix} \\
&= \begin{pmatrix} J_{-k} & J_{-k-1} + 2J_{-k-2} & 2J_{-k-1} \\ J_{-k-1} & J_{-k-2} + 2J_{-k-3} & 2J_{-k-2} \\ \frac{1}{2}J_{-k+1} - \frac{1}{2}J_{-k-1} - \frac{1}{2}J_{-k} & \frac{1}{2}J_{-k} + \frac{1}{2}J_{-k-1} - \frac{3}{2}J_{-k-2} - J_{-k-3} & J_{-k} - J_{-k-1} - J_{-k-2} \end{pmatrix} \\
&= \begin{pmatrix} J_{-k} & J_{-k-1} + 2J_{-k-2} & 2J_{-k-1} \\ J_{-k-1} & J_{-k-2} + 2J_{-k-3} & 2J_{-k-2} \\ J_{-k-2} & J_{-k-3} + 2J_{-k-4} & 2J_{-k-3} \end{pmatrix} \\
&= \begin{pmatrix} J_{-(k+1)+1} & J_{-(k+1)} + 2J_{-(k+1)-1} & 2J_{-(k+1)} \\ J_{-(k+1)} & J_{-(k+1)-1} + 2J_{-(k+1)-2} & 2J_{-(k+1)-1} \\ J_{-(k+1)-1} & J_{-(k+1)-2} + 2J_{-(k+1)-3} & 2J_{-(k+1)-2} \end{pmatrix} \\
&= \mathcal{J}_{-(k+1)}
\end{aligned}$$

Thus, by induction on $n$, this proves (g) for $n \geq 0$. Suppose now that $n \leq 0$. Say $m = -n$. Then (g) can be written as

$$\mathcal{J}_m = (\mathcal{J}_{-m})^{-1}$$

and we prove this. Since $m \geq 0$, from the first part of the proof, we have

$$\mathcal{J}_{-m} = (\mathcal{J}_m)^{-1}$$

and so

$$(\mathcal{J}_{-m})^{-1} = ((\mathcal{J}_m)^{-1})^{-1} = \mathcal{J}_m$$

which completes the proof.

(j)Taking $-n+1$ for $m$ and 1 for $n$ in $\mathcal{V}_0\mathcal{V}_{m+n} = \mathcal{V}_m\mathcal{V}_n$ which is given in (h), we obtain that

$$\mathcal{V}_0\mathcal{V}_{-n} = \mathcal{V}_{-n+1}\mathcal{V}_{-1}. \tag{10}$$

If we multiply both side of the equation (10) with $\mathcal{V}_0$ we have the relation

$$\begin{aligned}\mathcal{V}_0\mathcal{V}_0\mathcal{V}_{-n} &= \mathcal{V}_0\mathcal{V}_{-n+1}\mathcal{V}_{-1} \\ &= \mathcal{V}_{-n+2}\mathcal{V}_{-1}\mathcal{V}_{-1}.\end{aligned}$$

Repeating this process we then obtain

$$\mathcal{V}_0^{n-1}\mathcal{V}_{-n} = \mathcal{V}_{-1}^{n}.$$

Thus, it follows that

$$\mathcal{V}_{-n} = \mathcal{V}_0^{1-n}\mathcal{V}_{-1}^{n}.$$

This completes the proof. □

Note that using Lemma 19 (j) and (d), we obtain

$$\mathcal{V}_{-n} = (\mathcal{V}_0)^{1-n}(\mathcal{V}_{-1})^{n} = (\mathcal{V}_n\mathcal{J}_{-n})^{1-n}\mathcal{V}_{-1}^{n} = \mathcal{J}_{-n}^{1-n}\mathcal{V}_n^{1-n}\mathcal{V}_{-1}^{n}$$

and then by Lemma (i), we get

$$\mathcal{V}_{-n} = \mathcal{J}_n^{n-1}\mathcal{V}_n^{1-n}\mathcal{V}_{-1}^{n}.$$

Using Lemma 19 and comparing matrix entries, we have next result.

**Corollary 11.** *For generalized third-order Jacobsthal, third-order Jacobsthal, third-order Jacobsthal-Lucas and modified third-order Jacobsthal numbers, we have the following identities:*

(a)$V_{m+n} = J_mV_{n+1} + (J_{m-1} + 2J_{m-2})V_n + 2J_{m-1}V_{n-1} = J_{m+1}V_n + J_m(V_{n-1} + 2V_{n-2}) + 2J_{m-1}V_{n-1}$.
(b)$J_{m+n} = J_mJ_{n+1} + (J_{m-1} + 2J_{m-2})J_n + 2J_{m-1}J_{n-1} = J_{m+1}J_n + J_m(J_{n-1} + 2J_{n-2}) + 2J_{m-1}J_{n-1}$.
(c)$j_{m+n} = J_mj_{n+1} + (J_{m-1} + 2J_{m-2})j_n + 2J_{m-1}j_{n-1} = J_{m+1}j_n + J_m(j_{n-1} + 2j_{n-2}) + 2J_{m-1}j_{n-1}$.
(d)$K_{m+n} = J_mK_{n+1} + (J_{m-1} + 2J_{m-2})K_n + 2J_{m-1}K_{n-1} = J_{m+1}K_n + J_m(K_{n-1} + 2K_{n-2}) + 2J_{m-1}K_{n-1}$.
(e)$V_0V_{m+n+1} + (V_1 - V_0)V_{m+n} + (V_2 - V_1 - V_0)V_{m+n-1} = V_{m+1}V_n + (V_m + 2V_{m-1})V_{n-1} + 2V_mV_{n-2} = V_mV_{n+1} + (V_{m-1} + 2V_{m-2})V_n + 2V_{m-1}V_{n-1}$.
(f)$J_{m+n} = J_{m+1}J_n + (J_m + 2J_{m-1})J_{n-1} + 2J_mJ_{n-2} = J_mJ_{n+1} + (J_{m-1} + 2J_{m-2})J_n + 2J_{m-1}J_{n-1}$.
(g)$2j_{m+n+1} - j_{m+n} + 2j_{m+n-1} = j_{m+1}j_n + (j_m + 2j_{m-1})j_{n-1} + 2j_mj_{n-2} = j_mj_{n+1} + (j_{m-1} + 2j_{m-2})j_n + 2j_{m-1}j_{n-1}$.
(h)$3K_{m+n+1} - 2K_{m+n} - K_{m+n-1} = K_{m+1}K_n + (K_m + 2K_{m-1})K_{n-1} + 2K_mK_{n-2} = K_mK_{n+1} + (K_{m-1} + 2K_{m-2})K_n + 2K_{m-1}K_{n-}$

*Proof.* We prove (a) and (e) by using Lemma 19 (d) and (h). The others are special cases of (a) and (e). Lemma 19 (d), i.e., $\mathcal{J}_m\mathcal{V}_n = \mathcal{V}_n\mathcal{J}_m = \mathcal{V}_{m+n}$, can be writtten as

$$\begin{aligned} & \begin{pmatrix} J_{m+1} & J_m + 2J_{m-1} & 2J_m \\ J_m & J_{m-1} + 2J_{m-2} & 2J_{m-1} \\ J_{m-1} & J_{m-2} + 2J_{m-3} & 2J_{m-2} \end{pmatrix} \begin{pmatrix} V_{n+1} & V_n + 2V_{n-1} & 2V_n \\ V_n & V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2} + 2V_{n-3} & 2V_{n-2} \end{pmatrix} \\ = & \begin{pmatrix} V_{n+1} & V_n + 2V_{n-1} & 2V_n \\ V_n & V_{n-1} + 2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2} + 2V_{n-3} & 2V_{n-2} \end{pmatrix} \begin{pmatrix} J_{m+1} & J_m + 2J_{m-1} & 2J_m \\ J_m & J_{m-1} + 2J_{m-2} & 2J_{m-1} \\ J_{m-1} & J_{m-2} + 2J_{m-3} & 2J_{m-2} \end{pmatrix} \\ = & \begin{pmatrix} V_{m+n+1} & V_{m+n} + 2V_{m+n-1} & 2V_{m+n} \\ V_{m+n} & V_{m+n-1} + 2V_{m+n-2} & 2V_{m+n-1} \\ V_{m+n-1} & V_{m+n-2} + 2V_{m+n-3} & 2V_{m+n-2} \end{pmatrix} \end{aligned}$$

Now, by multiplying the matrices and then by comparing the 2nd rows and 1st columns entries, we get the required identities in (a).

Lemma 19 (h), i.e., $\mathcal{V}_n\mathcal{V}_m = \mathcal{V}_m\mathcal{V}_n = \mathcal{V}_0\mathcal{V}_{m+n}$, can be writtten as

$$\begin{pmatrix} V_{n+1} & V_n+2V_{n-1} & 2V_n \\ V_n & V_{n-1}+2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2}+2V_{n-3} & 2V_{n-2} \end{pmatrix}\begin{pmatrix} V_{m+1} & V_m+2V_{m-1} & 2V_m \\ V_m & V_{m-1}+2V_{m-2} & 2V_{m-1} \\ V_{m-1} & V_{m-2}+2V_{m-3} & 2V_{m-2} \end{pmatrix}$$

$$= \begin{pmatrix} V_{m+1} & V_m+2V_{m-1} & 2V_m \\ V_m & V_{m-1}+2V_{m-2} & 2V_{m-1} \\ V_{m-1} & V_{m-2}+2V_{m-3} & 2V_{m-2} \end{pmatrix}\begin{pmatrix} V_{n+1} & V_n+2V_{n-1} & 2V_n \\ V_n & V_{n-1}+2V_{n-2} & 2V_{n-1} \\ V_{n-1} & V_{n-2}+2V_{n-3} & 2V_{n-2} \end{pmatrix}$$

$$= \begin{pmatrix} V_1 & V_2-V_1 & 2V_0 \\ V_0 & V_1-V_0 & V_2-V_1-V_0 \\ \frac{1}{2}(V_2-V_1-V_0) & \frac{1}{2}(3V_0+V_1-V_2) & \frac{1}{2}(3V_1-V_0-V_2) \end{pmatrix}\begin{pmatrix} V_{m+n+1} & V_{m+n}+2V_{m+n-1} & 2V_{m+n} \\ V_{m+n} & V_{m+n-1}+2V_{m+n-2} & 2V_{m+n-1} \\ V_{m+n-1} & V_{m+n-2}+2V_{m+n-3} & 2V_{m+n-2} \end{pmatrix}$$

Now, by multiplying the matrices and then by comparing the 2nd rows and 1st columns entries, we get the required identities in (e). □

As an application of Lemma 19 (i) and Corollary 11 (b), we present the following example.

