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HOCHSTADT-LIEBERMAN TYPE THEOREMS FOR A DIFFUSION OPERATOR ON A TIME SCALE

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Abstract. In this paper, we consider a half inverse problem for a diffusion operator on a time scale which is the union of an interval and another arbitrary time scale such as $\mathbb{T} = [0, a_1] \cup \mathbb{T}_1$. We give a Hochstadt-Lieberman type theorem for this problem and some appropriate examples.

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1. Introduction. Inverse spectral problems involve reconstruction of an operator from its spectral characteristics. For inverse Sturm-Liouville problems, such characteristics are two given spectra, one spectrum and normalizing constants, spectral functions, nodal points, scattering data and the Weyl function. Such problems appear in mathematical physics, mechanics, electronics, geophysics and other branches of natural sciences. The first results on inverse theory of the classical Sturm-Liouville operator were given by Ambarzumyan and Borg [1, 6].

The half inverse Sturm-Liouville problem which is another important subject of the inverse spectral theory has been studied firstly by Hochstadt and Lieberman in 1978 [18]. Since then, this result has been generalized to various versions. Recently, some new uniqueness results in inverse spectral analysis with partial information on the potential for some classes of differential equations have been given (see [15, 19] and [27]). This kind results are known as Hochstadt and Lieberman-type theorems.

The time scale theory was introduced by Hilger in order to unify continuous and discrete analysis [17]. From then on this approach has received a lot of attention and has applied quickly to various area in mathematics. The literature for inverse problems for differential operators on a continuous interval is vast (see [2, 12] and [13]). However, differential operators defined on time scales are essentially more difficult for investigating, and nowadays there are only a few studies about this subject [3, 23, 25, 26]. Such problems is useful in many applied problems, for example in string theory, in dynamics of population, in spatial networks problems, etc.

A diffusion equation originated from the problem of describing interactions between colliding particles in physics [20]. In 1981, Gasymov and Guseinov [14] investigated the spectral theory of diffusion operators. They proved that the boundary value problem

$$\begin{aligned}&-y'' + [q(t) + 2\lambda p(t)]y = \lambda^2 y,\ t \in (0, l)\\&y'(0) - hy(0) = 0\\&y'(l) + Hy(l) = 0\end{aligned}$$

has countable many eigenvalues which are real, simple and different from zero, under the condition

$$h\,|y(0)|^2 + H\,|y(l)|^2 + \int_0^l \left(|y'(t)|^2 + q(t)\,|y(t)|^2\right) dt > 0.$$

Inverse problems for diffusion operator on an interval have been studied by many authors [7–11, 21, 22, 28–30]. However, there is no publication related to the inverse problem for this class of operators on time scales.

In this paper, we consider a half-inverse problem for a diffusion operator on a time scale and give a Hochstadt and Lieberman-type theorem.

We recall the some important concepts of the time scale theory.

If $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$ it called as a time scale. The jump operators σ, ρ and graininess operator on $\mathbb{T}$ are defined as follows:

$$\begin{aligned}&\sigma : \mathbb{T} \to \mathbb{T},\ \sigma(t) = \inf\{s \in \mathbb{T} : s > t\}\ \text{ if } t \neq \sup \mathbb{T},\\&\rho : \mathbb{T} \to \mathbb{T},\ \rho(t) = \sup\{s \in \mathbb{T} : s < t\}\ \text{ if } t \neq \inf \mathbb{T},\\&\sigma(\sup \mathbb{T}) = \sup \mathbb{T},\ \rho(\inf \mathbb{T}) = \inf \mathbb{T},\\&\mu : \mathbb{T} \to [0, \infty)\ \mu(t) = \sigma(t) - t.\end{aligned}$$

A point of $\mathbb{T}$ is called as left-dense, left-scattered, right-dense, right-scattered and isolated if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$ and $\rho(t) < t < \sigma(t)$, respectively.

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous on $\mathbb{T}$ if it is continuous at all right-dense points and has left-sided limits at all left-dense points in $\mathbb{T}$. The set of rd-continuous functions on $\mathbb{T}$ is denoted by $C_{rd}(\mathbb{T})$ or C_{rd}. If f is continuous on $\mathbb{T}$ it is also rd-continuous.

