Research Article

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On the p-integrable trajectories of the nonlinear control system described by the Urysohn-type integral equation

https://doi.org/10.1515/math-2022-0494
received February 3, 2021; accepted September 6, 2022

Abstract: The control systems described by the Urysohn-type integral equations and integral constraints on the control functions are considered. The functions from the closed ball of the space L_p, $p > 1$, with radius r, are chosen as admissible control functions. The trajectory of the system is defined as a p-integrable function, satisfying the system's equation almost everywhere. The boundedness and path-connectedness of the set of p-integrable trajectories are discussed. It is illustrated that the set of trajectories, in general, is not a closed subset of the space L_p. The robustness of a trajectory with respect to the fast consumption of the remaining control resource is established, and it is proved that every trajectory of the system can be approximated by the trajectory obtained by the full consumption of the control resource.

Keywords: nonlinear control system, integral equation, p-integrable trajectory, closedness, robustness

MSC 2020: 45G15, 93C23, 93C35

1 Introduction

In general, depending on the character of the control efforts, control systems can be classified as follows: control systems with geometric constraints on the control functions; control systems with integral constraints on the control functions; and control systems with mixed constraints on the control functions, which include both the geometric and integral constraints on the control functions. Geometric constraint characterizes the control efforts that are not exhausted by consumption, and these systems are a well-studied chapter of the control systems theory (see, e.g., [1–3]). They are also investigated in the framework of differential and integral inclusion theory (see, e.g., [4–7] and references therein). Integral constraint on the control functions appears in the case if the control resource is exhausted by consumption such as energy, fuel, finance, and food. For example, the motion of a flying object with rapidly changing mass is described by a control system with integral constraints on the control functions (see, e.g., [8–10]). Integrally boundedness of the function does not imply geometric boundedness, and therefore, additional difficulties are encountered in the investigation of these systems.

In theory and applications, the control systems described by various types of evolution equations are studied. One of the more interesting systems is the control system described by integral equations. The solution concepts for different types of initial and boundary value problems can be reduced to the appropriate solution notions for Volterra- and Urysohn-Hammerstein-type integral equations. It should be noted that integral models have some advantages over differential ones. For example, the trajectories for such systems can be defined as continuous, even as integrable functions. The integrable solution notion is an

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adequate way to describe the behavior of some physical processes (see, e.g., [11,12]). Therefore, the investigation of control systems described by the Urysohn-type integral equations is important for the theory and applications of control systems theory. Note that the controllability properties and existence of the optimal trajectories of the control system described by the Urysohn-type integral equation are investigated in [13–17] (see also references therein). The articles [18,19] consider the approximate construction of the set of trajectories of the control systems described by Urysohn-type integral equations and integral constraints on the control functions. In [13,14,19], it is assumed that the trajectory of the system is a continuous function and satisfies the system's equation everywhere. In articles [15–17], the integrable functions are chosen as the system's trajectory. In this article, the properties of the set of p-integrable trajectories of the control system described by the Urysohn-type integral equation with the integral constraint on the control functions are studied where the system's equation is nonlinear with respect to state and control vectors. The boundedness and path-connectedness of the set of trajectories are established, and it is illustrated that, in general, the set of trajectories is not a closed subset of the space of p-integrable functions. It is proved that the consumption of the remaining control resource on a domain with sufficiently small measure causes a nonessential change in the trajectory of the system. It is shown that the set of trajectories generated by a full consumption of the control resource is dense in the set of trajectories generated by all admissible control functions.

The article is organized as follows. In Section 2, the basic conditions, which are used in the following arguments, are given. In Section 3, the existence and uniqueness of p-integrable trajectories generated by a given admissible control function (Proposition 1), the boundedness of the set of trajectories (Proposition 2), the dependence of the trajectories on the generating admissible control functions (Proposition 3), and the path-connectedness of the set of trajectories (Theorem 1) are presented. In Section 4, it is shown that the set of trajectories, in general, is not a closed subset of the space L_p (Example 1). It is illustrated that not closedness of the set of trajectories can permit the appearance of sliding modes for approximate optimal trajectory. In Section 5, it is proved that every trajectory of the system is robust with respect to the fast consumption of the remaining control resource (Theorem 2) and that every trajectory of the system can be approximated by the trajectory obtained by full consumption of the control resource (Theorem 3).

2 The system

Consider a control system described by the Urysohn-type integral equation as follows:

$$x(\xi) = f(\xi, x(\xi)) + \int\limits_{\Omega} K(\xi, s, x(s), u(s))\mathrm{d}s, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control vector, $\xi \in \Omega$, $\Omega \subset \mathbb{R}^k$ is the Lebesgue measurable set such that $\mu(\Omega) < +\infty$, and $\mu(\Omega)$ is the Lebesgue measure of the set Ω.

For given $p > 1$ and $r > 0$, we set

$$U_{p,r} = \{u(\cdot) \in L_p(\Omega; \mathbb{R}^m) : \|u(\cdot)\|_p \leq r\},$$

where $L_p(\Omega; \mathbb{R}^m)$ is the space of the Lebesgue measurable functions $u(\cdot) : \Omega \to \mathbb{R}^m$ such that $\|u(\cdot)\|_p < +\infty$, and $\|u(\cdot)\|_p = \left(\int_{\Omega} \|u(s)\|^p \mathrm{d}s\right)^{\frac{1}{p}}$, $\|\cdot\|$ denotes the Euclidean norm.