**Example 20.** *For all integers $n$, we have the following identities.*

$$J_{-n} = \frac{1}{2^{n-1}}(J_{n-1}^2 - J_nJ_{n-2})$$

*and*

$$(J_{n+2}+J_{n+1}+J_n)(J_{n+2}^2+3J_{n+1}^2+4J_n^2-3J_{n+2}J_{n+1}-2J_{n+2}J_n) = 2^{n+1}.$$

Solution. Note that for all integers $n$, we have

$$J_{n-1} = \frac{1}{2}(J_{n+2}-J_{n+1}-J_n), \quad (11)$$

$$J_{n-2} = \frac{1}{4}\left(-J_{n+2}+3J_{n+1}-J_n\right), \quad (12)$$

$$J_{n-3} = \frac{1}{8}\left(-J_{n+2}-J_{n+1}+7J_n\right). \quad (13)$$

By using (taking $m=n$) Corollary 11 (b) and (11)-(12) we get

$$J_{2n} = \frac{1}{2}J_{n+2}^2 + \frac{1}{2}J_{n+1}^2 - \frac{1}{2}J_n^2 + 3J_{n+1}J_n - J_{n+2}J_n - J_{n+2}J_{n+1}. \quad (14)$$

In [[13], Corollary 12 (a)], the following formula is presented for $J_{-n}$ :

$$J_{-n} = \frac{1}{2^{n+1}}(3J_n^2 + 2J_{2n} + J_{n+2}J_n - 7J_{n+1}J_n).$$

Using (14), we obtain

$$J_{-n} = \frac{1}{2^{n+1}}(J_{n+2}^2 + J_{n+1}^2 + 2J_n^2 - 2J_{n+2}J_{n+1} - J_{n+2}J_n - J_{n+1}J_n). \quad (15)$$

By comparing the 2nd rows and 1st columns entries of both sides of the relation $\mathcal{J}_{-n} = (\mathcal{J}_n)^{-1}$ which is given in Lemma 19 (i), we get

$$J_{-n} = \frac{1}{2}\frac{J_{n-1}^2 - J_nJ_{n-2}}{J_{n-1}^3 + J_n^2J_{n-3} + J_{n-2}^2J_{n+1} - J_{n+1}J_{n-1}J_{n-3} - 2J_nJ_{n-1}J_{n-2}}. \quad (16)$$

Note that by using (11)-(13) we get

$$J_{n-1}^2 - J_nJ_{n-2} = \frac{1}{4}(J_{n+2}^2 + J_{n+1}^2 + 2J_n^2 - 2J_{n+2}J_{n+1} - J_{n+2}J_n - J_{n+1}J_n)$$

and

$$J_{n-1}^3 + J_n^2J_{n-3} + J_{n-2}^2J_{n+1} - J_{n+1}J_{n-1}J_{n-3} - 2J_nJ_{n-1}J_{n-2}$$

$$= \frac{1}{8}(4J_n^3 + 4J_n^2J_{n+1} + 2J_n^2J_{n+2} + 3J_nJ_{n+1}^2 - 5J_nJ_{n+1}J_{n+2} - J_nJ_{n+2}^2 + 3J_{n+1}^3 - 2J_{n+1}J_{n+2}^2 + J_{n+2}^3)$$

$$= \frac{1}{8}(J_{n+2}+J_{n+1}+J_n)(J_{n+2}^2+3J_{n+1}^2+4J_n^2-3J_{n+2}J_{n+1}-2J_{n+2}J_n)$$

and so (16) can be written as

$$J_{-n} = \frac{J_{n+2}^2 + J_{n+1}^2 + 2J_n^2 - 2J_{n+2}J_{n+1} - J_{n+2}J_n - J_{n+1}J_n}{(J_{n+2}+J_{n+1}+J_n)(J_{n+2}^2+3J_{n+1}^2+4J_n^2-3J_{n+2}J_{n+1}-2J_{n+2}J_n)}. \quad (17)$$

So the rights sides of the equations (15) and (17) must be equal. This completes the solution. □

**Theorem 21.** *For all integers $m$ and $n$, we have the following identities.*

(a)$4\mathcal{V}_m\mathcal{V}_n = \mathcal{V}_n\mathcal{V}_m = (V_0+V_1-V_2)^2\mathcal{J}_{m+n+4} - 2(3V_0+V_1-V_2)(V_0+V_1-V_2)\mathcal{J}_{m+n+3} + (11V_0^2-5V_1^2-V_2^2+2V_0V_1-6V_0V_2+6V_1V_2)\mathcal{J}_{m+n+2} - 2(V_0-3V_1+V_2)(3V_0+V_1-V_2)\mathcal{J}_{m+n+1} + (V_0-3V_1+V_2)^2\mathcal{J}_{m+n}.$
(b)$2\mathcal{V}_m\mathcal{V}_n = 2\mathcal{V}_n\mathcal{V}_m = (-V_0-V_1+V_2)\mathcal{V}_{m+n+2} + (3V_0+V_1-V_2)\mathcal{V}_{m+n+1} + (-V_0+3V_1-V_2)\mathcal{V}_{m+n}.$
(c)$2\mathcal{J}_m\mathcal{V}_n = 2\mathcal{V}_n\mathcal{J}_m = (-V_0-V_1+V_2)\mathcal{J}_{m+n+2} + (3V_0+V_1-V_2)\mathcal{J}_{m+n+1} + (-V_0+3V_1-V_2)\mathcal{J}_{m+n}.$
(d)$24\mathcal{J}_m\mathcal{V}_n = 24\mathcal{V}_n\mathcal{J}_m = (5V_0+5V_1-3V_2)\mathcal{M}_{m+n+2} + (-11V_0-3V_1+5V_2)\mathcal{M}_{m+n+1} + (5V_0-11V_1+5V_2)\mathcal{M}_{m+n}.$
(e)$147\mathcal{J}_m\mathcal{V}_n = 147\mathcal{V}_n\mathcal{J}_m = (-V_0+27V_1-8V_2)\mathcal{K}_{m+n+2} + 3(-5V_0-12V_1+9V_2)\mathcal{K}_{m+n+1} + (55V_0-15V_1-V_2)\mathcal{K}_{m+n}.$

*Proof.*

(a)It follows from Theorem 15 (c) and Lemma 19 (c).
(b)It follows from Theorem 15 (c) and Lemma 19 (d).
(c)It follows from Theorem 15 (c) and Lemma 19 (c).
(d)It follows from Theorem 16 (c) and Lemma 19 (e).
(e)It follows from Theorem 17 (c) and Lemma 19 (f). □

Note that in Theorem 21 we use (c)'s of Theorems 15, 16 and 17. Using (a),(b),(d),(e),(f),(g)(h),(i),(j)'s of Theorems 15, 16 and 17 we can establish other recurence relations.

Using Theorem 21 and comparing matrix entries, we have next result.

**Theorem 22.** *For generalized third-order Jacobsthal, third-order Jacobsthal, third-order Jacobsthal-Lucas and modified third-order Jacobsthal numbers, we have the following identities:*

(a)$4(V_mV_{n+1}+(V_{m-1}+2V_{m-2})V_n+2V_{m-1}V_{n-1}) = 4(V_{m+1}V_n+V_m(V_{n-1}+2V_{n-2})+2V_{m-1}V_{n-1}) = (V_0+V_1-V_2)^2J_{m+n+4} - (6V_0+2V_1-2V_2)(V_0+V_1-V_2)J_{m+n+3} - (-11V_0^2-2V_0V_1+6V_0V_2+5V_1^2-6V_1V_2+V_2^2)J_{m+n+2} - (2V_0-6V_1+2V_2)(3V_0+V_1-V_2)J_{m+n+1} + (V_0-3V_1+V_2)^2J_{m+n}.$
(b)$2(V_mV_{n+1}+(V_{m-1}+2V_{m-2})V_n+2V_{m-1}V_{n-1}) = 2(V_{m+1}V_n+V_m(V_{n-1}+2V_{n-2})+2V_{m-1}V_{n-1}) = (-V_0-V_1+V_2)V_{m+n+2} + (3V_0+V_1-V_2)V_{m+n+1} + (-V_0+3V_1-V_2)V_{m+n}.$
(c)$2(J_mV_{n+1}+((J_{m-1}+2J_{m-2})V_n+2J_{m-1}V_{n-1}) = 2(J_{m+1}V_n+J_m(V_{n-1}+2V_{n-2})+2J_{m-1}V_{n-1}) = (-V_0-V_1+V_2)J_{m+n+2} + (3V_0+V_1-V_2)J_{m+n+1} + (-V_0+3V_1-V_2)J_{m+n}.$
(d)$24(J_mV_{n+1}+(J_{m-1}+2J_{m-2})V_n+2J_{m-1}V_{n-1}) = 24(J_{m+1}V_n+J_m(V_{n-1}+2V_{n-2})+2J_{m-1}V_{n-1}) = (5V_0+5V_1-3V_2)j_{m+n+2} + (-11V_0-3V_1+5V_2)j_{m+n+1} + (5V_0-11V_1+5V_2)j_{m+n}.$
(e)$147(J_mV_{n+1}+(J_{m-1}+2J_{m-2})V_n+2J_{m-1}V_{n-1}) = 147(J_{m+1}V_n+J_m(V_{n-1}+2V_{n-2})+2J_{m-1}V_{n-1}) = (-V_0+27V_1-8V_2)K_{m+n+}$ $(-15V_0-36V_1+27V_2)K_{m+n+1} + (-15V_1+55V_0-V_2)K_{m+n}.$

*Proof.* By multiplying matrices and then by comparing the 2nd rows and 1st columns entries in Theorem 21 (a), we get the required identities in (a). The remaining of identities can be proved by considering again Theorem 21. □

Taking $V_n = J_n$ in Theorem 22, we obtain the following corollary.

**Corollary 12.** *For third-order Jacobsthal numbers, we have the following identities:*

(a)$J_mJ_{n+1}+(J_{m-1}+2J_{m-2})J_n+2J_{m-1}J_{n-1} = J_{m+1}J_n+J_m(J_{n-1}+2J_{n-2})+2J_{m-1}J_{n-1} = J_{m+n}.$
(b)$12(J_mJ_{n+1}+(J_{m-1}+2J_{m-2})J_n+2J_{m-1}J_{n-1}) = 12(J_{m+1}J_n+J_m(J_{n-1}+2J_{n-2})+2J_{m-1}J_{n-1}) = j_{m+n+2}+j_{m+n+1}-3j_{m+n}.$
(c)$147(J_mJ_{n+1}+(J_{m-1}+2J_{m-2})J_n+2J_{m-1}J_{n-1}) = 147(J_{m+1}J_n+J_m(J_{n-1}+2J_{n-2})+2J_{m-1}J_{n-1}) = 19K_{m+n+2}-9K_{m+n+1}-16K_{m+n}.$

Taking $V_n = j_n$ in Theorem 22, we get the following corollary.

**Corollary 13.** *For third-order Jacobsthal-Lucas numbers, we have the following identities:*

(a)$j_mj_{n+1}+(j_{m-1}+2j_{m-2})j_n+2j_{m-1}j_{n-1} = j_{m+1}j_n+j_m(j_{n-1}+2j_{n-2})+2j_{m-1}j_{n-1} = J_{m+n+4}+2J_{m+n+3}-3J_{m+n+2}-4J_{m+n+1}+4J_{m+n}.$
(b)$j_mj_{n+1}+(j_{m-1}+2j_{m-2})j_n+2j_{m-1}j_{n-1} = j_{m+1}j_n+j_m(j_{n-1}+2j_{n-2})+2j_{m-1}j_{n-1} = j_{m+n+2}+j_{m+n+1}-2j_{m+n}.$
(c)$J_mj_{n+1}+((J_{m-1}+2J_{m-2})j_n+2J_{m-1}j_{n-1} = J_{m+1}j_n+J_m(j_{n-1}+2j_{n-2})+2J_{m-1}j_{n-1} = J_{m+n+2}+J_{m+n+1}-2J_{m+n}.$
(d)$J_mj_{n+1}+(J_{m-1}+2J_{m-2})j_n+2J_{m-1}j_{n-1} = J_{m+1}j_n+J_m(j_{n-1}+2j_{n-2})+2J_{m-1}j_{n-1} = j_{m+n}.$
(e)$49(J_mj_{n+1}+(J_{m-1}+2J_{m-2})j_n+2J_{m-1}j_{n-1}) = 49(J_{m+1}j_n+J_m(j_{n-1}+2j_{n-2})+2J_{m-1}j_{n-1}) = 30K_{m+n}+23K_{m+n+1}-5K_{m+n+2}.$

Taking $V_n = K_n$ in Theorem 22, we obtain the following corollary.