Put $\mathbb{T}^\kappa := \begin{cases} \mathbb{T} - \{\sup \mathbb{T}\}, & \sup \mathbb{T} \text{ is left-scattered} \\ \mathbb{T}, & \text{the other cases}\end{cases}$, $\mathbb{T}^{\kappa^2} := (\mathbb{T}^\kappa)^\kappa$.

Let $t \in \mathbb{T}^\kappa$. Suppose that for given any $\varepsilon > 0$, there exist a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ such that

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| \leq \varepsilon\,|\sigma(t) - s|$$

for all $s \in U$, then f is called differentiable at $t \in \mathbb{T}^\kappa$. We call $f^\Delta(t)$ the delta derivative of f at t. The set

$$C^n_{rd}(\mathbb{T}) := \{f : f \text{ rd-continuously } n\text{-order delta-differentiable on } \mathbb{T}\}.$$

A function $F : \mathbb{T} \to \mathbb{R}$ defined as $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$ is called an antiderivative of f on $\mathbb{T}$. In this case the Δ-integral of f is defined by

$$\int_a^b f(t) \triangle t = F(b) - F(a)\ \text{ for } a, b \in \mathbb{T}.$$

We give some necessary relations in the following lemma. Their proofs can be found in [4], chapter1. Further knowledge about the concepts Δ-derivation, Δ-integration and dynamic equations on time scales can be found in [4] and [5].

LEMMA 1. *Let $f:\mathbb{T}\to\mathbb{R}$, $g:\mathbb{T}\to\mathbb{R}$ be two functions and $t\in\mathbb{T}^{\kappa}$.*
i) If $f^{\Delta}(t)$ exists, then f is continuous at t.
ii) If t is right-scattered and f is continuous at t, then f is Δ-differentiable at t and $f^{\Delta}(t)=\dfrac{f^{\sigma}(t)-f(t)}{\sigma(t)-t}$, where $f^{\sigma}(t)=f(\sigma(t))$.
iii) If $f^{\Delta}(t)$ exists, then $f^{\sigma}(t)=f(t)+\mu(t)f^{\Delta}(t)$.
iv) If $f^{\Delta}(t)$ and $g^{\Delta}(t)$ exist, then $(f\pm g)^{\Delta}(t)=f^{\Delta}(t)\pm g^{\Delta}(t)$, $(fg)^{\Delta}(t)=\left(f^{\Delta}g+f^{\sigma}g^{\Delta}\right)(t)$ and if $(gg^{\sigma})(t)\neq 0$, then $\left(\dfrac{f}{g}\right)^{\Delta}(t)=\left(\dfrac{f^{\Delta}g-fg^{\Delta}}{gg^{\sigma}}\right)(t)$.
v) If $f\in C_{rd}(\mathbb{T})$, then it has an antiderivative on $\mathbb{T}$.

Let $\mathbb{T}=[0,a_1]\cup\mathbb{T}_1$, where $\mathbb{T}_1$ is a bounded time scale such that $a_2:=\inf\mathbb{T}_1>a_1$ and $l:=\rho(\sup\mathbb{T}_1)$. Let us denote the following boundary value problem by $L=L(q,p,h,H)$

$$\ell y:=-y^{\Delta\Delta}(t)+[q(t)+2\lambda p(t)]y^{\sigma}(t)=\lambda^2 y^{\sigma}(t),\quad t\in\mathbb{T}^{\kappa^2} \tag{1}$$

$$U(y):=y^{\Delta}(0)-hy(0)=0 \tag{2}$$

$$V(y):=y^{\Delta}(l)+Hy(l)=0 \tag{3}$$

where $q(t)$ and $p(t)$ are real valued continuous functions on $\mathbb{T}$; $h,\ H\in\mathbb{R}$ and λ is a spectral parameter.

Together with L, we consider a boundary value problem $\widetilde{L}=L(\widetilde{q},\widetilde{p},h,H)$ of the same form but with different coefficients $\widetilde{q}(t)$ and $\widetilde{p}(t)$. We assume that if a certain symbol s denotes an object related to L , then $\widetilde{s}$ will denote an analogous object related to $\widetilde{L}$.

A function y, defined on $\mathbb{T}$, is called a solution of equation (1) if $y\in C^2_{rd}(\mathbb{T})$ and y satisfies (1) for all $t\in\mathbb{T}$. The values of the λ parameter for which (1)-(3) has nonzero solutions are called eigenvalues, and the corresponding nontrivial solutions are called eigenfunctions.

We state the main result of this article.