$U_{p,r}$ is called the set of admissible control functions, and every $u(\cdot) \in U_{p,r}$ is said to be an admissible control function.

Let $q > 1$ be such that $\frac{1}{q} + \frac{1}{p} = 1$. It is assumed that the functions $f(\cdot)$ and $K(\cdot)$ satisfy the following conditions.

A. The function $f(\cdot,x) : \Omega \to \mathbb{R}^n$ is Lebesgue measurable for each fixed $x \in \mathbb{R}^n$, $f(\cdot,0) \in L_p(\Omega; \mathbb{R}^n)$, and there exists $\gamma_0(\cdot) \in L_\infty(\Omega; \mathbb{R}^1)$ such that for almost all (a.a.) $\xi \in \Omega$, the inequality

$$\|f(\xi, x_1) - f(\xi, x_2)\| \le \gamma_0(\xi)\|x_1 - x_2\|$$

is verified for every $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^n$, where $L_\infty(\Omega; \mathbb{R}^{n_*})$ is the space of the Lebesgue measurable functions $w(\cdot) : \Omega \to \mathbb{R}^{n_*}$ such that $\|w(\cdot)\|_\infty < +\infty$, $\|w(\cdot)\|_\infty = \inf\{c > 0 : \|w(s)\| \le c$ for a.a. $s \in \Omega\}$;

B. The function $K(\cdot,\cdot,x, u) : \Omega \times \Omega \to \mathbb{R}^n$ is Lebesgue measurable for each fixed $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$ and $K(\cdot,\cdot,0, 0) \in L_p(\Omega \times \Omega; \mathbb{R}^n)$, and there exist $\gamma_i(\cdot,\cdot) : \Omega \times \Omega \to [0, +\infty)$ $(i = 1, 2)$ such that

$$\int_\Omega \left(\int_\Omega \gamma_i(\xi, s)^q \mathrm{d}s \right)^{\frac{p}{q}} \mathrm{d}\xi < +\infty, \quad i = 1, 2$$

and for a.a. $(\xi, s) \in \Omega \times \Omega$, the inequality

$$\|K(\xi, s, x_1, u_1) - K(\xi, s, x_2, u_2)\| \le \gamma_1(\xi, s)\|x_1 - x_2\| + \gamma_2(\xi, s)\|u_1 - u_2\|$$

is held for each $(x_1, u_1) \in \mathbb{R}^n \times \mathbb{R}^m$ and $(x_2, u_2) \in \mathbb{R}^n \times \mathbb{R}^m$.

C. The inequality $5^{p-1}[\alpha_0^p + \alpha_1^p] < 1$ is satisfied, where

$$\alpha_0 = \|\gamma_0(\cdot)\|_\infty, \tag{2}$$

$$\alpha_i = \left(\int_\Omega \left(\int_\Omega \gamma_i(\xi, s)^q \mathrm{d}s \right)^{\frac{p}{q}} \mathrm{d}\xi \right)^{\frac{1}{p}}, \quad i = 1, 2. \tag{3}$$

Let $u(\cdot) \in U_{p,r}$. The function $x(\cdot) \in L_p(\Omega; \mathbb{R}^n)$ satisfying equation (1) for a.a. $\xi \in \Omega$ is said to be a trajectory of system (1) generated by an admissible control function $u(\cdot) \in U_{p,r}$. The set of trajectories of system (1) generated by all admissible control functions is denoted by $\mathbf{X}_{p,r}$.

3 Basic properties of the set of trajectories

The following propositions characterize the basic properties of the set of trajectories. Denote

$$g_1 = \|f(\cdot,0)\|_p = \left(\int_\Omega \|f(\xi, 0)\|^p \mathrm{d}\xi \right)^{\frac{1}{p}}, \tag{4}$$

$$g_2 = \|K(\cdot,\cdot,0, 0)\|_p = \left(\int_\Omega \int_\Omega \|K(\xi, s, 0, 0)\|^p \mathrm{d}s\mathrm{d}\xi \right)^{\frac{1}{p}}. \tag{5}$$

Proposition 1. *Every admissible control function $u_*(\cdot) \in U_{p,r}$ generates a unique trajectory of system* (1).

Proof. For given $u_*(\cdot) \in U_{p,r}$, let us define a map $x(\cdot) \to F(x(\cdot))|(\cdot)$, $x(\cdot) \in L_p(\Omega; \mathbb{R}^n)$, setting

$$F(x(\cdot))|(\xi) = f(\xi, x(\xi)) + \int_\Omega K(\xi, s, x(s), u_*(s))\mathrm{d}s, \quad \text{for a.a. } \xi \in \Omega.$$

First, let us show that $F(x(\cdot))|(\cdot) \in L_p(\Omega; \mathbb{R}^n)$ for every $x(\cdot) \in L_p(\Omega; \mathbb{R}^n)$. Let us fix $x(\cdot) \in L_p(\Omega; \mathbb{R}^n)$. Since $x(\cdot)$ and $u_*(\cdot)$ are Lebesgue measurable functions, then conditions A, B and Theorem 1.4.22 of [20] imply that the map $\xi \to F(x(\cdot))|(\xi)$, $\xi \in \Omega$, is the Lebesgue measurable function.