**Corollary 14.** *For modified third-order Jacobsthal numbers, we have the following identities:*

(a)$4(K_mK_{n+1}+(K_{m-1}+2K_{m-2})K_n+2K_{m-1}K_{n-1}) = 4(K_{m+1}K_n+K_m(K_{n-1}+2K_{n-2})+2K_{m-1}K_{n-1}) = J_{m+n+4}-14J_{m+n+3}$ $55J_{m+n+2}-42J_{m+n+1}+9J_{m+n}.$
(b)$2(K_mK_{n+1}+(K_{m-1}+2K_{m-2})K_n+2K_{m-1}K_{n-1}) = 2(K_{m+1}K_n+K_m(K_{n-1}+2K_{n-2})+2K_{m-1}K_{n-1}) = -K_{m+n+2}+7K_{m+n+1}-3K_{m+n}.$
(c)$2(J_mK_{n+1}+((J_{m-1}+2J_{m-2})K_n+2J_{m-1}K_{n-1}) = 2(J_{m+1}K_n+J_m(K_{n-1}+2K_{n-2})+2J_{m-1}K_{n-1}) = -J_{m+n+2}+7J_{m+n+1}-3J_{m+n}.$

(d) $24(J_mK_{n+1}+(J_{m-1}+2J_{m-2})K_n+2J_{m-1}K_{n-1}) = 24(J_{m+1}K_n+J_m(K_{n-1}+2K_{n-2})+2J_{m-1}K_{n-1}) = 11j_{m+n+2}-21j_{m+n+1}+19j_{m+n}$.
(e) $J_mK_{n+1}+(J_{m-1}+2J_{m-2})K_n+2J_{m-1}K_{n-1} = J_{m+1}K_n+J_m(K_{n-1}+2K_{n-2})+2J_{m-1}K_{n-1} = K_{m+n}$.

The next two theorems provide us the convenience to obtain the powers of generalized third-order Jacobsthal, third-order Jacobsthal, third-order Jacobsthal-Lucas and Naraya-perrin matrix sequences.

**Theorem 23.** *For all integers $m, n$ and $r$, the following identities hold:*

(a) $\mathcal{J}_n^m = \mathcal{J}_{mn}$,
(b) $\mathcal{J}_{n+1}^m = \mathcal{J}_1^m \mathcal{J}_{mn}$,
(c) $\mathcal{J}_{n-r}\mathcal{J}_{n+r} = \mathcal{J}_n^2 = \mathcal{J}_2^n$.

*Proof.* We prove for $m, n, r \geq 0$. The other cases can be proved similarly.

(a) We can write $\mathcal{J}_n^m$ as

$$\mathcal{J}_n^m = \mathcal{J}_n\mathcal{J}_n...\mathcal{J}_n \text{ } (m \text{ times}).$$

Using Theorem 19 (c) iteratively, we obtain the required result:

$$\begin{aligned} \mathcal{J}_n^m &= \underbrace{\mathcal{J}_n\mathcal{J}_n...\mathcal{J}_n}_{m \text{ times}} \\ &= \mathcal{J}_{2n}\underbrace{\mathcal{J}_n\mathcal{J}_n...\mathcal{J}_n}_{m-1 \text{ times}} \\ &= \mathcal{J}_{3n}\underbrace{\mathcal{J}_n\mathcal{J}_n...\mathcal{J}_n}_{m-2 \text{ times}} \\ &\vdots \\ &= \mathcal{J}_{(m-1)n}\mathcal{J}_n \\ &= \mathcal{J}_{mn}. \end{aligned}$$

(b) As a similar approach in (a) we have

$$\mathcal{J}_{n+1}^m = \mathcal{J}_{n+1}.\mathcal{J}_{n+1}...\mathcal{J}_{n+1} = \mathcal{J}_{m(n+1)} = \mathcal{J}_m\mathcal{J}_{mn} = \mathcal{J}_1\mathcal{J}_{m-1}\mathcal{J}_{mn}.$$

Using Theorem 19 (c), we can write iteratively $\mathcal{J}_m = \mathcal{J}_1\mathcal{J}_{m-1}, \mathcal{J}_{m-1} = \mathcal{J}_1\mathcal{J}_{m-2}, ..., \mathcal{J}_2 = \mathcal{J}_1\mathcal{J}_1$. Now it follows that

$$\mathcal{J}_{n+1}^m = \underbrace{\mathcal{J}_1\mathcal{J}_1...\mathcal{J}_1}_{m \text{ times}}\mathcal{J}_{mn} = \mathcal{J}_1^m\mathcal{J}_{mn}.$$

(c) Theorem 19 (c) gives

$$\mathcal{J}_{n-r}\mathcal{J}_{n+r} = \mathcal{J}_{2n} = \mathcal{J}_n\mathcal{J}_n = \mathcal{J}_n^2$$

and also

$$\mathcal{J}_{n-r}\mathcal{J}_{n+r} = \mathcal{J}_{2n} = \underbrace{\mathcal{J}_2\mathcal{J}_2...\mathcal{J}_2}_{n \text{ times}} = \mathcal{J}_2^n.$$

We have analogues results for the matrix sequence $\mathcal{V}_n$.

**Theorem 24.** *For all integers $m, n$ and $r$, the following identities hold:*

(a) $\mathcal{V}_{n-r}\mathcal{V}_{n+r} = \mathcal{V}_n^2$,
(b) $\mathcal{V}_n^m = \mathcal{V}_0^m\mathcal{J}_{m+n}$.

*Proof.*

(a) We use Binet's formula of generalized third-order Jacobsthal sequence which is given in Theorem 5. So

$$\begin{aligned} & \mathcal{V}_{n-r}\mathcal{V}_{n+r} - \mathcal{V}_n^2 \\ = & (A\alpha^{n-r}+B\beta^{n-r}+C\gamma^{n-r})(A\alpha^{n+r}+B\beta^{n+r}+C\gamma^{n+r}) - (A\alpha^n+B\beta^n+C\gamma^n)^2 \\ = & AB\alpha^{n-r}\beta^{n-r}(\alpha^r-\beta^r)^2 + AC\alpha^{n-r}\gamma^{n-r}(\alpha^r-\gamma^r)^2 + BC\beta^{n-r}\gamma^{n-r}(\beta^r-\gamma^r)^2 \\ = & 0 \end{aligned}$$

since $AB = AC = BC = 0$ (see Lemma 18). Now we get the result as required.

(b)By Theorem 23, we have

$$\mathcal{V}_0^m \mathcal{J}_{mn} = \underbrace{\mathcal{V}_0 \mathcal{V}_0 ... \mathcal{V}_0}_{m \text{ times}} \underbrace{\mathcal{J}_n \mathcal{J}_n ... \mathcal{J}_n}_{m \text{ times}}.$$

When we apply Lemma 19 (b) iteratively, it follows that

$$\begin{aligned} \mathcal{V}_0^m \mathcal{J}_{mn} &= (\mathcal{V}_0 \mathcal{J}_n)(\mathcal{V}_0 \mathcal{J}_n)...(\mathcal{V}_0 \mathcal{J}_n) \\ &= \mathcal{V}_n \mathcal{V}_n ... \mathcal{V}_n = \mathcal{V}_n^m. \end{aligned}$$

This completes the proof. □

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# A Price Adjustment Model in Market Equilibrium with Conformable Laplace Method

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**Abstract:** In this study, we consider a price adjustment model which is a common and very important tool in market equilibrium. We provide the fundamental solutions of the model by an analytical/approximate method defined with the conformable derivative operator. Also, we take the Laplace transform into account to be able to obtain accurate and analytical solution. We represent our results by illustrative figures to point out the efficiency of fractional parameter. We prove the efficiency and accuracy of the Laplace transform and the series method constructed with the conformable operator in providing the solution to the mentioned financial model by considering the theoretical results and illustrative applications. It can be pointed out that the proposed method is an accurate way to solve such problems that include fractional-order parameter. One of the prominent properties of the method is the possibility of using it in solving the similar equations including fractional derivatives having different types of kernels.

**Keywords:** Analytical solution, Conformable derivative, Financial interpret, Laplace transform, Price adjustment model.

## 1 Introduction

Applications of fractional derivatives have attracted the attention of researchers in the last half-century, and today there is a lot of work on it. With the development of technology, it is seen from recent studies that integer-order derivatives are sometimes not sufficient in modelling events that occur in nature. As a result of this situation, a number of mathematicians focused on fractional derivative strategies. In 2014, the conformable fractional derivative procedure, which was defined by Khalil et al. [1] with the help of the limit definition of the derivative, provides features that other derivative approaches do not provide. The fact that the conformable fractional derivative approach provides the chain rule is of great importance in the analytical solution of partial differential equations. Therefore, recently analytical solutions of partial differential equations with conformable fractional derivatives have been obtained using different methods.

In 2014, Khalil *et al.* [1] defined a new derivative operator which was named as "conformable". This derivative has been used extensively to solve some real-life problems by various scientists such as [3–11].

On the other hand, numerical and analytical solution methods are attracting attention all over the world. In all sciences, it is important to establish the system of equations that models a process. However, solving that model analytically and at least numerically, is highly significant. In recent years, a large number of new solution methods have been developed and some existing methods have been improved [12–21].

Many economic models are presented with the mathematical equipments, which show to consider the problems in market aiming the equilibrium. We can see lots of implementations related to these models. The view of expense timing is based on the sketchily competitious establishment in which the request inclines to be lower and at multifarious expense rather than a certain expense. For more details see [22]. A competitive market is with the competitive equilibrium that comes to mean the amount of belongings requested by receivers equals to the amount of belongings prepared by vendors. The amount requested and the amount prepared can be given by request function $q_a(t)$ and provision function $q_b(t)$ by:

$$q_a(t) = a_0 - a_1\psi(t), \quad q_b(t) = -b_0 + b_1\psi(t).$$

In the above equations, we define $\psi(t)$ as the expense of belongings. Additionally, $a_0, b_0, a_1, b_1$ are positive constants which are the elements impressing the amount requested and amount prepared. We get the equilibrium expense as:

$$\psi^* = \frac{a_0 + b_0}{a_1 + b_1},$$

for $q_a(t) = q_b(t)$, when the requested amount equals the provided amount. Therefore, the expense inclines to stand stable and there is no lack and surplus in economics in a such case. We take into consideration the expense timing equation as [23]:

$$\psi'(t) = k(q_a - q_b),$$

where $k > 0$ is the speed of timing constant. If we put the $q_a(t)$ and $q_b(t)$ into Eq. (1), then we have

$$\psi'(t) + k(a_1 + b_1)\psi(t) = k(a_0 + b_0).$$

Then, we can obtain:

$$\psi(t) = \frac{a_0 + b_0}{a_1 + b_1} - \left[\psi(0) + \frac{a_0 + b_0}{a_1 + b_1}\right] \exp\left(-k(a_1 + b_1)t\right).$$

We describe the $\psi(0)$ as the expense at the time $t = 0$. We define $q_a$ and $q_b$ by:

$$q_a(t) = a_0 - a_1\psi(t) + a_2\psi'(t), \quad q_b(t) = -b_0 + b_1\psi(t) - b_2\psi'(t).$$

We equalize $q_a(t)$, $q_b(t)$ and obtain:

$$\psi'(t) - \frac{a_1 + b_1}{a_2 + b_2}\psi(t) = -\frac{a_0 + b_0}{a_2 + b_2}. \tag{1}$$

Then, we obtain

$$\psi(t) = \frac{a_0 + b_0}{a_1 + b_1} - \left[\psi(0) + \frac{a_0 + b_0}{a_1 + b_1}\right] \exp\left(\frac{(a_1 + b_1)}{(a_2 + b_2)}t\right). \tag{2}$$

## 2 Preliminaries

For the concept of fractional derivative, we adopt the conformable derivative operator. This derivative is very good at properly handling initial value problems where the initial conditions are given.

