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Self-adjoint extensions for a class of singular operators

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Abstract: In this study, we consider the domains of the minimal and maximal operators generated of singular differential-expression-type Sturm–Liouville and obtain all self-adjoint extensions of the operator in terms of boundary conditions.

Key words: Sturm–Liouville equation, minimal and maximal operators, self-adjoint extensions

1. Introduction

One of the fundamental problems in the theory of symmetric operators is to construct all those extensions of a given symmetric operator $\mathcal{A}$ which are themselves symmetric operators. A special case of this is the problem of setting up conditions under which an operator shall have a self-adjoint extension and to construct all the self-adjoint extensions when these conditions hold.

If $\mathcal{B}$ is a symmetric extension of a symmetric operator $\mathcal{A}$, then $\mathcal{A} \subset \mathcal{B}$, so we $\mathcal{B}^* \subset \mathcal{A}^*$. However, $\mathcal{B}$ is a symmetric operator, i.e. $\mathcal{B} \subset \mathcal{B}^*$, so we have $\mathcal{A} \subset \mathcal{B} \subset \mathcal{B}^* \subset \mathcal{A}^*$, i.e. every symmetric extension of an operator $\mathcal{A}$ is a restriction of the operator $\mathcal{A}^*$. A symmetric operator $\mathcal{A}$ is said to be maximal if it has no proper symmetric extension. Every self-adjoint operator $\mathcal{A}$ is a maximal symmetric operator. For, in this case, $\mathcal{A}^* = \mathcal{A}$, and the $\mathcal{A} \subset \mathcal{B} \subset \mathcal{B}^* \subset \mathcal{A}^*$ implies that $\mathcal{B} = \mathcal{A}$, i.e. every symmetric extention $\mathcal{B}$ of $\mathcal{A}$ coincides with $\mathcal{A}$. There are maximal operators which are not self-adjoint.

Boundary value problems for the differential operators with integrable coefficient in the finite interval were well learned (see for exp., [1, 6]). In the last years, learning the differential operators with the nonintegrable or singular coefficient has become even more important according to physical science needs.

In the present paper, in interval $(0, 1)$, all self-adjoint extensions have been considered for Sturm–Liouville operator which is generated by differential expression

$$l\,[u] = -u''(x) + cx^{-\alpha}u(x) + l(l+1)x^{-2}u(x) + q(x)u(x) \tag{1.1}$$

where c, α, and l are real numbers, $\alpha \in [1, 2)$, $l \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and $q(x)$ is a real-valued measurable bounded function and define $\sigma(x) = \int q(x)dx$.

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Note that a similar problem was completely studied in [1, 4, 6, 7] for the Sturm–Liouville equation with integrable potential and with Bessel potential $(c = 0)$. Boundary value problems with another regularization were given in [2] for $\alpha < \frac{3}{2}$ $(l = 0)$. Self-adjoint extensions of the operator generated by differential expression (1.1) in the case $l = 0$ with potential $\sigma'(.)$, $\sigma(.) \in L_2(0,1)$ were given in [5]. Regularizations used in the paper coincide with those in [5] only for $\alpha < \frac{3}{2}$ in the case $l = 0$. More generally than studies [2] and [5], that is, in the case of $\alpha \in [1,2)$ and $\ell = 0$, the operators self-adjoint extensions produced by (1.1) differential expression were obtained in [3].

2. Main results

On the set $D_0 = C_0^\infty(0,1)$, we introduce an operator L_0' by the formula $L_0'u \equiv l\,[u]$. Obviously, L_0' is a symmetric operator in $L_2(0,1)$. Its closure L_0 is called the minimal operator generated by differential expression (1.1). The adjoint operator L_0^* is called the maximal operator. Since the operators L_0 and L_0' generated by the expression (1.1) differ from the corresponding operators generated by the expression (1.1) with $q(x) \equiv 0$ by a bounded operator, in the following, we assume that

$$l\,[u] = -u''(x) + cx^{-\alpha}u(x) + l(l+1)x^{-2}u(x)$$

For the determination of the domains of the minimal and maximal operators we need a fundamental (principal) solution system for the equation

$$-u''(x) + cx^{-\alpha}u(x) + l(l+1)x^{-2}u(x) = 0 \tag{2.1}$$

First of all, let us give the following lemma:

Lemma 2.1 *If $M \neq n-1$, $\forall n \in \mathbb{N}$, then the differential equation*

$$u''(t) - \frac{M}{t}u'(t) = \frac{Q}{t}u(t),\ \ Q \neq 0 \tag{2.2}$$

has only one solution as the following:

$$u(t) = \sum_{k=0}^{+\infty} c_k t^k$$

If $M = n-1$ for any $n \in \mathbb{N}$, then the differential equation (2.2) has only one solution as the following:

$$u(t) = \sum_{k=0}^{n-1} c_k t^k + t^n \frac{Q}{n} c_{n-1} \ln t + \sum_{k=n+1}^{+\infty} \left(\widetilde{A}_k + \widetilde{B}_k \ln t\right) t^k$$

where $c_k = Q^k (k!)^{-1} \prod_{p=0}^{k-1} (p-M)^{-1}$, $c_0 = 1$, $\widetilde{B}_{k+1}(k+1)(k-n+1) = \widetilde{B}_k$, $k \geq n$, $\widetilde{B}_n = \frac{Q}{n}c_{n-1}$,
$\widetilde{A}_{n+1}(n+1) + \widetilde{B}_{n+1}(n+2) = 0$,
$Q\widetilde{A}_k = (k+1)(k-n+1)\widetilde{A}_{k+1} + (2k+2-n)\widetilde{B}_{k+1}$, $k > n$.

It is clearly seen that the function $u(t)$ given in the expression of the lemma satisfies equation (2.2) in both $M \neq n-1$ and $M = n-1$ cases.

Let us investigate the fundamental system of solution of differential equation (2.1) as

$$u_1(x) = x^{l+1}\psi(x),\ u_2(x) = x^{-l}\varphi(x).$$

Then, we obtain the differential equations

$$\psi''(x) + \frac{2(l+1)}{x}\psi'(x) = \frac{c}{x^{\alpha}}\psi(x),\ \varphi''(x) - \frac{2l}{x}\varphi'(x) = \frac{c}{x^{\alpha}}\varphi(x)$$

for functions $\psi(x)$ and $\varphi(x)$.

If the transformation $t = x^{2-\alpha}$ in the obtained differential equations for the functions $\psi(x)$ and $\varphi(x)$ and representation $\widetilde{\psi}(t) = \psi\left(t^{\frac{1}{2-\alpha}}\right)$, $\widetilde{\varphi}(t) = \varphi\left(t^{\frac{1}{2-\alpha}}\right)$ are used, we get the following differential equations:

$$\begin{gathered}\widetilde{\psi}''(t) + \frac{2(l+1)+1-\alpha}{2-\alpha}.\frac{1}{t}\widetilde{\psi}'(t) = \frac{c}{(2-\alpha)^2}.\frac{1}{t}\widetilde{\psi}(t)\\ \widetilde{\varphi}''(t) - \frac{2l-1+\alpha}{2-\alpha}.\frac{1}{t}\widetilde{\varphi}'(t) = \frac{c}{(2-\alpha)^2}.\frac{1}{t}\widetilde{\varphi}(t)\end{gathered}$$

Therefore, it was shown that functions $\widetilde{\psi}(t)$ and $\widetilde{\varphi}(t)$ satisfy the differential equation with the type (2.2). By using the solutions of differential equation (2.2), we obtained the representations in Lemma2.1 for the functions $\widetilde{\psi}(t)$ and $\widetilde{\varphi}(t)$, so also functions $u_1(x)$ and $u_2(x)$ are fundamental solution of equation (2.1).

Lemma 2.2 *Equation (2.1) has a solution $u_1(x)$ with the initial conditions*

$$u_1(x) = x^{l+1}\left[1 + o(1)\right],\ u_1'(x) = (l+1)\,x^l + o(x^l)\ x \to +0$$

and a solution $u_2(x)$ that can be represented in the form

$$u_2(x) = x^l \begin{cases} \sum\limits_{k=0}^{+\infty} a_k x^{(2-\alpha)k} & \text{if } \alpha \neq \dfrac{2n-1-2l}{n},\ (n \in \mathbb{N}) \\ \sum\limits_{k=0}^{n-1} a_k x^{(2-\alpha)k} + \dfrac{c}{2l+1} a_{n-1} x^{2l+1} \ln x + \sum\limits_{k=n+1}^{+\infty} \left(A_k + B_k \ln x\right) x^{(2-\alpha)k} & \text{if } \alpha = \dfrac{2n-1-2l}{n},\ (n \in \mathbb{N}) \end{cases} \tag{2.3}$$

where $a_0 = 1,\ a_k = \dfrac{1}{k!}\dfrac{c^k}{(2-\alpha)^k}\prod\limits_{p=1}^{k}(2p-1-\alpha p-2l)^{-1}$.