Let $x(\cdot) \in L_p(\Omega; \mathbb{R}^n)$ be fixed and $\|x(\cdot)\|_p = \beta_* < +\infty$. Conditions A and B, inclusion $u_*(\cdot) \in U_{p,r}$, Hölder's inequality, and (2) yield

$$\|F(x(\cdot))|(\xi)\| \leq \gamma_0(\xi)\|x(\xi)\| + \|f(\xi, 0)\| + \int_\Omega [\gamma_1(\xi, s)\|x(s)\| + \gamma_2(\xi, s)\|u_*(s)\| + \|K(\xi, s, 0, 0)\|]\mathrm{d}s$$
$$\leq \alpha_0\|x(\xi)\| + \|f(\xi, 0)\| + \beta_*\left(\int_\Omega \gamma_1(\xi, s)^q \mathrm{d}s\right)^{\frac{1}{q}} + r\left(\int_\Omega \gamma_2(\xi, s)^q \mathrm{d}s\right)^{\frac{1}{q}}$$
$$+ [\mu(\Omega)]^{\frac{1}{q}}\left(\int_\Omega \|K(\xi, s, 0, 0)\|^p \mathrm{d}s\right)^{\frac{1}{p}}$$

for a.a. $\xi \in \Omega$, and hence

$$\|F(x(\cdot))|(\xi)\|^p \leq 5^{p-1}\left[\alpha_0^p\,\|x(\xi)\|^p + \|f(\xi, 0)\|^p + \beta_*^p\left(\int_\Omega \gamma_1(\xi, s)^q \mathrm{d}s\right)^{\frac{p}{q}} + r^p\left(\int_\Omega \gamma_2(\xi, s)^q \mathrm{d}s\right)^{\frac{p}{q}}\right.$$
$$\left. + [\mu(\Omega)]^{\frac{p}{q}}\int_\Omega \|K(\xi, s, 0, 0)\|^p \mathrm{d}s\right]$$

for a.a. $\xi \in \Omega$. Integrating the obtained inequality on the set Ω and taking into consideration (3), (4), and (5), we finally have

$$\|F(x(\cdot))|(\cdot)\|_p \leq 5^{\frac{p-1}{p}}\left[\alpha_0^p\beta_*^p + g_1^p + \beta_*^p\alpha_1^p + r^p\alpha_2^p + [\mu(\Omega)]^{\frac{p}{q}}g_2^p\right]^{\frac{1}{p}} < +\infty.$$

From the last inequality, we conclude that $F(x(\cdot))|(\cdot) \in L_p(\Omega; \mathbb{R}^n)$. Now, let us show that the map $F(x(\cdot))|(\cdot) : L_p(\Omega; \mathbb{R}^n) \to L_p(\Omega; \mathbb{R}^n)$ is contractive one. Let us choose $x_1(\cdot) \in L_p(\Omega; \mathbb{R}^n)$ and $x_2(\cdot) \in L_p(\Omega; \mathbb{R}^n)$. From conditions A and B, Hölder's inequality, and (2), it follows that

$$\|F(x_2(\cdot))|(\xi) - F(x_1(\cdot))|(\xi)\| \leq \alpha_0\|x_2(\xi) - x_1(\xi)\| + \left(\int_\Omega \gamma_1(\xi, s)^q \mathrm{d}s\right)^{\frac{1}{q}} \|x_2(\cdot) - x_1(\cdot)\|_p,$$

and consequently,

$$\|F(x_2(\cdot))|(\xi) - F(x_1(\cdot))|(\xi)\|^p \leq 2^{p-1}\left[\alpha_0^p\,\|x_2(\xi) - x_1(\xi)\|^p + \left(\int_\Omega \gamma_1(\xi, s)^q \mathrm{d}s\right)^{\frac{p}{q}} \|x_2(\cdot) - x_1(\cdot)\|_p^p\right]$$

for a.a. $\xi \in \Omega$. Integrating the last inequality on the set Ω and taking into consideration (3), we obtain

$$\|F(x_2(\cdot))|(\cdot) - F(x_1(\cdot))|(\cdot)\|_p \leq 2^{\frac{p-1}{p}}[\alpha_0^p + \alpha_1^p]^{\frac{1}{p}}\|x_2(\cdot) - x_1(\cdot)\|_p.$$

According to the condition C, we have $2^{\frac{p-1}{p}}[\alpha_0^p + \alpha_1^p]^{\frac{1}{p}} < 1$, and thus the map $F(x(\cdot))|(\cdot) : L_p(\Omega; \mathbb{R}^n) \to L_p(\Omega; \mathbb{R}^n)$ is contractive. According to the Banach fixed point theorem, this map has unique fixed point $x_*(\cdot) \in L_p(\Omega; \mathbb{R}^n)$, which is a unique function satisfying the equation

$$x_*(\xi) = f(\xi, x_*(\xi)) + \int_\Omega K(\xi, s, x_*(s), u_*(s))\mathrm{d}s$$

for a.a. $\xi \in \Omega$. □

The following proposition characterizes boundedness of the set of trajectories in the space $L_p(\Omega; \mathbb{R}^n)$. Let us set

$$\sigma_* = \left(\frac{5^{p-1}\left[g_1^p + r^p \alpha_2^p + [\mu(\Omega)]^{\frac{p}{q}} g_2^p \right]}{1 - 5^{p-1}[\alpha_0^p + \alpha_1^p]} \right)^{\frac{1}{p}}, \tag{6}$$

where α_0, α_1, α_2, g_1, and g_2 are defined by (2)–(5), respectively.

Proposition 2. *For every $x(\cdot) \in \mathbf{X}_{p,r}$, the inequality $\|x(\cdot)\|_p \leq \sigma_*$ is satisfied, where $\sigma_* > 0$ is defined by equality* (6).