**Definition 1.** *The conformable derivative is given by [1]*

$$ {}_0^C T_\tau^\beta \{h(\tau)\} = \lim_{\eta \to 0} \frac{h\left(\tau + \eta\tau^{1-\beta}\right) - h(\tau)}{\eta}, \tag{3}$$

*for all* $\tau > 0, \beta \in (0, 1]$ *and where* $h \in L_1(a, b)$.

**Definition 2.** *The Laplace transform (LT) related to the conformable derivative* ${}_0^C T_\tau^\beta\{h(\tau)\}$ *is given by [2]*

$$\mathbb{L}_\beta\{{}_\rho^C T_\tau^\beta\{h(\tau)\}\}(\kappa) = \kappa\mathbb{L}_\beta\{h(\tau)\} - h(\rho), \tag{4}$$

*where* $\rho \in \mathbb{R}, \beta \in (0, 1]$ *and* $h : (\rho, \infty) \to \mathbb{R}$ *is a differentiable real valued function.*

**Definition 3.** *Let* $h$ *be an* $n-$*times differentiable at* $\tau$. *Then the conformable derivative of* $h$ *order* $\beta$ *is defined as [1]:*

$$ {}_0^C T_\tau^\beta \{h(\tau)\} = \lim_{\eta \to 0} \frac{h^{(\lceil\beta\rceil - 1)}\left(\tau + \eta\tau^{(\lceil\beta\rceil - \beta)}\right) - h^{(\lceil\beta\rceil - 1)}(\tau)}{\eta}, \tag{5}$$

*for all* $\tau > 0, \beta \in (n, n + 1]$.

**Lemma 1.** *Let* $h$ *be an* $n-$*times differentiable at* $\tau$. *Then*

$$ {}_0^C T_\tau^\beta \{h(\tau)\} = \tau^{\lceil\beta\rceil - \beta} h^{\lceil\beta\rceil}(\tau), \tag{6}$$

*for all* $\tau > 0, \beta \in (n, n + 1]$ *[1].*

## 3 Modified Laplace Decomposition Method

We define the solution method by using the conformable derivative operator and we compare the behaviors of the solutions according to types of methods which have been defined in the previous sections. Here we suggest an infinite series solution method namely modified Laplace decomposition method (MLDM). Firstly, we consider the general case of the price adjustment model which is defined in Eq. (1):

$$ {}_0^C T_\tau^\beta \psi + F[\psi] + G[\psi] = H(\tau), \tag{7}$$

subject to the initial condition

$$\psi(0) = g(\tau), \tag{8}$$

where $0 < \beta \leq 1$, $g(\tau)$ is a known function and ${}_0^C T_\tau^\beta \psi$ is the time-conformable operator of order $\beta$ of given function $\psi(\tau)$. In Eq. (7) we denote with the function $F[\psi]$ the linear part, with the function $G[\psi]$ the nonlinear term and with the function $H(\tau)$ the nonhomogeneous part.

Using the Laplace transforms of the conformable operator which is given in Definition 2, we define the $\mathcal{L}\{\psi(\tau)\} = \tilde{\psi}(\kappa)$. The mentioned method supposes an infinite series solution for wanted function given by

$$\psi(\tau) = \sum_{m=0}^{\infty} \psi_m(\tau). \tag{9}$$

In Eq. (7), the nonlinear terms shown by $G[\psi]$ are derived as

$$G[\psi] = \sum_{m=0}^{\infty} \Theta_m(\psi_0, \psi_1, \ldots, \psi_m), \tag{10}$$

where $\Theta_m$ is the Adomian polynomial which can be obtained by

$$\Theta_m(\psi_0, \psi_1, \ldots, \psi_m) = \frac{1}{m!}\left[\frac{d^m}{d\omega^m} G\left[\sum_{\zeta=0}^{\infty} \omega^{\zeta} \psi_{\zeta}\right]\right]_{\omega=0}, \quad m \geq 0. \tag{11}$$

It is easy to compute Adomian polynomials of the mentioned nonlinear part in this study by setting a code.

### 3.1 *Modified Laplace Decomposition Series for the Conformable Operator*

In this subsection, we define the modified decomposition components constructed by the conformable operator for Eq. (7). Taking the Laplace transform of both sides of Eq. (7) and taking the fact that in Lemma 1, we get

$$\tilde{\psi}(\kappa) = \frac{1}{\kappa}\mathcal{L}\left\{\tau^{\beta-1} H(\tau)\right\} - \frac{1}{\kappa}\mathcal{L}\left\{\tau^{\beta-1}\left[F[\psi] + G[\psi]\right]\right\} + \frac{1}{\kappa}\psi(0). \tag{12}$$

Then we apply the inverse LT of Eq. (12), we have

$$\psi(\tau) = \Phi(\tau) - \mathcal{L}^{-1}\left\{\frac{1}{\kappa}\mathcal{L}\left\{\tau^{\beta-1} F[\psi]\right\}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{\kappa}\mathcal{L}\left\{\tau^{\beta-1} G[\psi]\right\}\right\}, \tag{13}$$

where $\Phi(\tau) = \phi(\xi, 0) + \mathcal{L}^{-1}\left\{\frac{1}{\kappa}\mathcal{L}\left\{\tau^{\beta-1} H(\tau)\right\}\right\}$. If the term $\Phi(\tau)$ can be supposed as $\Phi(\tau) = \Phi_0(\tau) + \Phi_1(\tau)$, then one can construct the recursive algorithm for the first component $\psi_0(\tau)$ and the general term $\psi_{m+1}(\tau)$ with respect to the modified Laplace decomposition method (MLDM) as

$$\psi_0(\tau) = \Phi_0(\tau) \tag{14}$$

and

$$\psi_1(\tau) = \Phi_1(\tau) - \mathcal{L}^{-1}\left\{\frac{1}{\kappa}\mathcal{L}\left\{\tau^{\beta-1} F[\psi_0]\right\}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{\kappa}\mathcal{L}\left\{\tau^{\beta-1} \Theta_0(\tau)\right\}\right\}, \tag{15}$$

respectively. As a result, the recurrence relation becomes

$$\psi_{m+1}(\xi, \zeta) = -\mathcal{L}^{-1}\left\{\frac{1}{\kappa}\mathcal{L}\left\{\tau^{\beta-1} F[\psi_m]\right\}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{\kappa}\mathcal{L}\left\{\tau^{\beta-1} \Theta_m(\tau)\right\}\right\}. \tag{16}$$

Therefore, it can be approximated the solution $\psi(\tau)$ by considering the series $\psi(\tau) = \sum_{m=0}^{\infty} \psi_m(\tau)$.

## 4 Solution to the price adjustment model

In this sub-part of the study we obtain the solution of the model by using conformable operator. Then, we consider the mentioned problem in Eqs. (1) which is constructed by conformable operator. Then we get the following steps:

$$\begin{aligned}
\psi_0(\tau) &= \frac{a_1+b_1}{a_2+b_2}, \\
\psi_1(\tau) &= \frac{\left(\frac{(a_1+b_1)^2}{(a_2+b_2)^2}-\frac{a_0+b_0}{a_2+b_2}\right)\Gamma(\beta)t^\beta}{\Gamma(\beta+1)}, \\
\psi_2(\tau) &= -\frac{\Gamma(\beta)t^\beta\left(a_2^2b_0\Gamma(2\beta+1)-3a_1^2b_1\Gamma(2\beta)t^\beta-a_1^3\Gamma(2\beta)t^\beta\right)}{(a_2+b_2)^3\Gamma(\beta+1)\Gamma(2\beta+1)} \\
&\quad -\frac{\Gamma(\beta)t^\beta\left(2a_2b_0b_2\Gamma(2\beta+1)+a_2b_0b_1\Gamma(2\beta)t^\beta+b_1^3(-\Gamma(2\beta))t^\beta\right)}{(a_2+b_2)^3\Gamma(\beta+1)\Gamma(2\beta+1)} \\
&\quad -\frac{\Gamma(\beta)t^\beta\left(a_0(a_2+b_2)\left(a_2\Gamma(2\beta+1)+b_2\Gamma(2\beta+1)+b_1\Gamma(2\beta)t^\beta\right)+b_0b_2^2\Gamma(2\beta+1)+b_0b_1b_2\Gamma(2\beta)t^\beta\right)}{(a_2+b_2)^3\Gamma(\beta+1)\Gamma(2\beta+1)} \\
&\quad -\frac{\Gamma(\beta)t^\beta\left(a_1\left(a_2b_0+a_0(a_2+b_2)-3b_1^2+b_0b_2\right)\Gamma(2\beta)t^\beta\right)}{(a_2+b_2)^3\Gamma(\beta+1)\Gamma(2\beta+1)}, \\
&\quad \vdots
\end{aligned}$$

In this way, we can have other parts of the series. Then we get the solution to the stated problem as $\psi(\tau)=\psi_0(\tau)+\psi_1(\tau)+\psi_2(\tau)+\psi_3(\tau)+\cdots$.

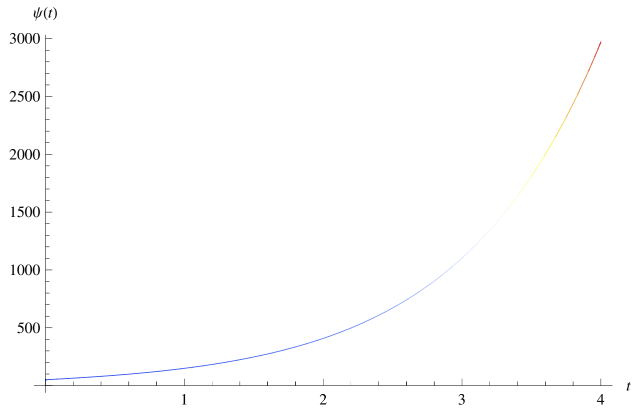


**Fig. 1**: Numerical simulation with conformable operator $\alpha = 0.9$.

## 5 Conclusion

In this paper, we have investigated and provided an approximate-analytical solution to the price adjustment model in detail by Laplace transform coupled with the modified decomposition method. Moreover, we have used the conformable derivative. We have proved the efficiency of the mentioned integral transformation for the price adjustment problem. In addition to these effective results, we have demonstrated our findings with a figure. We have constituted and constructed the mentioned solution with the suggested method for the first time in this work. Another major advantage of the method presented in this article is that it can be used to solve other similar linear/nonlinear problems by using this suggested method. This point can be considered as research works in the future by researchers in this field.

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# Babesiosis Disease Modeling of Fractional Order and Investigation of its Dynamics

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**Abstract:** In this study, we investigate the dynamics of the babesiosis transmission on bovine populations and ticks. The most prominent role in the transmission of the parasite is the ticks from the ixodidae family. The vector tick takes factors (merozoites in erythrocytes) from the diseased animal while sucking blood. To model and investigate the transmissions of this parasite and address this important issue, we consider the disease in a fractional epidemiological model. This paper, therefore, discusses the mechanisms of transmission of babesiosis defined in the Caputo fractional derivative sense to study the propagation mechanisms of babesiosis. The application of fixed-point theory is used to derive the concept of the qualitative properties of the mentioned model. The solution is obtained by using the Adams-Bashforth type predictor-corrector scheme. Numerical simulations are performed, and the effect of the fractional order derivatives are investigated graphically.