Let $\Lambda:=\{\lambda_n,\ n\in\mathbb{Z}\}$ and $\widetilde{\Lambda}:=\left\{\widetilde{\lambda}_n,\ n\in\mathbb{Z}\right\}$ be the eigenvalues sets of L and $\widetilde{L}$, respectively. Put $G_{\delta}:=\{\lambda:|\lambda-\lambda_n|\geq\delta \text{ for all } n\}$, where δ is a sufficiently small number. We denote characteristic function of L by $\Delta(\lambda)$ and assume

$$h\left|y(0)\right|^2+H\left|y(l)\right|^2+\int_0^l\left(\left|y^{\Delta}(t)\right|^2+q(t)\left|y^{\sigma}(t)\right|^2\right)\Delta t>0. \tag{4}$$

THEOREM 1. *Assume that for given any $\delta>0$, there exist some $C_{\delta}>0$ such that*

$$|\Delta(\lambda)|\geq C_{\delta}\left|\lambda\right|^{\alpha}\exp\left|\tau\right|2a_1,\ \ \lambda\in G_{\delta},\ \ |\lambda|>\lambda^{*} \tag{5}$$

where $\alpha>2$, λ^{*} is a sufficiently large number and $\tau=\operatorname{Im}\lambda$.

If $\Lambda=\widetilde{\Lambda}$, $p(t)=\widetilde{p}(t)$ on $\mathbb{T}_1\cup\{a_1\}$ and $q(t)=\widetilde{q}(t)$ on $\mathbb{T}_1$ then $q(t)=\widetilde{q}(t)$ and $p(t)=\widetilde{p}(t)$ on $\mathbb{T}$.

We note that, if it is taken $\mathbb{T} = [0, \ell]$ in L, the classical diffusion operator is obtained. One of the most important results related to the half-inverse problem for the classical diffusion operator is proven by Yang and Zettl [30]. On the other hand, a similar uniqueness theorem for the Sturm-Liouville operator on time scales is given in [26]. This paper is a continuation of these investigations. We hope that our results will contribute to the development of inverse spectral theory on time scales and to obtain stronger results in some applied sciences.

2. Preliminaries.

Let $\varphi(t,\lambda)$, $s(t,\lambda)$, $c(t,\lambda)$ and $\psi(t,\lambda)$ be the solutions of (1) under the initial conditions

$$\varphi(0,\lambda) = 1,\ \varphi^{\Delta}(0,\lambda) = h \tag{6}$$

$$s(0,\lambda) = 0,\ s^{\Delta}(0,\lambda) = 1 \tag{7}$$

$$c(0,\lambda) = 1,\ c^{\Delta}(0,\lambda) = 0 \tag{8}$$

$$\psi(l,\lambda) = 1,\ \psi^{\Delta}(l,\lambda) = -H \tag{9}$$

respectively. It is clear that the Wronskian

$$W[c(t,\lambda), s(t,\lambda)] = c(t,\lambda)s^{\Delta}(t,\lambda) - c^{\Delta}(t,\lambda)s(t,\lambda)$$

does not depend on t. From (7) and (8) we have $W[c(t,\lambda), s(t,\lambda)] = 1$ and so $s(t,\lambda)$ and $c(t,\lambda)$ are linear independent. Moreover, it can be seen that the relation $\varphi(t,\lambda_n) = \chi_n \psi(t,\lambda_n)$ is valid for each eigenvalue λ_n, where $\chi_n = \varphi(l,\lambda_n) \neq 0$ for all n. The following theorem can be proven by using a method similar to that in [24].

THEOREM 2. *$s(t,\lambda)$, $c(t,\lambda)$, $\varphi(t,\lambda)$, $\psi(t,\lambda)$ and their Δ-derivatives are entire functions on λ for each fixed t.*

It is clear that the characteristic function of L can be given as follows

$$\Delta(\lambda) = W[\psi, \varphi] = \varphi^{\Delta}(l,\lambda) + H\varphi(l,\lambda). \tag{10}$$

It follows from Theorem 2 that $\Delta(\lambda)$ is also an entire function and so L has a discrete spectrum.