Proof. Choose an arbitrary $x(\cdot) \in \mathbf{X}_{p,r}$ generated by the control function $u(\cdot) \in U_{p,r}$. By virtue of conditions A and B, Hölder's inequality, and the inclusion $u(\cdot) \in U_{p,r}$ and (2), we have

$$\begin{aligned} \|x(\xi)\| \leq \alpha_0 \|x(\xi)\| + \|f(\xi, 0)\| + \left(\int_\Omega \gamma_1(\xi, s)^q \mathrm{d}s \right)^{\frac{1}{q}} \cdot \|x(\cdot)\|_p + r \left(\int_\Omega \gamma_2(\xi, s)^q \mathrm{d}s \right)^{\frac{1}{q}} \\ + [\mu(\Omega)]^{\frac{1}{q}} \left(\int_\Omega \|K(\xi, s, 0, 0)\|^p \mathrm{d}s \right)^{\frac{1}{p}} \end{aligned}$$

for a.a. $\xi \in \Omega$, and hence

$$\begin{aligned} \|x(\xi)\|^p \leq 5^{p-1} \Bigg[\alpha_0^p \|x(\xi)\|^p + \|f(\xi, 0)\|^p + \left(\int_\Omega \gamma_1(\xi, s)^q \mathrm{d}s \right)^{\frac{p}{q}} \cdot \|x(\cdot)\|_p^p + r^p \left(\int_\Omega \gamma_2(\xi, s)^q \mathrm{d}s \right)^{\frac{p}{q}} \\ + [\mu(\Omega)]^{\frac{p}{q}} \int_\Omega \|K(\xi, s, 0, 0)\|^p \mathrm{d}s \Bigg] \end{aligned}$$

for a.a. $\xi \in \Omega$. Integrating the last inequality on the set Ω, we obtain

$$\|x(\cdot)\|_p^p \leq 5^{p-1}[\alpha_0^p + \alpha_1^p] \cdot \|x(\cdot)\|_p^p + 5^{p-1}\left[g_1^p + r^p \alpha_2^p + [\mu(\Omega)]^{\frac{p}{q}} g_2^p \right],$$

where α_i, $(i = 1, 2)$, g_1, and g_2, are defined by (3)–(5), respectively. From the obtained inequality, condition C, and (6), we conclude that

$$\|x(\cdot)\|_p \leq \left(\frac{5^{p-1}\left[g_1^p + r^p \alpha_2^p + [\mu(\Omega)]^{\frac{p}{q}} g_2^p \right]}{1 - 5^{p-1}[\alpha_0^p + \alpha_1^p]} \right)^{\frac{1}{p}} = \sigma_*. \qquad \square$$

Let us set

$$k_* = \frac{3^{\frac{p-1}{p}} \alpha_2}{[1 - 3^{p-1}(\alpha_0^p + \alpha_1^p)]^{\frac{1}{p}}}. \tag{7}$$

In the following, we give an evaluation between the trajectories of system (1) generated by different admissible control functions.

Proposition 3. *Let $x_1(\cdot) \in \mathbf{X}_{p,r}$ and $x_2(\cdot) \in \mathbf{X}_{p,r}$ be the trajectories of system* (1) *generated by the admissible control functions $u_1(\cdot) \in U_{p,r}$ and $u_2(\cdot) \in U_{p,r}$, respectively. Then, the inequality*

$$\|x_1(\cdot) - x_2(\cdot)\|_p \leq k_* \cdot \|u_1(\cdot) - u_2(\cdot)\|_p$$

is held where $k_* > 0$ is defined by (7).

Proof. From conditions A and B and Hölder's inequality, we obtain that

$$\|x_1(\xi) - x_2(\xi)\| \le \alpha_0 \|x_1(\xi) - x_2(\xi)\| + \left(\int_{\Omega} \gamma_1(\xi, s)^q \mathrm{d}s \right)^{\frac{1}{q}} \cdot \|x_1(\cdot) - x_2(\cdot)\|_p + \left(\int_{\Omega} \gamma_2(\xi, s)^q \mathrm{d}s \right)^{\frac{1}{q}} \cdot \|u_1(\cdot) - u_2(\cdot)\|_p$$

for a.a. $\xi \in \Omega$, where α_0 is defined by (2). Raising to the power p and integrating the last inequality on the set Ω, we have

$$\|x_1(\cdot) - x_2(\cdot)\|_p^p \le 3^{p-1} [\alpha_0^p \|x_1(\cdot) - x_2(\cdot)\|_p^p + \alpha_1^p \|x_1(\cdot) - x_2(\cdot)\|_p^p + \alpha_2^p \|u_1(\cdot) - u_2(\cdot)\|_p^p], \tag{8}$$

where α_i, $(i = 1, 2)$, is defined by (3). Finally, condition C, (7), and (8) imply that

$$\|x_1(\cdot) - x_2(\cdot)\|_p \le \frac{3^{\frac{p-1}{p}} \alpha_2}{[1 - 3^{p-1}(\alpha_0^p + \alpha_1^p)]^{\frac{1}{p}}} \cdot \|u_1(\cdot) - u_2(\cdot)\|_p = k_* \cdot \|u_1(\cdot) - u_2(\cdot)\|_p. \qquad \square$$

From Proposition 3, it follows that every solution depends on its generating control function Lipschitz continuously.

Now, let us give a definition of path-connectedness.