**Keywords:** Babesiosis disease, Caputo fractional derivative, Epidemiology, Adams-Bahsforth scheme.

## 1 Introduction

Bovine babesiosis (BB) is transmitted through tick bites and is one of the most common diseases in humid areas to assault bovine populations. There is a substantial financial loss in hot and warm regions due to the bovine depletion of BB with a reduction of bovine products and by-products. Furthermore, environmental conditions in all these regions encourage the reproduction and survival of ticks so that bovine animals continue to interact with these vectors [1]. Therefore, when parasites infect the ovaries of the female ticks, a vertical spread in the bovines and ticks is probably to occur [1]. The origins of behavior in syndromes have been known for a long time and are an important real-life problem. The susceptible-infected-recovered (SIR) model was developed by Kermack and Mc Kendrick [2]. It is the most relevant model to use to understand infectious diseases. A particular case of an ordinary differential equation system is used to study various types of diseases. Aranda et al. [3] presented the bovine babesiosis and tick infection epidemiological model.

Differential equations are often used to explain the behavior of real-life phenomena. However, sometimes describing the real-life phenomenon modeled by differential equations with an integer order may be difficult. If someone is interested in knowing the behaviors of the problem at a fractional-order derivative, it may not be possible. Also, exploring the dynamics between two points can be difficult when the derivatives are defined in an integer-order. Fractional calculus has been (FC) introduced in science and engineering to overcome these forms of limitations found in integer-order derivatives. Because of its inherited properties and the definition of memory [4, 5], fractional–order models are more practical and better suited to specific phenomena than integer–order models. Many papers related to FC have recently been published using different methods [6–9]. Fractional derivatives and integrals have nonlocal characteristics, which means that the next state in a model depends not only on the present state but also on all previous states. Caputo and Fabrizio [10], have recently proposed a new definition of fractional differentiation with an exponential kernel rather than a power-law that is commonly used in the Liouville-Caputo sense. Such newly defined derivatives remove any singularity and effectively describe the memory effect [11–17].

The fractional differential equations (FDEs) are usually associated with systems with multiple time scale dynamics and memory effects arising in different biological systems [18–25]. Mathematical models with FDEs have proven essential to understand the dynamics of the epidemic models of BB. In this regard, various authors have analyzed the dynamics of the non-integer order model of BB diseases. Zafar et al. [26] studied the stability and existence of steadiness points and investigated the stability of the endemic equilibrium (EE) point locally and asymptotically. Aranda et al. [3] suggested a discrete model for the transmission of BB diseases. Saad-Roy et al. [28] introduced the BB disease model on Juvenile Cattle.

## 2 Preliminaries

Here, we propose the basic definitions that we will use in the paper.

**Definition 1.** *[29] The Riemann-Liouville form of fractional integral operator of order $\vartheta > 0$ of a function $f : (0, \infty) \to \mathcal{R}$ is defined by*

$$ {}_0^{RL}D_t^{-\vartheta} f(t) = \frac{1}{\Gamma(\vartheta)} \int_0^t (t-\tau)^{\vartheta-1} f(\tau) d\tau, \ \ t > 0, \tag{1}$$

*or*

$$ {}_0^{RL}\mathbb{I}_t^{\vartheta} f(t) = \frac{1}{\Gamma(\vartheta)} \int_0^t (t-\tau)^{\vartheta-1} f(\tau) d\tau, \ \ t > 0, \tag{2}$$

*where $\vartheta > 0$ and $\Gamma\left(.\right)$ is Gamma function.*

**Definition 2.** *[29] The Riemann-Liouville form of fractional derivative of order $\vartheta > 0$ of a function $f : (0, \infty) \to \mathcal{R}$ is given by*

$$ {}_0^{RL}D_t^{\vartheta} f(t) = \begin{cases} \frac{1}{\Gamma(n-\vartheta)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(\tau)}{(t-\tau)^{\vartheta-n+1}} d\tau, & 0 \le n-1 < \vartheta < n, n = [\vartheta], \\ \left(\frac{d}{dt}\right)^n f(t), & \vartheta = n \in N. \end{cases} \tag{3}$$

**Definition 3.** *[29] The Caputo fractional derivative of order $\vartheta > 0$ of the function that has been given in Definition 2 is presented as*

$$ {}_0^{C}D_t^{\vartheta} f(t) = \begin{cases} \frac{1}{\Gamma(n-\vartheta)} \int_0^t \frac{(d/d\tau)^n f(\tau)}{(t-\tau)^{\vartheta-n+1}} d\tau, & 0 \le n-1 < \vartheta < n, n = [\vartheta], n \in N, \\ \left(\frac{d}{dt}\right)^n f(t), & \vartheta \ = n, n \in N. \end{cases} \tag{4}$$

For the convenience, we use the notation of $\mathbb{I}^{\vartheta} f(t)$ to represent the Caputo fractional integral operator ${}_0^{C}\mathbb{I}_t^{\vartheta} f(t)$.

**Definition 4.** *[29] The Laplace transform (LT) of the Caputo operator of a function $f(t)$ of order $\vartheta > 0$ is defined as*

$$ \mathcal{L}\left[{}_0^{C}D_t^{\vartheta} f\left(t\right)\right] = \varrho^{\vartheta} f(\varrho) - \sum_{v=0}^{n-1} f^{(v)}(0) \varrho^{\vartheta-v-1}. \tag{5}$$

**Definition 5.** *[29] The LT of the function $t^{\vartheta_1 - 1} E_{\vartheta,\vartheta_1}(\pm \mu t^{\vartheta})$ is defined as*

$$ \mathcal{L}\left[t^{\vartheta_1-1} E_{\vartheta,\vartheta_1}(\pm \mu t^{\vartheta})\right] = \frac{\varrho^{\vartheta-\vartheta_1}}{\varrho^{\vartheta} \mp \mu}, \tag{6}$$

*where $E_{\vartheta,\vartheta_1}$ is the MLF with two-parameter where $\vartheta, \ \vartheta_1 > 0$.*

## 3 Model Formulation

The creation of mathematical models of diseases under certain assumptions provides important information about the future course of these diseases. Thanks to this information, different types of strategies can be developed for the spread or prevention of the related diseases to a lesser population. This situation has also been taken into account for the babesiosis disease, and many mathematical models have been developed for this disease. In this section, we present the biological model of babesiosis. The assumptions mentioned in Aranda et al. [27] are used.

Based on the assumptions, the following integer-order nonlinear system of equations may be presented to model the dynamics of transmission of the suggested disease [27, 30]:

$$\begin{aligned} X'(t) &= (\mu_B^{\vartheta} + \alpha^{\vartheta})(1 - X(t) - Y(t)) - \beta_B^{\vartheta} X(t) V(t), \\ Y'(t) &= \beta_B^{\vartheta} X(t) V(t) - \lambda_B^{\vartheta} Y(t), \\ V'(t) &= \beta_T^{\vartheta}(1 - V(t)) Y(t) - \mu_T^{\vartheta} p V(t). \end{aligned}$$

The parameters of the model are identified in Table 1.
Although integer order equations give some successful results, it is obvious that FDEs give more realistic results to real phenomenons than integer order equations. The main feature that distinguishes FDEs from integer order is that FDEs nonlocal property that is not found in integer differential equations. In addition, many epidemic diseases naturally related to with memory and hereditary properties. This event can be successfully mirrored using FDEs. Moreover, FDEs minimizes the errors caused by parameters that we have to neglect while modeling. Because of these useful features of FDEs, many disease models have been studied using fractional order differential equations.

In the fractional systems, dimensionally consistent is a very important tool, in which the units of measurement from the left- and right-hand sides of the equations are coherent. This consistent can be provided by modifying the parameters involved in the right-hand side of the equations, e.g. raising them to power $\vartheta$. In this context, we have extended the model given in (7) to the fractional-order which is presented in the following system:

$$\begin{aligned} {}_0^C D_t^{\vartheta} X &= (\mu_B^{\vartheta} + \alpha^{\vartheta})(1 - X(t) - Y(t)) - \beta_B^{\vartheta} X(t) V(t), \\ {}_0^C D_t^{\vartheta} Y &= \beta_B^{\vartheta} X(t) V(t) - \lambda_B^{\vartheta} Y(t), \\ {}_0^C D_t^{\vartheta} V &= \beta_T^{\vartheta}(1 - V(t)) Y(t) - \mu_T^{\vartheta} p V(t). \end{aligned}$$

with initial conditions

$$X(0) = X_0 \geq 0, Y(0) = Y_0 \geq 0, V(0) = V_0 \geq 0. \tag{7}$$

**Table 1** The biological meanings of parameters along with their taken values

| Par. | Meaning | Value | Sources |
|---|---|---|---|
| $\mu_B^{\vartheta}$ | Bovine birth rate | 0.0002999 | [30, 31] |
| $\alpha^{\vartheta}$ | Fraction of those that are controlled and moved to the susceptible population | 0.001 | [30, 31] |
| $\beta_B^{\vartheta}$ | Infected tick rate | 0.006 | [30, 31] |
| $\lambda_B^{\vartheta}$ | Rate of those that are treated from the parasite | 0.000265 | [30, 31] |
| $\beta_T^{\vartheta}$ | At rate infection of a susceptible tick may occur when the infected bovine bites | 0.00048 | [30, 31] |
| $\mu_T^{\vartheta}$ | Natural birth rate | 0.0016091 | [30, 31] |
| $p^{\vartheta}$ | The possibility that a susceptible tick may have been born from an infected one | 0.1 | [30, 31] |
| $X(0)$ | The rate of infected with symptoms who have been in quarantine total | 0.3756 | [30, 31] |
| $Y(0)$ | Mortality rate due to complications | 0.5184 | [30, 31] |
| $V(0)$ | Mortality rate due to complications | 0.6 | [30, 31] |

## 4 Numerical Solution of the Model

In this part of the study, numerical solutions have been addressed to check the reliability and efficiency of the present technique. All the numerical calculations are obtained by the Adams-Bashforth numerical scheme. Moreover, the variation of each sub-population over time has been simulated for different values of the fractional parameter $\vartheta$ by using the taken parameter values given in Table 1. In addition, considering the parameters that significantly change the direction of the disease, graphics have been obtained for different values of these parameters. We have shown dynamical behavior of each state variable from the proposed babesiosis model in Figures 1-3 for varying values of the fractional order parameter $\vartheta$.

The values of the initial conditions and the parameters are taken from the references [30, 31]. From Fig. 1, one may observe that a susceptible group that may turn into infected decreases with increasing in time and decreasing values of $\vartheta$. Fig. 2 indicates that the infected bovine group infected by the Babesia parasite increases with time as the value of $\vartheta$ decreases. From, Fig. 3, we may find that the infected ticks group infected by the Babesia parasite increases with time as $\vartheta$ takes decreasing values. But subsequently, its nature becomes the opposite, which means it decreases over time. The graphical depictions represent that the model is mainly dependent on the fractional order.