THEOREM 3. *The eigenvalues of the problem (1)-(3) are real and algebraicly simple.*

Proof. It is proved in [16] that all eigenvalues of the problem (1)-(3) are real numbers. Let us prove that they are also algebraicly simple. Let λ_n be an eigenvalue of (1)-(3). Obviously, $\dfrac{d\Delta(\lambda)}{d\lambda} = \varphi_{\lambda}^{\Delta}(l,\lambda) + H\varphi_{\lambda}(l,\lambda)$, where $\varphi_{\lambda} = \dfrac{\partial \varphi}{\partial \lambda}$ and $\varphi_{\lambda}^{\Delta} = \dfrac{\partial \varphi^{\Delta}}{\partial \lambda}$. It will be sufficient to see $\left.\dfrac{d\Delta(\lambda)}{d\lambda}\right|_{\lambda=\lambda_n} \neq 0$.

Write the equation (1) for $\varphi(t,\lambda)$

$$-\varphi^{\Delta\Delta}(t,\lambda)+[q(t)+2\lambda p(t)]\varphi^{\sigma}(t,\lambda)=\lambda^{2}\varphi^{\sigma}(t,\lambda). \tag{11}$$

By derivation according to λ, we get

$$-\varphi_{\lambda}^{\Delta\Delta}(t,\lambda)+2p(t)\varphi^{\sigma}(t,\lambda)+[q(t)+2\lambda p(t)]\varphi_{\lambda}^{\sigma}(t,\lambda)=\lambda^{2}\varphi_{\lambda}^{\sigma}(t,\lambda)+2\lambda\varphi^{\sigma}(t,\lambda). \tag{12}$$

Now by using the standard procedure of multiplying (11) by $\varphi_{\lambda}^{\sigma}$, (12) by φ^{σ}, subtracting and $\Delta-$integrating, we obtain

$$\left[\varphi(t,\lambda)\varphi_{\lambda}^{\Delta}(t,\lambda)-\varphi^{\Delta}(t,\lambda)\varphi_{\lambda}(t,\lambda)\right]_{0}^{l}=\int_{0}^{l}\left[\varphi^{\sigma}(t,\lambda)\right]^{2}\left(2p(t)-2\lambda\right)\Delta t. \tag{13}$$

Taking into account the initial conditions (6) we get

$$\int_{0}^{l}\left[\varphi^{\sigma}(t,\lambda)\right]^{2}\left(2p(t)-2\lambda\right)\Delta t=\varphi(l,\lambda)\varphi_{\lambda}^{\Delta}(l,\lambda)-\varphi^{\Delta}(l,\lambda)\varphi_{\lambda}(l,\lambda). \tag{14}$$

Putting λ_n instead of λ in (14) and making necessary calculations we get

$$\int_{0}^{l}\left[\varphi^{\sigma}(t,\lambda_n)\right]^{2}\left(2p(t)-2\lambda_n\right)\Delta t=\varphi(l,\lambda_n)\varphi_{\lambda}^{\Delta}(l,\lambda_n)-\varphi^{\Delta}(l,\lambda_n)\varphi_{\lambda}(l,\lambda_n)$$

$$=\quad \varphi(l,\lambda_n)\left.\frac{d\Delta(\lambda)}{d\lambda}\right|_{\lambda=\lambda_n}.$$

On the other hand, from (4) we have

$$-\lambda_n\int_{0}^{l}\left[\varphi^{\sigma}(t,\lambda_n)\right]^{2}\left(2p(t)-2\lambda_n\right)\Delta t$$

$$=h\left|\varphi(0,\lambda_n)\right|^{2}+H\left|\varphi(l,\lambda_n)\right|^{2}+\int_{0}^{l}\left(\left|\varphi^{\Delta}(t,\lambda_n)\right|^{2}+q(t)\left|\varphi^{\sigma}(t,\lambda_n)\right|^{2}\right)\Delta t>0.$$

Hence

$$\left.\frac{d\Delta(\lambda)}{d\lambda}\right|_{\lambda=\lambda_n}\neq 0$$

This completes the proof. □

It is known from [14] that $\varphi(t,\lambda)$ satisfies the following representation on $(0,a_1)$

$$\varphi(t,\lambda)=\cos\left(\lambda t-\alpha(t)\right)+\int_{0}^{t}A(t,x)\cos\lambda x dx+\int_{0}^{t}B(t,x)\sin\lambda x dx \tag{15}$$

where the kernels $A(t,x)$ and $B(t,x)$ are the solution of the problem

$$\frac{\partial^2 A(t,x)}{\partial t^2} - 2p(t)\frac{\partial B(t,x)}{\partial x} - q(t)A(t,x) = \frac{\partial^2 A(t,x)}{\partial x^2},$$
$$\frac{\partial^2 B(t,x)}{\partial t^2} + 2p(t)\frac{\partial A(t,x)}{\partial x} - q(t)B(t,x) = \frac{\partial^2 B(t,x)}{\partial x^2},$$
$$A(0,0) = h,\ B(t,0) = 0,\ \left.\frac{\partial A(t,x)}{\partial x}\right|_{x=0} = 0$$

where

$$q(t) + p^2(t) = 2\frac{d}{dt}\left[A(t,t)\cos\alpha(t) + B(t,t)\sin\alpha(t)\right],\ \alpha(t) = \int_0^t p(x)dx.$$