Definition 1. Let $(E, d(\cdot, \cdot))$ be a metric space. The set $Q \subset E$ is said to be path-connected if for each $x \in Q$ and $y \in Q$, there exists a continuous function $\psi(\cdot) : [0, 1] \to Q$ such that $\psi(0) = x$ and $\psi(1) = y$.

Let $\Phi(u(\cdot))|(\cdot)$ be the trajectory of system (1) generated by the admissible control function $u(\cdot) \in U_{p,r}$. Then, according to Proposition 3, we have that the map $\Phi(u(\cdot))|(\cdot) : U_{p,r} \to L_p(\Omega; \mathbb{R}^n)$ is Lipschitz continuous. Since the set $U_{p,r} \subset L_p(\Omega; \mathbb{R}^m)$ is convex and

$$\mathbf{X}_{p,r} = \{\Phi(u(\cdot))|(\cdot) : u(\cdot) \in U_{p,r}\},$$

we conclude that the set $\mathbf{X}_{p,r}$ is a path-connected subset of the space $L_p(\Omega; \mathbb{R}^n)$. So, the following theorem is valid.

Theorem 1. *The set of trajectories $\mathbf{X}_{p,r}$ of system* (1) *is a path-connected subset of the space $L_p(\Omega; \mathbb{R}^n)$.*

4 Closedness of the set of trajectories

In this section, it is shown that the set of trajectories $\mathbf{X}_{p,r}$ is not closed in the space $L_p(\Omega; \mathbb{R}^n)$. To illustrate not closedness of the set of trajectories of system (1), the modified version of the example given in [7], p. 62, is used, where not closedness of the set of solutions of the differential inclusion with nonconvex right-hand side is presented. Let us give an auxiliary proposition that will be used in the discussion of the closedness of the set of trajectories.

Proposition 4. *Let $g_1(\cdot) : [a, b] \to \mathbb{R}$ and $g_2(\cdot) : [a, b] \to \mathbb{R}$ are continuous functions and $g_1(\xi) = g_2(\xi)$ for a.a. $\xi \in [a, b]$. Then, $g_1(\xi) = g_2(\xi)$ for every $\xi \in [a, b]$. In addition, if the functions $g_1(\cdot)$ and $g_2(\cdot)$ are differentiable for a.a. $\xi \in [0, 1]$, then $g_1'(\xi) = g_2'(\xi)$ for a.a. $\xi \in [a, b]$.*

Example 1. Consider the control system described by the following system of integral equations:

$$\begin{cases} x_1(\xi) = f(\xi) + \int_0^1 \chi(\xi, s)[-|x_2(s)|^p + |u(s)|^p] \mathrm{d}s, \\ x_2(\xi) = \int_0^1 \chi(\xi, s) u(s) \mathrm{d}s, \end{cases} \tag{9}$$

where $p > 1$, $u(\cdot) \in U^*_{p,1}$, $U^*_{p,1} = \{u(\cdot) \in L_p([0,1];\mathbb{R}) : \|u(\cdot)\|_p \le 1\}$, $\xi \in [0,1]$,

$$f(\xi) = \begin{cases} 0 & \text{if} \quad \xi \in [0,1] \quad \text{is irrational}, \\ 1 & \text{if} \quad \xi \in [0,1] \quad \text{is rational}, \end{cases} \tag{10}$$

$$\chi(\xi, s) = \begin{cases} 1 & \text{if} \quad s \le \xi, \\ 0 & \text{if} \quad s > \xi. \end{cases} \tag{11}$$

The set of trajectories of system (9) generated by all admissible control functions $u(\cdot) \in U^*_{p,1}$ is denoted by symbol $\mathbf{X}^*_{p,1}$.

First of all, let us show that $\mathbf{X}^*_{p,1}$ is a bounded subset of the space $L_p([0,1];\mathbb{R})$. Let $(x_1(\cdot), x_2(\cdot)) \in \mathbf{X}^*_{p,1}$ be an arbitrarily chosen trajectory generated by admissible control function $u(\cdot) \in U^*_{p,1}$. Then, (9), (11), and the inclusion $u(\cdot) \in U^*_{p,1}$ imply that

$$|x_2(\xi)| \le \int_0^1 |u(s)|\mathrm{d}s \le \left(\int_0^1 |u(s)|^p \mathrm{d}s \right)^{\frac{1}{p}} \le 1 \tag{12}$$

for a.a. $\xi \in [0,1]$, and consequently,

$$|x_1(\xi)| \le |f(\xi)| + \int_0^1 [|x_2(s)|^p + |u(s)|^p]\mathrm{d}s \le 1 + 1 = 2 \tag{13}$$

for a.a. $\xi \in [0,1]$. (12) and (13) yield that

$$\|(x_1(\cdot), x_2(\cdot))\|_p \le \left(\int_0^1 (\sqrt{4+1})^p \mathrm{d}\xi \right)^{\frac{1}{p}} = \sqrt{5}. \tag{14}$$

Inequality (14) means that the set of trajectories $\mathbf{X}^*_{p,1}$ is a bounded subset of the space $L_p([0,1];\mathbb{R})$. Now, let us show that the set of trajectories $\mathbf{X}^*_{p,1}$ is not a closed subset of the space $L_p([0,1];\mathbb{R})$.