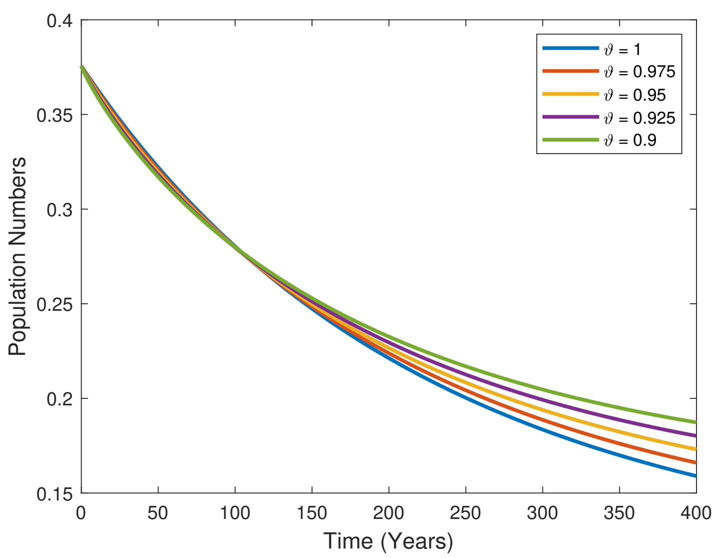


**Fig. 1**: Solution plot for $X(t)$ vs. time $t$ at different values of $\vartheta$

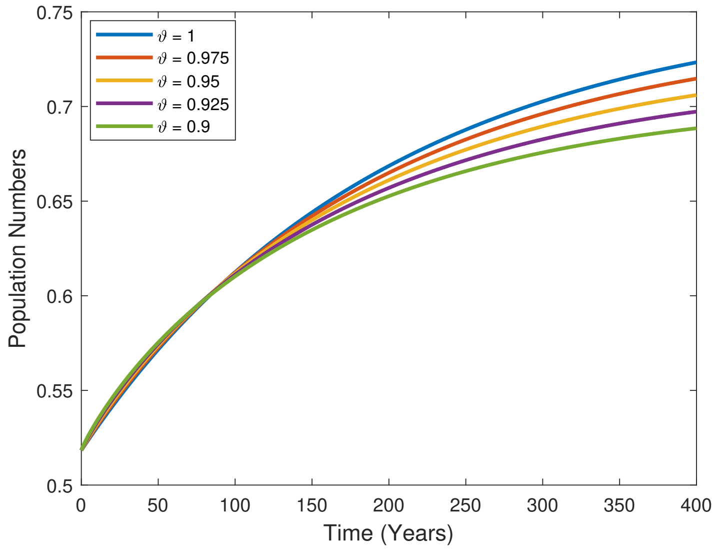


**Fig. 2**: Solution plot for $Y(t)$ vs. time $t$ at different values of $\vartheta$

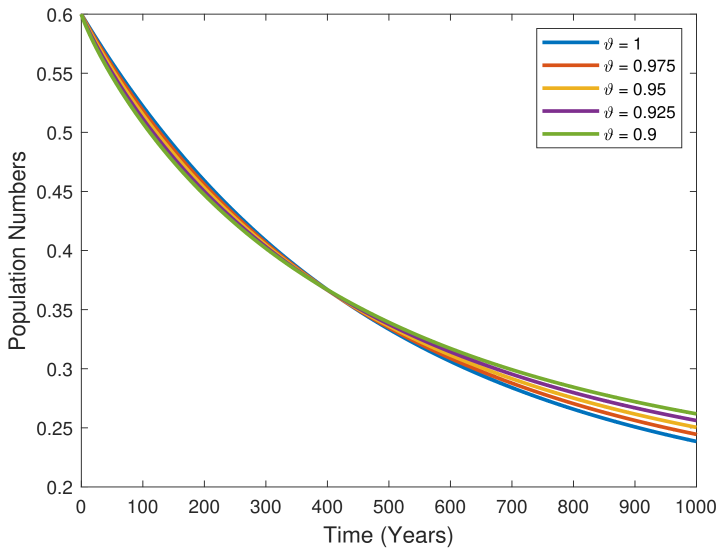


**Fig. 3**: Solution plot for $V(t)$ vs. time $t$ at different values of $\vartheta$

## 5 Conclusion

In this study, a new fractional order mathematical model of the babesiosis disease in bovine and tick populations has been proposed and analyzed. Firstly, we have reconstructed a new babesiosis model of integer-order. Then, we have considered the Caputo type fractional derivative instead of integer derivative, so that the system (7) is dimensionally consistent: the units of measurement from the left- and right-hand sides of the equations agree. It has been achieved by modifying the parameters involved in the right-hand side of the equations, e.g. raising them to power $\vartheta$.
The solution for the fractional order model is obtained through the implementation of the Adams-Bashforth predictor-corrector method which is given in [32]. Related simulations are performed to reflect the effect of the fractional-order. Through simulation, it is obvious that the Caputo derivative exhibits more interesting behavior when appears to be 1. Finally, this study shows that the mentioned method along with the Caputo fractional derivative is an efficient way of handling nonlinear FDEs.

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# MATHEMATICAL EDUCATION

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# Possibilities and Constraints of History of Mathematics for Cultural Diversity

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**Abstract:** Despite the benefits of history of mathematics in mathematics teacher education programs are well known in the literature, few studies examine the issues of cultural diversity. This article explores the possibilities and limits of mathematics history for cultural diversity in mathematics based on prospective teachers perspectives. Twelve middle school prospective mathematics teachers participated in face-to-face interviews following the completion of a History of Mathematics course that focuses on multicultural dimension of knowledge development processes through the history. According to the results, the course has provided a space to recognize and become aware of the contribution of diverse cultures through the perceived need of solving daily life problems across civilizations. The possibilities, nevertheless, have been constrained by ranking the mathematical needs from survival to intellectual. In the light of these promises and limits, implications of the findings for History of Mathematics courses are discussed.

**Cultural diversity, History of mathematics, Multicultural mathematics, Mathematics Teacher education**

## 1 Introduction

Calls for inclusion of History of Mathematics [HOM] into mathematics teacher education programs are not new. In addition to improve teachers' pedagogical repertoire to teach mathematics effectively [1], HOM is a promising space for prospective teachers to engage with the cultural dimension of mathematics [2]. Experiencing the historical construction of mathematical knowledge can build a cultural understanding of the subject [1], and can bring an awareness to the growing nature mathematics in diverse settings [3]. Despite the commonly held perception of mathematics as a product of Western culture [4], HOM can provide an opportunity know and appreciate the other approaches or ways of thinking to do mathematics that could be found in diverse cultural settings [5].

Opening a chapter for cultural diversity in mathematics history is important as the recent research confirms the need to recognize the contribution of non-European groups to mathematical knowledge, which is often portrayed as an achievement of selected groups of people in mathematics textbooks [6]. Differences in history, geographic location, culture and belief do have a significant role in the historical development of scientific and mathematical knowledge. As each cultural group has its own ways to count things or reorient themselves in their living space [7], studying HOM with an emphasis on cultural diversity potentially unpack mathematical diversity in the history and enable an understanding of mathematics as a product of social interactions and human culture.

The aim of this study is to investigate how PMTs recognize and become aware of the contribution of diverse cultures to the development of mathematical knowledge. The focus on cultural diversity in the context of history considers HOM not only as a pedagogical tool to improve mathematical learning, but also as a goal [8]. History-as-a-goal argument emphasizes the development of mathematics within and across and different cultural setting and considers mathematics as a cultural enterprise. When mathematics history attends to the cultural dimension of mathematics, one can respect and value the work of others, recognize the different context, needs, and purposes, and realize that each society significantly contributes to the body of mathematical knowledge [9].

HOM is considered as an epistemic endeavor that provides an analysis of development of mathematical knowledge within a culture or across different cultures [10]. Looking at various mathematical practices across different geographies and periods reveal a more localized and specific set of mathematical ideas that are different from those constructed as mainstream mathematics [7]. Studying and recognizing of diverse mathematical systems in a specific socio-cultural environment is considered as a way to challenge mainstream mathematics that is often associated with the Western civilizations [11]. Activities such as counting, locating, measuring, designing and playing in diverse cultures provide concrete evidence from diverse cultures approving that all of these activities exist [7]. Analysis of diverse mathematical contributions brings an alternative trajectory for HOM that takes into account significant developments in mathematics in diverse cultures.

Given the range of conceptual arguments for cultural diversity in the context of HOM, there is a lack of empirical studies examining how the cultural aspects of the historical development of mathematical knowledge is taken in educational settings such as curriculum, teaching or teacher education. Within this scarcity, there is a recent study that examines mathematics textbooks in terms multiculturalism, which is conceptualized as the representation of different sociocultural practices in mathematics history [6]. The examples that the authors find in the textbooks are promising to attain the cultural diversity in the development of mathematical knowledge, yet they are lack of effective integration of different cultures since HOM is only presented as anecdotes or as side notes. Authors conclude that there is a need for more systemic research since the Eurocentric perspective of mathematics history is still dominant in the textbooks despite the recognition of diverse cultures. To contribute the HOM scholarship that intersects with the issues of cultural diversity, this paper investigates how a HOM course that is grounded in social and cultural elements are perceived by PMTs, including what the emphasis on cultural diversity might offer for teacher candidates and what the constraints are in these processes.

The empirical research on HOM in teacher education programs mainly focuses on the views and preparedness of PMTs to use HOM in their future mathematics teaching [12]. Overarching theme of the research indicates that prospective teachers are generally positive about using HOM in their future mathematics teaching [13-15], meaning that they are open to learn HOM and to gain experience in the integration of it [17]. Studies reported the contribution of HOM to improve mathematical knowledge of prospective teachers [16] and to develop positive attitudes towards mathematics [18], but they have not attained issues of cultural diversity. Although empirical studies in teacher education programs in the context of HOM are increasing, the body of scholarship has yet to give insight to cultural and humanistic aspects of historical development of mathematical knowledge. In this context, following research question is asked: What are the potentials and limitations of the emphasis on cultural diversity in the context of a HOM course for PMTs?

## 2 Methods

### 2.1 Participants

Participants were 12 PMTs enrolled in an elective history of mathematics course at a large research university in Turkey. All participants were in elementary mathematics education degree program for middle schools at bachelor's level. Five of them were in their third year and seven of them in the fourth year.

### 2.2 History of Mathematics Course

This study was conducted in an elective two-credit history of mathematics course for prospective mathematics teachers. The main purpose of the course was to provide necessary knowledge and skills for PMTs regarding the historical development of mathematical concepts and thinking. As suggested by the Higher Education Council in Turkey, mathematics history was not merely portrayed through the Eurocentric trajectory. Instead, the course provided an account of histories of mathematical work through acknowledging and representing the ways in which various civilizations engaged in different cultures and geographies [19].

The course offered a wide range of practices for PMTs to engage with the cultural dimension of HOM. Following the study of different civilizations, mathematical problems from each civilization were solved. PMTs were instructed to solve mathematics problems as if they were in that particular time and space so that they were expected to attain the diverse ways of mathematical reasoning of those social and cultural groups. Additionally, prospective teachers watched mathematicians' experiences, opportunities and struggles in various sociocultural contexts. Throughout the course, there was also focus on the pedagogical dimension of HOM as an instructional strategy to teach mathematics.

### 2.3 Data Collection and Analysis

Data source was the face-to-face interviews, asking how PMTs made meaning of the existence of diverse cultures and epistemologies throughout mathematics history. After transcriptions, the instances that PMTs talked about issues that relate culture, diversity and humanity in the context of historical development of mathematical knowledge were coded with inductive qualitative methods [20]. Through constant comparison method, the themes are identified.

## 3 Findings

Two main findings emerged from the data. First, the HOM course offers effective spaces for prospective teachers to understand cultural diversity in mathematics where they affirm the diverse needs and acknowledge the diverse contribution and human involvement in construction of mathematical knowledge. Second, these possibilities are constrained by two main issues: PMTs differentiate mathematical needs and rank the human contribution and involvement on a hierarchy.