On the other hand, $\varphi^{\Delta}(t,\lambda)$ is continuous at a_1, and so the relation

$$a\varphi'(a_1 - 0, \lambda) = \varphi(a_2,\lambda) - \varphi(a_1,\lambda) \tag{16}$$

holds, where $a := a_2 - a_1$. Therefore we obtain that if $a > 0$, then the following asymptotic formula holds for $|\lambda| \to \infty$;

$$\varphi(a_2,\lambda) = -a\lambda\sin(\lambda a_1 - \alpha(a_1)) + O(\exp|\tau|\,a_1), \tag{17}$$

where $\tau = \operatorname{Im}\lambda$.

Now we are ready to prove our main result.

3. Proof of the main result.

Proof of Theorem 1. Let us write the equation (1) for φ and $\widetilde{\varphi}$

$$-\varphi^{\Delta\Delta}(t,\lambda) + [q(t) + 2\lambda p(t)]\varphi^{\sigma}(t,\lambda) = \lambda\varphi^{\sigma}(t,\lambda) \tag{18}$$

$$-\widetilde{\varphi}^{\Delta\Delta}(t,\lambda) + [\widetilde{q}(t) + 2\lambda\widetilde{p}(t)]\widetilde{\varphi}^{\sigma}(t,\lambda) = \lambda\widetilde{\varphi}^{\sigma}(t,\lambda). \tag{19}$$

It can be obtained from (18) and (19) that

$$\begin{aligned}\left[\varphi^{\Delta}(t,\lambda)\widetilde{\varphi}(t,\lambda) - \varphi(t,\lambda)\widetilde{\varphi}^{\Delta}(t,\lambda)\right]^{\Delta} \\ = \left[q(t) - \widetilde{q}(t) + 2\lambda(p(t) - \widetilde{p}(t))\right]\varphi^{\sigma}(t,\lambda)\widetilde{\varphi}^{\sigma}(t,\lambda).\end{aligned} \tag{20}$$

Taking $P(t) = p(t) - \widetilde{p}(t)$, $Q(t) = q(t) - \widetilde{q}(t)$ and $\Delta-$integrating both sides of (20) on $[0,l]\cap\mathbb{T}$, we get

$$\begin{aligned}\left[\varphi^{\Delta}(t,\lambda)\widetilde{\varphi}(t,\lambda)-\varphi(t,\lambda)\widetilde{\varphi}^{\Delta}(t,\lambda)\right]_0^l &= \int_0^l \left[Q(t)+2\lambda P(t)\right]\varphi^{\sigma}(t,\lambda)\widetilde{\varphi}^{\sigma}(t,\lambda)\Delta t \\ &= \int_0^{a_1} \left[Q(t)+2\lambda P(t)\right]\varphi(t,\lambda)\widetilde{\varphi}(t,\lambda)dt \\ &\quad+\int_{a_1}^{a_2} \left[Q(t)+2\lambda P(t)\right]\varphi^{\sigma}(t,\lambda)\widetilde{\varphi}^{\sigma}(t,\lambda)\Delta t \\ &\quad+\int_{a_2}^{l} \left[Q(t)+2\lambda P(t)\right]\varphi^{\sigma}(t,\lambda)\widetilde{\varphi}^{\sigma}(t,\lambda)\Delta t.\end{aligned}$$