Choose a sequence of uniform partitions $\{\Delta^{(k)}\}_{k=1}^{\infty}$ of the closed interval $[0,1]$, where $\Delta^{(k)} = \left\{0, \frac{1}{2k}, \frac{2}{2k}, \dots, \frac{2k-1}{2k}, 1\right\}$, $k = 1, 2, \dots$ Let the functions $u_k(\cdot) : [0,1] \to \mathbb{R}$ be defined as follows:

$$u^{(k)}(s) = \begin{cases} 1 & \text{if} \quad s \in \left[\frac{2i}{2k}, \frac{2i+1}{2k}\right), \\ -1 & \text{if} \quad s \in \left[\frac{2i+1}{2k}, \frac{2i+2}{2k}\right), \end{cases} \tag{15}$$

where $i = 0, 1, \dots, k-1$. It is not difficult to verify that $u^{(k)}(\cdot) \in U^*_{p,1}$ for every $k = 1, 2, \dots$. Let $(x_1^{(k)}(\cdot), x_2^{(k)}(\cdot)) \in \mathbf{X}^*_{p,1}$ be the trajectory of system (9) generated by admissible control function $u^{(k)}(\cdot) \in U^*_{p,1}$. Then,

$$\begin{cases} x_1^{(k)}(\xi) = f(\xi) + \int_0^1 \chi(\xi, s)[-|x_2^{(k)}(s)|^p + |u^{(k)}(s)|^p]\mathrm{d}s, \\ x_2^{(k)}(\xi) = \int_0^1 \chi(\xi, s) u^{(k)}(s)\mathrm{d}s \end{cases} \tag{16}$$

for a.a. $\xi \in [0,1]$. From (11), (15), and (16), it follows that

$$\begin{aligned} &\left(1 - \frac{1}{(2k)^p}\right)\xi \le x_1^{(k)}(\xi) \le \xi, \\ &0 \le x_2^{(k)}(\xi) \le \frac{1}{2k} \end{aligned} \tag{17}$$

for a.a. $\xi \in [0, 1]$. Setting $x_1^*(\xi) = \xi$, $x_2^*(\xi) = 0$, and $\xi \in [0, 1]$, from (17), we obtain

$$|x_1^{(k)}(\xi) - x_1^*(\xi)| \le \frac{1}{(2k)^p}, \; |x_2^{(k)}(\xi) - x_2^*(\xi)| \le \frac{1}{2k}$$

for a.a. $\xi \in [0, 1]$, and hence

$$\|(x_1^{(k)}(\cdot), x_2^{(k)}(\cdot)) - (x_1^*(\cdot), x_2^*(\cdot))\|_p \le \frac{1}{k} \tag{18}$$

for every $k = 1, 2, \ldots$. From (18), it follows that $(x_1^{(k)}(\cdot), x_2^{(k)}(\cdot)) \to (x_1^*(\cdot), x_2^*(\cdot))$ as $k \to \infty$ in the space $L_p([0, 1];\mathbb{R})$, where $(x_1^{(k)}(\cdot), x_2^{(k)}(\cdot)) \in \mathbf{X}_{p,1}^*$ for every $k = 1, 2, \ldots$ Finally, let us show that $(x_1^*(\cdot), x_2^*(\cdot)) \notin \mathbf{X}_{p,1}^*$. Assume the contrary. Let $(x_1^*(\cdot), x_2^*(\cdot)) \in \mathbf{X}_{p,1}^*$. Then, there exists $u^*(\cdot) \in U_{p,1}^*$ such that

$$\begin{cases} \xi = f(\xi) + \displaystyle\int_0^1 \chi(\xi, s)|u^*(s)|^p \mathrm{d}s, \\ 0 = \displaystyle\int_0^1 \chi(\xi, s)u^*(s)\mathrm{d}s \end{cases} \tag{19}$$

for a.a. $\xi \in [0, 1]$. (11) and (19) yield that

$$\begin{cases} \xi = f(\xi) + \displaystyle\int_0^\xi |u^*(s)|^p \mathrm{d}s, \\ 0 = \displaystyle\int_0^\xi u^*(s)\mathrm{d}s \end{cases} \tag{20}$$

for a.a. $\xi \in [0, 1]$. Since $u^*(\cdot) \in U_{p,1}^* \subset L_p([0, 1]; \mathbb{R})$, then we have that the function $\xi \to \int_0^\xi u^*(s)\mathrm{d}s$, $\xi \in [0, 1]$, is absolutely continuous, and therefore, it is differentiable for a.a. $\xi \in [0, 1]$. So, Proposition 4 implies that the second equality in system (20) is satisfied for every $\xi \in [0, 1]$, and hence $u^*(\xi) = 0$ for a.a. $\xi \in [0, 1]$. Thus, from the first equation of (20), we obtain that $\xi = f(\xi)$ for a.a. $\xi \in [0, 1]$, which contradicts the equality (10). This means that our assumption is not true, and $(x_1^*(\cdot), x_2^*(\cdot)) \notin \mathbf{X}_{p,1}^*$, which yields that the set of trajectories $\mathbf{X}_{p,1}^*$ is not a closed subset of the space $L_p([0, 1]; \mathbb{R})$.

Note that along with compactness, the closedness of the set of trajectories $\mathbf{X}_{p,r}$ is important for the existence of the optimal trajectories in the optimal control problem with continuous payoff functional, even if there exists an ε-approximate trajectory for every $\varepsilon > 0$.