The existence of the solution $u_1(x)$ is well known (see [8]). The assertion concerning the solution $u_2(x)$ can be obtained by a straightforward substitution of the expression (2.3) into equation (2.1).

Lemma 2.3 *The domain $D(L_0^*)$ of the maximal operator L_0^* generated by the expression (1.1) is the set of functions $u(x)$ that can be represented in the form*

$$u(x) = u_1(x)h_1 + u_2(x)h_2 + \int_0^x \left[u_2(x)u_1(t) - u_1(x)u_2(t)\right] f(t)dt$$

where $h_1, h_2 \in \mathbb{C}$ and $f \in L_2(0,1)$ ($\mathbb{C}$ is the set of complex numbers).

It follows from this lemma that the function $u \in L_2(0,1)$ belongs to $D(L_0^*)$ if and only if $u'(x)$ is absolutely continuous on $(0,1)$ and $l\,[u] \in L_2(0,1)$. Next, on $D(L_0^*)$, the operator L_0^* acts by the formula $L_0^* u = l\,[u]$.

It follows from Lemma2.2 and Lemma2.3 that the function $u'(x)$, where $u \in D(L_0^*)$, does not necessarily have a boundary value at the point $x = 0$. However, it has a regularized boundary value. We set $b_1 = a_1$ and $b_k = ka_k - b_1 a_{k-1} - \ldots - b_{k-1}a_1,\ k > 1$ and introduce the functions

$$\sigma_{\alpha,l}(x) = \begin{cases} 0 & \alpha < 1-2l \\ c\ln x & \alpha = 1-2l \\ \begin{array}{c}(2-\alpha)\sum\limits_{k=1}^{n-1} b_k x^{2k-1-\alpha k-2l} \\ +ca_{n-1}\ln x\end{array} & \alpha = \dfrac{2n-1-2l}{n},\ n = 2,3,\ldots \\ (2-\alpha)\sum\limits_{k=1}^{n} b_k x^{2k-1-\alpha k-2l} & \begin{array}{l}\alpha \in \left(\dfrac{2n-1-2l}{n}, \dfrac{2n+1-2l}{n+1}\right), \\ n = 1,2,\ldots\end{array} \end{cases}$$

$$(\Gamma_1 y)(x) = x^l y(x), (\Gamma_2 y)(x) = x^{-l-1}\left[xy'(x) + ly(x)\right] - \sigma_{\alpha,l}(x)x^l y(x)$$

Lemma 2.4 *Let $u \in D(L_0^*)$. Then the functions $(\Gamma_1 u)(x)$ and $(\Gamma_2 u)(x)$ has limits*

$$\lim_{x\to 0^+} (\Gamma_1 u)(x) = (\Gamma_1 u)(0), \quad \lim_{x\to 0^+} (\Gamma_2 u)(x) = (\Gamma_2 u)(0).$$

Now one can readily show (see [1, 6]) that the domain $D(L_0)$ of the minimal operator L_0 generated by expression (1.1) is the set of functions $u \in D(L_0^*)$ such that

$$(\Gamma_1 u)(0) = (\Gamma_2 u)(0) = u(1) = u'(1) = 0.$$

For each function $u \in D(L_0^*)$, we define an ordered pair $\{U, U'\}$ of vectors of the two-dimensional euclidean space $\mathbb{C}^2$ as follows:

$$U = \{-(\Gamma_1 u)(0), u(1)\},\ U' = \{(\Gamma_2 u)(0), u'(1)\} \tag{2.4}$$

The Lagrange formula for functions $u, v \in D(L_0^*)$ can be written out in the form

$$(L_0^* u, v) - (u, L_0^* v) = \langle U, V' \rangle - \langle U', V \rangle$$

where the symbols $(.,.)$ and $\langle .,. \rangle$ stand for the inner products in the spaces $L_2(0,1)$ and $\mathbb{C}^2$, respectively. Now, by using theorem on the representation of Hermitian relations [1, 7], one can prove the following theorem:

Theorem 2.5 *For each self-adjoint extensions of operator L_0^**

$$(CosA).U' - (\mathrm{Si}\,nA).U = 0 \tag{2.5}$$

$$(\grave{U} - I)U' + (\grave{U} + I)U = 0 \tag{2.6}$$

any of boundary conditions have been determined, where A, $\grave{U}$, and I are self-adjoint, uniter, and identical operators respectively. Conversely, helping any of boundary conditions (2.5) and (2.6), the self-adjoint extensions can be generated.

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