Since $p(t)=\widetilde{p}(t)$ on $\mathbb{T}_1\cup\{a_1\}$, $q(t)=\widetilde{q}(t)$ on $\mathbb{T}_1$ and

$$\int_{a_1}^{a_2} \left[Q(t)+2\lambda P(t)\right]\varphi^{\sigma}(t,\lambda)\widetilde{\varphi}^{\sigma}(t,\lambda)\Delta t = Q(a_1)\varphi^{\sigma}(a_1,\lambda)\widetilde{\varphi}^{\sigma}(a_1,\lambda)\left(a_2-a_1\right), \tag{21}$$

it is obvious that

$$\varphi^{\Delta}(l,\lambda)\widetilde{\varphi}(l,\lambda)-\varphi(l,\lambda)\widetilde{\varphi}^{\Delta}(l,\lambda)=\int_0^{a_1}\left[Q(t)+2\lambda P(t)\right]\varphi(t,\lambda)\widetilde{\varphi}(t,\lambda)dt \tag{22}$$
$$+aQ(a_1)\varphi(a_2,\lambda)\widetilde{\varphi}(a_2,\lambda).$$

By using (15),

$$\begin{aligned}\varphi(t,\lambda)\widetilde{\varphi}(t,\lambda) &= \cos\left(\lambda t-\alpha(t)\right)\cos\left(\lambda t-\widetilde{\alpha}(t)\right)+\int_0^t A(t,x)\cos\lambda x\cos\left(\lambda t-\widetilde{\alpha}(t)\right)dx \\ &+\int_0^t \widetilde{A}(t,x)\cos\lambda x\cos\left(\lambda t-\alpha(t)\right)dx+\int_0^t B(t,x)\sin\lambda x\cos\left(\lambda t-\widetilde{\alpha}(t)\right)dx \\ &+\int_0^t \widetilde{B}(t,x)\sin\lambda x\cos\left(\lambda t-\alpha(t)\right)dx+\int_0^t A(t,x)\cos\lambda x dx\int_0^t \widetilde{A}(t,x)\cos\lambda x dx \\ &+\int_0^t B(t,x)\sin\lambda x dx\int_0^t \widetilde{B}(t,x)\sin\lambda x dx+\int_0^t A(t,x)\cos\lambda x dx\int_0^t \widetilde{B}(t,x)\sin\lambda x dx \\ &+\int_0^t \widetilde{A}(t,x)\cos\lambda x dx\int_0^t B(t,x)\sin\lambda x dx.\end{aligned}$$

By extending the range of $A(t,x)$ and $\widetilde{A}(t,x)$ evenly and $B(t,x)$ and $\widetilde{B}(t,x)$ oddly with respect to the argument x, we can write

$$\varphi(t,\lambda)\widetilde{\varphi}(t,\lambda)=\frac{1}{2}\left[\cos(\alpha(t)-\widetilde{\alpha}(t))+\cos\left(2\lambda t-\theta(t)\right)\right] \tag{23}$$
$$+\frac{1}{2}\left[\int_0^t H_c(t,x)\cos\left(2\lambda x-\theta(x)\right)dx-\int_0^t H_s(t,x)\sin\left(2\lambda x-\theta(x)\right)dx\right]$$

where

$$\begin{aligned}H_c(t,x)&=2A(t,t-2x)\cos\left(\theta(x)-\widetilde{\alpha}(t)\right)+2\widetilde{A}(t,t-2x)\cos\left(\theta(x)-\alpha(t)\right)\\&-2B(t,t-2x)\sin\left(\theta(x)-\widetilde{\alpha}(t)\right)-2\widetilde{B}(t,t-2x)\sin\left(\theta(x)-\alpha(t)\right)\\&+A_1(x)\cos\theta(x)+A_2(x)\cos\theta(x)+B_1(x)\sin\theta(x)+B_2(x)\sin\theta(x),\\\theta(x)&=\alpha(x)+\widetilde{\alpha}(x),\end{aligned}$$

and

$$\begin{aligned}H_s(t,x)&=2A(t,t-2x)\sin\left(\theta(x)-\widetilde{\alpha}(t)\right)+2\widetilde{A}(t,t-2x)\sin\left(\theta(x)-\alpha(t)\right)\\&+2B(t,t-2x)\cos\left(\theta(x)-\widetilde{\alpha}(t)\right)+2\widetilde{B}(t,t-2x)\cos\left(\theta(x)-\alpha(t)\right)\\&+A_1(x)\sin\theta(x)+A_2(x)\sin\theta(x)-B_1(x)\cos\theta(x)-B_2(x)\cos\theta(x)\end{aligned}$$