Consider the following optimal control problem. Let the dynamics of the control system be described by system of integral equations (9), where $p > 1$ and the admissible control functions are chosen from the set $U_{p,1}^*$. It is required to minimize the functional

$$J(x_1(\cdot), x_2(\cdot)) = \int_0^1 [|x_1(\xi) - \xi|^p + |x_2(\xi)|^p]\mathrm{d}\xi \tag{21}$$

on the set of trajectories $\mathbf{X}_{p,1}^*$. It is obvious that $J(x_1(\cdot), x_2(\cdot)) \ge 0$ for every $(x_1(\cdot), x_2(\cdot)) \in \mathbf{X}_{p,1}^*$. The trajectory $(\tilde{x}_1(\cdot), \tilde{x}_2(\cdot)) \in \mathbf{X}_{p,1}^*$ is called an optimal iff

$$J(\tilde{x}_1(\cdot), \tilde{x}_2(\cdot)) \le J(x_1(\cdot), x_2(\cdot))$$

for every $(x_1(\cdot), x_2(\cdot)) \in \mathbf{X}^*_{p,1}$. For given $\varepsilon > 0$, the trajectory $(x_1^{\varepsilon}(\cdot), x_2^{\varepsilon}(\cdot)) \in \mathbf{X}^*_{p,1}$ is said to be ε-approximate optimal iff

$$J(x_1^{\varepsilon}(\cdot), x_2^{\varepsilon}(\cdot)) \leq \varepsilon + \inf\{J(x_1(\cdot), x_2(\cdot)) : (x_1(\cdot), x_2(\cdot)) \in \mathbf{X}^*_{p,1}\}.$$

Let $(x_1^{(k)}(\cdot), x_2^{(k)}(\cdot)) \in \mathbf{X}^*_{p,1}$ $(k = 1, 2, \dots)$ be the trajectory of system (9) defined by (16). The convergence $(x_1^{(k)}(\cdot), x_2^{(k)}(\cdot)) \to (x_1^*(\cdot), x_2^*(\cdot))$ as $k \to \infty$ in the space $L_p([0, 1]; \mathbb{R}^2)$, where $x_1^*(\xi) = \xi$, $x_2^*(\xi) = 0$ for a.a. $\xi \in [0, 1]$, yields

$$\inf\{J(x_1(\cdot), x_2(\cdot)) : (x_1(\cdot), x_2(\cdot)) \in \mathbf{X}^*_{p,1}\} = 0. \tag{22}$$

Since $J(x_1^{(k)}(\cdot), x_2^{(k)}(\cdot)) \to J(x_1^*(\cdot), x_2^*(\cdot))$ as $k \to \infty$, $J(x_1^*(\cdot), x_2^*(\cdot)) = 0$, $(x_1^*(\cdot), x_2^*(\cdot)) \notin \mathbf{X}^*_{p,1}$, then (22) implies that in the optimal control problem ((9), (21)) $\to$ inf, the optimal trajectory does not exist, but for every $\varepsilon > 0$, there exists an ε-approximate trajectory.

5 Robustness of the trajectories

In this section, in addition to the conditions **A**–**C** it will be assumed that the function $\gamma_2(\cdot,\cdot) : \Omega \times \Omega \to [0, +\infty)$ given in condition **B** belongs to the space $L_\infty(\Omega \times \Omega; [0, +\infty))$, i.e., we assume that $\gamma_2(\cdot,\cdot) \in L_\infty(\Omega \times \Omega; [0, +\infty))$ and let

$$\|\gamma_2(\cdot,\cdot)\|_\infty = \nu_*. \tag{23}$$

Theorem 2. *Let $\varepsilon > 0$ be a given number, $x(\cdot) \in \mathbf{X}_{p,r}$ be a trajectory of the system* (1) *generated by the admissible control function $u(\cdot) \in U_{p,r}$, $\|u(\cdot)\|_p = r_1 < r$, $\Omega_* \subset \Omega$ be a Lebesgue measurable set, the control function*

$$v(\xi) = \begin{cases} u(\xi) & \text{if } \xi \in \Omega \setminus \Omega_*, \\ u_*(\xi) & \text{if } \xi \in \Omega_* \end{cases} \tag{24}$$

be such that $\|v(\cdot)\|_p = r$ and $y(\cdot) : \Omega \to \mathbb{R}^n$ be the trajectory of system (1) *generated by the admissible control function $v(\cdot) \in U_{p,r}$. If*

$$\mu(\Omega_*) \leq \left\{ \frac{[1 - 3^{p-1}(\alpha_0^p + \alpha_1^p)]^{\frac{1}{p}}}{2 \cdot 3^{\frac{p-1}{p}} r \nu_* [\mu(\Omega)]^{\frac{1}{p}}} \cdot \varepsilon \right\}^{\frac{p}{p-1}}, \tag{25}$$

then $\|x(\cdot) - y(\cdot)\|_p \leq \varepsilon$.

Proof. From conditions **A** and **B**, the inclusion of $\gamma_2(\cdot,\cdot) \in L_\infty(\Omega \times \Omega; [0, +\infty))$, and $u(\cdot) \in U_{p,r}$, $v(\cdot) \in U_{p,r}$, Hölder's inequality, (2), (23), and (24), it follows that

$$\|x(\xi) - y(\xi)\| \leq \alpha_0 \|x(\xi) - y(\xi)\| + \left(\int_\Omega \gamma_1(\xi, s)^q \mathrm{d}s \right)^{\frac{1}{q}} \|x(\cdot) - y(\cdot)\|_p + 2r\nu_* [\mu(\Omega_*)]^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$, α_0 is defined by (2). Since $p > 1$, then raising to the power p and integrating the obtained inequality on the set Ω, we have from the last inequality

$$\|x(\cdot) - y(\cdot)\|_p^p \leq 3^{p-1}[\alpha_0^p \|x(\cdot) - y(\cdot)\|_p^p + \alpha_1^p \|x(\cdot) - y(\cdot)\|_p^p + 2^p r^p \nu_*^p \mu(\Omega)[\mu(\Omega_*)]^{p-1}], \tag{26}$$

where α_1 is defined by (3).