### 3.1 Possibilities of Mathematics History for Cultural Diversity in Mathematics

During the HOM course, PMTs had the opportunity to study mathematics history in diverse cultures and civilizations across the world. Solving the problems of daily life was one of the fundamental issues for people who needed mathematics and so developed mathematical knowledge and practices. For example, PMT8 stated that: "Actually, some of them tried to do something as a result of their daily life problems. And in different places, everyone did something according to the situations they encountered, depending on their culture and environment."

Solving the problems of daily life with mathematics indicated the situated nature of mathematics, which was influenced by diverse localities, cultures and environment. For PMTs, mathematics emerged and was used by different civilizations all over the world. Diverse communities developed mathematical knowledge to respond their needs in everyday life. Given the diverse geographies that were included in the alternative trajectory of HOM, at the end of the course, PMTs generally affirmed the existence of diverse mathematical needs that were specific to each culture. They were able to discuss the conditions mathematical knowledge development and involvement of humans in these processes. As PMT2 explained: "We all saw how much mathematics has undergone change from past to present. I think that people's own needs were the main factors in the movement of this knowledge [...] The problems caused by the Nile river pushed them to use mathematics and it has progressed day by day. As new needs and new problems emerged, people tried to [apply] the knowledge they had to other situations and they set sail for new horizons at the points where they were insufficient."

Mathematical needs, ranging from agricultural issues to the social problems had been a way to affirm the role of humans through mathematics history. PMT7 provided an example: "For example, in Egypt, mathematics was developed based on agriculture, and in another place, mathematics has been constructed to meet different needs. That is why, it may be the reason why mathematics was emerged in different civilizations in different ways and people oriented towards mathematical sciences based on their own needs in that particular period."

Learning HOM across diverse cultures and civilizations opened up possibilities for PMTs to recognize diverse range of mathematical needs while solving daily life problems and also humanistic dimension of mathematics. For most of the PMTs, development of mathematical knowledge was situated in a specific context and it was shaped by needs of daily life in that particular period and culture.

 

When PMTs were asked for their thoughts, feelings and ideas about learning HOM from a non-European trajectory, all of them positively responded. PMT1, for instance, considered mathematics had multiple points of emergence rather than single point of birth: “Actually, mathematics does not have a single basis… thousands… actually [HOM from a non-European trajectory] shows that [mathematics] has a lot of foundations. From this side as an advantage, this is very nice [to know].”

Learning HOM across diverse cultures enabled PMTs to further contextualize how mathematical knowledge emerged in a particular time and place in addition to mathematical needs. PMTs had opportunity to gain multiple perspectives regarding how mathematical knowledge developed through the history. PMT3 provided an explanation of her growth: "How mathematics was discovered in Greece, according to what? At that time, what were people’s views and how were they oriented towards mathematics? Or, in Egypt? [Cultural diversity in HOM] enabled me to look in multiple directions, actually… I was looking only through a single dimension according to what I have seen and needed in my own life. I have been doing mathematics in a single logic. But when I look back history, when I see different civilizations, it could be Indian, Egypt or Greeks or Europe. They all looked from different perspectives and this helped me to gain a broader perspective without classifying. How did diverse civilizations think?"

As seen in the quote above, PMT3 broadened her perspective about HOM when studying other cultures and civilizations. Particularly, she decentered her own life and mathematical experiences to make meaning of cultural diversity in mathematics history. Connecting her own and different historical experiences to the front, she considered multiple and diverse contributions of each culture without classifying. She continued: "For example, I am a person from a particular culture and I look at things in relation to what I have in my mind. But, in [HOM], since we were able to see how different cultures view mathematics… How can I say? We have seen the views of different civilizations as a religion, as a culture, without making a classification."

Bringing diverse civilizations together in the HOM course stimulated PMTs thinking about cultures that were different from theirs. In this sense, PMT9 highlighted the hidden civilizations in the mathematics history such as Chinese and Indian. She appreciated studying those cultures and their contributions to the mathematical knowledge: “If we do not look, explore or investigate, [the Chinese mathematics] remains closed. Similarly, I was interested in Indian mathematics. If we do not look at [that culture], we would not be aware of them.”

Learning about multiple emergence points of mathematics provided a historical awareness of diverse mathematical contributions. For PMTs, mathematics became a knowledge that each culture studied and used. Also, PMTs learned about historical integrity and connectedness of diverse contributions. That is, diverse cultures were not isolated places that gave birth to mathematics. Instead, as PMT2 pointed out, emergence and development of mathematics was a process where each culture was influenced by one another: "These are interconnected or interrelated processes. In some presentations, [for instance], the interaction between Arab and Indian civilizations. Because of something, this might be trades or other issues situated at that time, there is an interaction. Like introduction of Indian and Arab civilizations to the West."

This quote indicated that the alternative trajectory of HOM enabled a focus not only on how mathematical knowledge was developed, but also how mathematical knowledge bases were connected with one another. In this way, it became possible to problematize one single point of birth of mathematics. To conclude, prospective mathematics teachers recognized the cultural diversity embedded in mathematical knowledge by learning diverse contributions made by several societies and communities.

### 3.2 Constraints of Mathematics History for Cultural Diversity in Mathematics

Once PMTs considered the mathematical needs of daily life as a way to reflect upon the cultural diversity in the context of HOM, those needs were also ranked in a hierarchical continuum. Mathematics, at first, was used to respond to daily life needs such as food, agriculture or water sewing. Provided that the survival needs were met, mathematics was for conducting science. As PMT7 argued: "As I examined the civilizations, once the origin point is the daily needs. After that, as in the case of Hypatia, there was a little more contribution to the science. Isn’t it? As science improves, the comfort of people increases in daily life. For example, more needs can be answered […] and the difficulties they experience become a little easier and new needs are emerged […] So, the first point of departure is to answer the questions in their mind. In other words, we see that all scientists are looking for answers to those problems and this enable science to progress. As they find answers, new questions are emerged and I think [the science] is progressing in this way."

In these explanations, HOM became not a flat platform to understand mathematical discoveries and inventions in their own historical contexts. Rather, the needs were differentiated on a range of continuum from survival to intellectual according to how mathematics was used. PMT3 provided an example on this issue: “For example, Greeks [used] the mathematical formulas that they found to study astronomical sciences, but the Egyptians for field and agriculture…”

In addition to differentiation processes, diverse needs of humans were taken as an instrument to rank the kinds of mathematics in which specific cultures and civilizations were engaged. That reasoning was transferred to rank people, civilizations and so cultures on a hierarchical continuum. For example, PMT11 stated that doing mathematics for scientific purposes was reserved for the “distinguished” people: “Mathematics is the job of distinguished people in Rome or in Greece. When we teach mathematics we can say so… thinking that only distinguished people can do that. I think so, too.”

Despite PMTs recognized the diverse needs while developing mathematical knowledge throughout mathematics history, the civilizations were differentiated in relation with their diverse needs. In other words, provided that humans fulfill their daily life needs, social groups in the history were considered as ready to do mathematics for scientific purposes. This was highly related with the progressive and linear conception of mathematical knowledge development without considering dynamic nature of social environments.

## 4 Discussion

The emphasis on cultural diversity in a HOM course has provided a space for PMTs to recognize and become aware of the contribution of different civilizations to the historical development of scientific and mathematical knowledge. In this sense, history can enable PMTs to think and reflect on the humanistic and multicultural facet of mathematical knowledge production through an affirmation of diverse needs and an acknowledgement of diverse contributions. Nevertheless, these possibilities have been constrained by the ranking the needs from survival to intellectual, that distinguishes the civilizations in terms of their purposes of doing mathematics.

Needs of daily life have been a way to affirm the cultural diversity and the role of humans in the historical development of mathematical knowledge. The recognition of diverse needs throughout the history becomes a way to encounter with the humanistic aspects of the development of mathematical knowledge such as the effects of experience, lives and senses of human beings [21]. However, the involvement of humans and their mathematical needs has also been ranked from their survival to intellectual. That is, while PMTs try to understand the cultural diversity in mathematical practices of different civilizations, they simultaneously compare their needs in a context where humans are considered to be able to deal with their intellectual needs provided that they meet their survival needs. This logic becomes a way to position diverse cultures on a hierarchical continuum and confirms Grugnetti and Rogers’ argument, indicating that cultural dimension of HOM cannot put everyone in equal position while recognizing the diversity in mathematical knowledge [9]. Needs as an affirmation of cultural diversity and as an instrument to rank different civilizations suggest to bring issues of equity and social justice in the processes of mathematical knowledge development. That is, this finding indicates the necessity to examine history of science and mathematics with larger social debates, concerning with how relations of power play out in the knowledge making processes across the history.

HOM with a non-Eurocentric trajectory has stimulated a conversation with PMTs about multicultural facet of mathematics. Instead of taking a narrow Eurocentric perspective [9], most of PMTs have started to think about multiple locations in which mathematics has been emerged, discovered and developed. Examination of mathematics history with an emphasis on cultural diversity provides a space for PMTs to reflect on the mathematical contributions of diverse cultures that are not usually acknowledged in contemporary mathematics textbooks or curriculum [6]. Developed an understanding of multicultural nature of mathematical knowledge, PMTs start to problematize the single origin of mathematical knowledge by contextualizing mathematical knowledge and practices in diverse places and times.

## 5 Conclusions and Implications

The aim of this paper is to investigate the cultural and humanistic aspects in the historical development of mathematical knowledge. The focus on the cultural diversity in mathematical knowledge development can unpack the constraints and opportunities afforded by social context, including opportunities, difficulties and struggles that mathematicians have experienced over the history. This can reveal a more connected image of mathematics and mathematicians. The emphasis on cultural diversity could be a way of demystifying how mathematical knowledge is accumulated over time and also demythologizing the “white male myth” discourses in the mainstream mathematics education [4]. As Radford and his colleagues put, HOM is a place to overcome “one-sidedness” of meanings, to enter into a conversation with others cultures and to explore the actual experiences of mathematicians and their diverse needs and processes of mathematical knowledge development [10]. That is, HOM that emphasizes humanistic and cultural aspects of knowledge production can enable affirmative notions of mathematics ability that recognize every human being capable to do mathematics and related mathematical practices such as counting, ordering, classifying, locating or designing.

Cultural diversity in the context of HOM is, at the same time, can only be taken as the celebration of diversity rather than a contestation of mythical discourses that reproduce inequalities. As seen in the ranking of mathematical needs, the hierarchical gradation of cultures and civilizations are reproduced in the interpretative accounts of prospective mathematics teachers. In future HOM courses, these issues need to be addressed so that historical development of mathematical knowledge would not be perceived as belonging to particular groups of people and civilizations.

To conclude, learning HOM together with its cultural and humanistic aspect can potentially provide possibilities to respond the issues of diversity in the mathematical knowledge production throughout the history. That said, further empirical and conceptual work is needed to address those constrains so that history of science and mathematics scholarship becomes responsible to the current landscape of education.