with

$$\begin{aligned}A_1(x)&=\int_{-t}^{t-2x}A(t,s)\widetilde{A}(t,s+2x)ds+\int_{-t+2x}^{t}A(t,s)\widetilde{A}(t,s-2x)ds,\\A_2(x)&=-\int_{-t}^{t-2x}B(t,s)\widetilde{B}(t,s+2x)ds-\int_{-t+2x}^{t}B(t,s)\widetilde{B}(t,s-2x)ds,\\B_1(x)&=\int_{-t}^{t-2x}A(t,s)\widetilde{B}(t,s+2x)ds+\int_{-t+2x}^{t}A(t,s)\widetilde{B}(t,s-2x)ds,\\B_2(x)&=\int_{-t}^{t-2x}B(t,s)\widetilde{A}(t,s+2x)ds+\int_{-t+2x}^{t}B(t,s)\widetilde{A}(t,s-2x)ds.\end{aligned}$$

Let

$$H(\lambda):=\int_0^{a_1}\left[Q(t)+2\lambda P(t)\right]\varphi(t,\lambda)\widetilde{\varphi}(t,\lambda)dt \tag{24}$$
$$+aQ(a_1)\varphi(a_2,\lambda)\widetilde{\varphi}(a_2,\lambda).$$

Since $\varphi(l,\lambda_n)\widetilde{\varphi}^{\Delta}(l,\lambda_n)-\varphi^{\Delta}(l,\lambda_n)\widetilde{\varphi}(l,\lambda_n)=0$, $H(\lambda_n)=0$ for all $\lambda_n\in\Lambda$ and so $\chi(\lambda):=\dfrac{H(\lambda)}{\Delta(\lambda)}$ is an entire function on λ.

On the other hand, from (17) and (23), we obtain

$$H(\lambda) = O(\lambda^2 \exp 2|\tau| a_1)$$

for all complex λ. From assumptions of the theorem it can be calculated that

$$|\chi(\lambda)| \leq C|\lambda|^{2-\alpha}, \ \lambda \in G_\delta, \ \ |\lambda| > \lambda^*. \tag{25}$$

By the Liouville theorem $\chi(\lambda) = 0$ for all λ. Hence, $H(\lambda) \equiv 0$.

It follows from (17) that the equalities

$$\varphi(a_2, \lambda) = -a\lambda \sin(\lambda a_1 - \alpha(a_1)) + O(1),$$
$$\widetilde{\varphi}(a_2, \lambda) = -a\lambda \sin(\lambda a_1 - \widetilde{\alpha}(a_1)) + O(1)$$

are valid for sufficiently large λ on the real axis. From (17), (23), (24) and well-known Riemann-Lebesgue lemma, the following equality obtained for $\lambda \in \mathbb{R}$ as $\lambda \to \infty$,

$$\int_0^{a_1} [Q(t) + 2\lambda P(t)] [\cos(\alpha(t) - \widetilde{\alpha}(t)) + \cos(2\lambda t - \theta(t)) + o(1)] \, dt \tag{26}$$
$$+2aQ(a_1) \left[a^2\lambda^2 \sin(\lambda a_1 - \alpha(a_1)) \sin(\lambda a_1 - \widetilde{\alpha}(a_1)) + O(\lambda)\right] = 0.$$

Therefore, we have $q(a_1) = \widetilde{q}(a_1)$ and so $\int_0^{a_1} [Q(t) + 2\lambda P(t)] \varphi(t, \lambda)\widetilde{\varphi}(t, \lambda) dt = 0$. By integrating again both sides of the equality (20) on $(0, a_1)$, we get

$$\varphi'(a_1, \lambda)\widetilde{\varphi}(a_1, \lambda) = \varphi(a_1, \lambda)\widetilde{\varphi}'(a_1, \lambda). \tag{27}$$

Put $\psi(t, \lambda) := \varphi(a_1 - t, \lambda)$. It is clear that $\psi(t, \lambda)$ is the solution of the following initial value problem

$$-y'' + [q(a_1 - t) + 2\lambda p(a_1 - t)]y = \lambda y, \ t \in (0, a_1)$$
$$y(a_1) = 1, \ y'(a_1) = -h \ .$$

It follows from (27) that

$$\psi'(0, \lambda)\widetilde{\psi}(0, \lambda) = \psi(0, \lambda)\widetilde{\psi}'(0, \lambda). \tag{28}$$

Taking into account Theorem 4.1. in [28] it is concluded that $q(t) = \widetilde{q}(t)$ and $p(t) = \widetilde{p}(t)$ on $[0, a_1]$. This completes the proof. □

REMARK 1. If $\ell y := -y^{\Delta\Delta}(t) + q(t)y^\sigma(t) = \lambda y^\sigma(t)$ on $\mathbb{T}$ and $a_1 < a_2$, it is sufficient to take $\alpha > 1$ instead of $\alpha > 2$ in condition (5). This assertion is proven in [26].