By virtue of the condition **C**, (25), and (26), we obtain that

$$\|x(\cdot) - y(\cdot)\|_p \le \frac{2 \cdot 3^{\frac{p-1}{p}} r\nu_*[\mu(\Omega)]^{\frac{1}{p}}}{[1 - 3^{p-1}(\alpha_0^p + \alpha_1^p)]^{\frac{1}{p}}} \cdot [\mu(\Omega_*)]^{\frac{p-1}{p}} \le \varepsilon. \qquad \square$$

From Theorem 2, it follows that consuming a big quantity of the control resource on the domain with a sufficiently small measure is not an effective way to change the system's trajectory. In addition, Theorem 2 implies that if we have a superfluous control resource and we want to get rid of it, then consuming the all remaining control resource on the domain with sufficiently small measure, we will obtain a minor deviation of the system's trajectory.

Denote

$$V_{p,r} = \{u(\cdot) \in L_p(\Omega; \mathbb{R}^m) : \|u(\cdot)\|_p = r\}$$

and let $\mathbf{Z}_{p,r}$ be the set of trajectories of system (1) generated by all admissible control functions $v(\cdot) \in V_{p,r}$. Note that Example 1 also illustrates that the set $\mathbf{Z}_{p,r} \subset L_p(\Omega; \mathbb{R}^n)$ is not a closed set.

Theorem 3. *The equality $h_p(\mathbf{X}_{p,r}, \mathbf{Z}_{p,r}) = 0$ is held, where $h_p(\cdot,\cdot)$ stands for Hausdorff distance between the subsets of the space $L_p(\Omega;\mathbb{R}^n)$.*

Proof. Let us choose an arbitrary $\sigma > 0$ and trajectory $x(\cdot) \in \mathbf{X}_{p,r}$ generated by admissible control function $u(\cdot) \in U_{p,r}$ and let $\|u(\cdot)\|_p = r_1 < r$, $E_* \subset \Omega$ be a Lebesgue measurable set,

$$\mu(E_*) \le \left\{ \frac{[1 - 3^{p-1}(\alpha_0^p + \alpha_1^p)]^{\frac{1}{p}}}{2 \cdot 3^{\frac{p-1}{p}} r\nu_*[\mu(\Omega)]^{\frac{1}{p}}} \cdot \sigma \right\}^{\frac{p}{p-1}} \tag{27}$$

and $\int_{\Omega \setminus E_*} \|u(s)\|^p ds = r_*^p \le r_1^p$. Define new control function $v(\cdot) : \Omega \to \mathbb{R}^m$ setting

$$v(\xi) = \begin{cases} u(\xi) & \text{if} \quad \xi \in \Omega \setminus E_*, \\ \left[\dfrac{r^p - r_*^p}{\mu(E_*)} \right]^{\frac{1}{p}} \cdot b_* & \text{if} \quad \xi \in E_*, \end{cases}$$

where $b_* \in S_m = \{b \in \mathbb{R}^m : \|b\| = 1\}$ is an arbitrarily chosen vector. It is not difficult to verify that $\|v(\cdot)\|_p = r$, i.e., $v(\cdot) \in V_{p,r}$. Let $y(\cdot)$ be the trajectory of system (1) generated by admissible control function $v(\cdot) \in V_{p,r}$. From (27) and Theorem 2, it follows that $\|x(\cdot) - y(\cdot)\|_p \le \sigma$, which yields

$$\mathbf{X}_{p,r} \subset \mathbf{Z}_{p,r} + \sigma B_{L_p}(1) \tag{28}$$

where $B_{L_p}(1) = \{z(\cdot) \in L_p(\Omega; \mathbb{R}^n) : \|z(\cdot)\|_p \le 1\}$.

Taking into consideration that $\mathbf{Z}_{p,r} \subset \mathbf{X}_{p,r}$, we obtain from (28) that

$$h_p(\mathbf{X}_{p,r}, \mathbf{Z}_{p,r}) \le \sigma. \tag{29}$$

Since $\sigma > 0$ is arbitrarily chosen, inequality (29) gives the proof of the theorem. $\square$

Theorem 3 implies the validity of the following corollary.

Corollary 1. *The equality $cl(\mathbf{X}_{p,r}) = cl(\mathbf{Z}_{p,r})$ is verified, where cl denotes the closure of a set.*

6 Conclusion

This article investigates the boundedness, path-connectedness, and closedness properties of the set of p-integrable trajectories of the control system described by the Urysohn-type integral equation. The control functions have an integral constraint, i.e., the control resource is exhausted by consumption, which often arises in different problems of theory and applications. The specified path-connectedness property does not permit splitting of the set of trajectories, and not closedness of the set of trajectories in the considered example allows the existence of the sliding modes for approximate optimal trajectories. The robustness of the trajectory with respect to the fast consumption of the remaining control resource is shown. This means that in domains with sufficiently small measures, it is efficient to consume energy like control resources in small portions. If there is an excess control resource and it is required to get rid of it, then by consuming all the remaining control resources in the domain with a sufficiently small measure, it is possible to obtain a minor variation of the system's trajectory.

Conflict of interest: The author states no conflict of interest.

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