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# An In-Service Primary Teacher's Responses to Unexpected Student Questions About the Measurement of Length

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**Abstract:** In our observations of an in-service primary teacher's lessons on the measurement of length, we were interested by her responses to some unexpected questions from the students. Therefore we proceeded to answer the following research questions: What kind of unexpected moments occurred during the in-service primary teacher's lessons on the measurement of length? How did the teacher respond to these unexpected moments? In our reflections, we draw on the Knowledge Quartet [1], a theoretical framework for the analysis of mathematical knowledge in teaching. In particular, we draw on the contributory codes of the Contingency dimension of the Knowledge Quartet. Our study was located in a school in Turkey, where the in-service primary teacher-participant taught a fourth-grade class with 36 students. She taught nine 40-minute lessons to cover the objectives regarding the measurement of length in the curriculum. These lessons were observed and video-recorded by one researcher, who made field notes in order to identify what seemed to be unexpected contributions – ideas and questions - from the students. Ten unexpected moments were identified in the observed lessons, and relevant contributory codes of the Contingency dimension of the Knowledge Quartet framework [1] were assigned to each of these moments. Semi-structured interviews were conducted with the teacher after each lesson, in order to interpret and understand the reasons for her responses to these contingent moments. Three particular moments arising from students' unexpected questions in the lessons were identified for further analysis. The nature of the teacher's response to each of these events was classified as one of three response-types identified by Rowland et al.[2].

**Keywords:** Contingency, Knowledge, Measurement, Teachers.

## 1 Introduction

Since measurement, one of the content strands of mathematics curriculum in many countries, is commonly used in daily life, everybody should learn about measures and measurement [3]. Because of its importance, students in Turkey start to learn measurement concepts from first grade, in order to make sense of the world around them.

Despite this early introduction of measurement concepts, students score lower grades in the measurement content strand when compared with the other content strands in international tests such as Trends in International Mathematics and Science Study (TIMSS) and the Programme for International Student Assessment [PISA] [4]. In order to help students understand measurement concepts conceptually, and to increase their grades in tests of attainment, "we need much stronger measurement instruction in the early years" ([5], p. 299).

The very first measurement concept learned by students, and one with which students have some difficulty, is the measurement of length. Fennema and Franke [6] assert that how teachers teach a concept influences what students learn. Similarly, the way in which teachers respond to their students' ideas and questions is important for students' conceptual understanding [7]. Therefore, it is reasonable to expect that students' difficulties of length measurement can be attributed to how the topic is taught, and in particular, how teachers respond to their students' ideas and questions.

Every (mathematics) class begins with a series of imagined scenarios in the teacher's mind. Leinhardt [8] refers to the series of imagined scenarios is termed as the lesson 'agenda' - the agenda is the intended lesson structure. Another term, 'lesson image', is used by Morine-Dershimer [9] to capture a general picture of what the teacher expects to happen as the lesson progresses. The lesson image includes the content and associated learning objectives, the way the content would be taught, the tasks that the students would engage in, and how teachers would respond to students [10]. However, sometimes events do not go according to the intended agenda, or lesson image, and the teacher then encounters surprises – some unexpected events. Rowland, Turner, Thwaites, and Huckstep [2] refer to these unplanned events as 'contingent' moments. These researchers found that in mathematics classrooms, such contingent moments are located in, and triggered by, three types of situations or events: (1) responding to students' responses and questions, (2) teacher insight, and (3) the unexpected (un)availability of tools and resources [10]. The focus of this paper, responding to students' responses and questions, is the moments that are initiated by a student's unexpected response to a stimulus within the lesson, requiring a response from the teacher to the student.

In response to this trigger, teachers' responses can be one of three kinds: (1) to ignore, (2) to acknowledge but put aside, and (3) to acknowledge and incorporate [10]. If the teacher just ignores or dismisses, or does not give the student's response or question further consideration, then his/her response type is regarded as ‘ignoring’. That is, the teacher seems as if the student’s response or question was not noticed. The second kind of response - acknowledging but continuing the plan - refers to moments in which the teacher recognises the student’s response or question, but does not deviate from the lesson plan. If the teacher goes beyond accepting the student’s response or question by incorporating it into the lesson and adjusting the lesson agenda, then the teacher’s response is one of ‘acknowledging and incorporating’. The purpose of this paper is to identify the unexpected moments initiated by students’ questions and ideas, and to analyse the teacher’s responses to these student contributions. We address the following research questions:

- What kind of unexpected moments occurs during an in-service primary teacher’s lessons on measurement of length?
- How does the teacher respond to these unexpected moments?

## 2 Methodology

A case study design was used to identify the unexpected moments that occurred during an in-service primary teacher’s lessons on the measurement of length, and her responses to these unexpected moments. This study was conducted in a public school in Turkey, where the in-service primary teacher-participant, Ozge, taught a fourth-grade class (age 10-11) with 36 students. Ozge taught nine 40-minute lessons to cover the learning objectives of length measurement in the curriculum. All of these lessons were observed and video-recorded by the first researcher, who also took field notes. To interpret and understand Ozge’s responses to these unexpected ideas and questions, semi-structured interviews were conducted following her lessons. Unexpected moments were identified in the observed lessons, and relevant contributory codes of the Contingency dimension of the Knowledge Quartet framework [11] were assigned to each of these moments.

## 3 Findings

Ten unexpected moments were identified in Ozge’s length measurement lessons, three of which were coded as responding to a student’s question. We will consider both the trigger of the unexpected moments and Ozge’s responses to them. We shall include transcripts from the dialogues between Ozge and her students, or between the students.

**Unexpected Moment 1:**

The first unexpected moment of the length measurement instruction triggered by a student’s question is about metre to kilometre (m-km) conversion. The students were told that they have to divide a given length by 1000 to convert it from metres into kilometres. Ozge explained that “there are some numbers that cannot be divided by 1000 to give a whole number. For instance, there is an example in Figure 1 in the textbook:”


**Fig. 1**: Example for m-km conversion textbook

Ozge continued:

Ozge:We said that we could write 1560 km as 1 km 560 m. We can also write this as 1.560 km. That is, while the part before the decimal point gives km, the part after the decimal point gives m.
Student:But, it is written as 1.56 km, isn’t it?
Ozge:As you may remember, I said that the zeros may be omitted.

Ozge only considered the student’s idea in a cursory fashion. Therefore, whether Ozge’s explanation was always correct or not was questioned during the post-interview.

Researcher:What do you think about why the result was written as 1.56 km?
Ozge:The zeros after the decimal point may not be written.
Researcher:You said that the fraction part after the decimal point gives us metres. There is 56 after the decimal point. So, can we accept this as 56m?
Ozge:hmm. Actually, we can accept; however, it will be wrong. Then, why was it written like that in the textbook?
Researcher:Can the information written in the textbook be wrong?
Ozge:Maybe.
Researcher:If you asked a question similar to this in your exam and one of your students had written 1 km 56 m, and another one had written 1 km 560 m, how would you assess their solutions?
Ozge:Since the student solved it according to my explanations, I would accept it as correct. However, it should be 1056 m in order to be written as 1 km 56 m. Why did the textbook give this way? There may be something that I don’t know.

As can be seen from the dialogue, Ozge does not know how best to respond to the student's question. That is, if Ozge could have gone back in time and retaught this part, she still could not offer a clear explanation about it. Telling the students that the part before the decimal point gives the km and the part after the decimal point gives may result in some misconceptions before the students had sufficient opportunity to understand the reason for the relationship between a metre and a kilometre. Since Ozge could not understand why her explanation was wrong, her response to this unexpected moment, triggered by a student's question, was coded as acknowledging but sidelining the question from the student.

**Unexpected Moment 2:**
Another unexpected moment was related to conversion between centimetres and millimetres (cm-mm).
Before telling the students a rule for converting a length from cm to mm, Ozge wanted to investigate her students' own ideas.

Ozge:Suppose that a question asks us to convert 3.4 cm to mm. How can we do it?
Student1:We will turn 3.4 into 34, multiply by 10.
Ozge:Will we turn into 34 and then multiply by 10?
Student1:Yes.
Ozge:No.
Student2:We will multiply the number to the right of the decimal point by 10.
Ozge:No. Ok, last chance, come on.
Student3:We put 4 aside, then multiply 3 by 10 and add 4.
Ozge:No.

The problem requires students to recognise that 3 cm = 30 mm. So then 3 cm 4 mm = 30 mm + 4 mm. In effect, Student1 offers a correct, albeit procedural, solution – "multiply by 10". Ozge seems to mishear the student's suggestion, and she inserts "and" before "multiply by 10". Unfortunately, Student1 does not hear the "and", and he confirms what Ozge has said: Ozge's "No" tells him that her answer is/was incorrect. Student2 seems to be suggesting that only the 4 is multiplied by 10. Ozge says "No" but does not ask Student2 what he intended. Student3 provides a correct idea and correctly converts the cm to mm – "We put 4 aside, then multiply 3 by 10 and add 4" – but this is not what Ozge expected – and she says "No".

After Ozge watched this part of the video-recording during the interview, she was asked to reflect on these students' answers. During the interview, Ozge explained that although Student3's answer was actually correct, she did not notice that it was correct. Furthermore, during the lesson following the above dialogue, Ozge explained that the decimal point should be shifted one step to the left to convert a length given in mm to a length given in cm. Then, one of her students stated that "I understand now, 3 is the quotient and 4 is the remainder." As a reply to the student's statement, Ozge just stated that "Yes, 3 is the quotient and 4 is the remainder." As can be understood from the above excerpt and statement, Ozge acknowledged her students' ideas; however, could not incorporate them into her lesson effectively.

**Unexpected Moment 3:**
The last unexpected moment occurred while the students were solving the following question, given in the textbook.

*There are three ropes of different lengths, 4 mm, 12 cm, and 17 mm, respectively. If these ropes are joined together end to end, what will be the length of the new rope?*

One of the students came to the board to solve the problem. First, he added the lengths of ropes in similar units - 4 mm and 17 mm - and presented the answer as 12 cm 21 mm. When the student was about to return to his desk, another student asked Ozge if they could write the answer as 14 cm 1 mm. Following this question, the dialogue played out between Ozge and the student standing at the board:

Ozge:How many centimetres are there in 21 mm?
Student:2.
Ozge:2cm. Can we add 12 cm to this 2 cm?
Student:Yes.
Ozge:Then, what will be the total?
Student:14 cm.
Ozge:14 cm. What is left of 21 mm?
Student:1 mm.
Ozge:Yes, 1 mm. Let's write our answer, 14 cm 1 mm.

As she had done in the two unexpected moments presented above, Ozge acknowledged the student's question. However, there is a difference in the response type of Ozge compared with the previous ones. Ozge not only acknowledged the student's idea - working with the mm first - Ozge followed up the idea and asked the student questions. In this case, we see that Ozge incorporated the student's idea into the lesson.

## 4 Conclusion

In this paper, three unexpected moments coded as "responding to a student's question/idea" were focused. All three questions were related to conversion between metric units, such as cm to mm. In two of the three events, Ozge's response was to acknowledge the student's idea - but to put it aside. In neither of those events did she question the student in order to understand their meaning, or somehow incorporate it into her lesson. But in the third unexpected event, Ozge acknowledged the student's idea and then followed it up in the lesson. In short, Ozge's responses were classified as two of the kinds identified by Rowland et al. [10]: 'to acknowledge but set aside'; and 'to acknowledge and incorporate'. In order to make mathematics instruction more meaningful, teachers need to consider how to respond to students' unexpected ideas or questions [2]. To respond effectively to these ideas or questions, teachers need, first of all, to try to understand what students are suggesting or asking [12]. If teachers pause to consider the ideas suggested by students in class, and the questions they ask, they would be more likely to discover how to build upon these ideas or questions in the light of the lesson objectives [13]. The reason why Ozge was not able to acknowledge and incorporate some of her students' ideas and questions in this study may be that she did not really consider what the students suggested or asked, and so she could not decide how to use them to fulfil the learning objectives related to the measurement of length. Our approach to the analysis of this case could be applied to other cases: teachers' post-lesson reflections (with a mentor/colleague) on their responses to unexpected events in their lessons are likely to contribute to the development of their mathematics teaching.


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