EXAMPLE 1. Let us consider the problem (1)-(3) on the time scale $\mathbb{T} = [0, a_1] \cup [a_2, l]$, where $a_1 < a_2$ and $l - a_2 = a_1$.

Since $\varphi(t,\lambda)$ satisfies the equation (1) for $t = a_1$, it follows that

$$(a^2\lambda^2 + b\lambda + c)\varphi(a_2,\lambda) + a\varphi'(a_2+0,\lambda) + \varphi(a_1,\lambda) = 0, \tag{29}$$

where $a = a_2 - a_1$, $b = -2a^2p(a_1)$, $c = -a^2q(a_1) - 1$.

It can be calculated from (16) and (29) that the following asymptotic formulas hold for $|\lambda| \to \infty$:

(30)

$$\varphi(t,\lambda) = \begin{cases} \cos\lambda t + \frac{h}{\lambda}\sin\lambda t + O\left(\frac{1}{\lambda}\exp|\tau|\,t\right), & t \in [0,a_1], \\ a^2\lambda^2 \sin\lambda a_1 \sin\lambda(t-a_2) + O\left(\lambda\exp|\tau|\,(t-a_2+a_1)\right), & t \in [a_2,l], \end{cases}$$

(31)

$$\varphi^{\Delta}(t,\lambda) = \begin{cases} -\lambda\sin\lambda t + h\cos\lambda t + O\left(\exp|\tau|\,t\right), & t \in [0,a_1), \\ a^2\lambda^3 \sin\lambda a_1 \cos\lambda(t-a_2) + O\left(\lambda^2\exp|\tau|\,(t-a_2+a_1)\right), & t \in [a_2,l]. \end{cases}$$

It follows from $\Delta(\lambda) = \varphi^{\Delta}(l,\lambda) + H\varphi(l,\lambda)$ that the asymptotic relation

$$\Delta(\lambda) = \frac{a^2}{2}\lambda^3 \sin 2\lambda a_1 + O\left(\lambda^2 \exp|\tau|\,2a_1\right) \tag{32}$$

is valid for $|\lambda| \to \infty$.

Hence $\Delta(\lambda)$ satisfies the condition (5) and so Theorem 1 can be given as follows on this time scale.

THEOREM 4. *If* $l - a_2 = a_1$, $\Lambda = \widetilde{\Lambda}$, $p(t) = \widetilde{p}(t)$ *on* $\{a_1\} \cup [a_2,l]$ *and* $q(t) = \widetilde{q}(t)$ *on* $[a_2,l]$, *then* $q(t) = \widetilde{q}(t)$ *and* $p(t) = \widetilde{p}(t)$ *on* $[0,a_1] \cup [a_2,l]$.

EXAMPLE 2. Consider the problem (1)-(3) on the time scale $\mathbb{T} = \cup_{j=0}^{m}[a_{2j},a_{2j+1}]$, where $a_i \neq a_k$ for $i \neq k$, $a_0 = 0$ and $\sum_{j=1}^{m}(a_{2j+1} - a_{2j}) = a_1$. It can be proven by using similar methods that

$$\Delta(\lambda) = (-1)^{m+1}\lambda^{2m+1}\sin\lambda a_1 \prod_{j=1}^{m}(a_{2j} - a_{2j-1})^2 \cos\lambda(a_{2j+1} - a_{2j}) + O(\lambda^{2m}\exp 2|\tau|\,a_1).$$

Therefore, the condition (5) is valid. Hence Theorem 1 can be given as follows on this time scale.

THEOREM 5. *If* $\Lambda = \widetilde{\Lambda}$, $p(t) = \widetilde{p}(t)$ *on* $\{a_1\} \cup \cup_{j=1}^{m}[a_{2j},a_{2j+1}]$ *and* $q(t) = \widetilde{q}(t)$ *on* $\cup_{j=1}^{m}[a_{2j},a_{2j+1}]$, *then* $q(t) = \widetilde{q}(t)$ *and* $p(t) = \widetilde{p}(t)$ *on* $\mathbb{T}$.

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