DE GRUYTER J. Appl. Anal. 2023; 29(1): 151–159

Research Article

Nesir Huseyin* and Anar Huseyin

On the continuity properties of the L_p balls

https://doi.org/10.1515/jaa-2022-1008
Received December 27, 2021; revised August 17, 2022; accepted September 21, 2022

Abstract: In this paper the right upper semicontinuity at $p = 1$ and continuity at $p = \infty$ of the set-valued map $p \to B_{\Omega,\mathcal{X},p}(r)$, $p \in [1, \infty]$, are studied where $B_{\Omega,\mathcal{X},p}(r)$ is the closed ball of the space $L_p(\Omega, \Sigma, \mu; \mathcal{X})$ centered at the origin with radius r, (Ω, Σ, μ) is a finite and positive measure space, $\mathcal{X}$ is a separable Banach space. It is proved that the considered set-valued map is right upper semicontinuous at $p = 1$ and continuous at $p = \infty$. An application of the obtained results to the set of integrable outputs of the input-output system described by the Urysohn-type integral operator is discussed.

Keywords: Continuity, semicontinuity, set-valued map, Urysohn integral operator, input-output system

MSC 2010: 26E25, 46T20, 93C35

1 Introduction

In the problems of theory and applications sometimes there is a need to define the distance between the subsets of different metric spaces (see, e.g. [10, 13] and references therein). For this aim often the Hausdorff–Gromov distance concept is applied which is a generalization of the Hausdorff distance notion, widely used in the set-valued analysis (see, e.g., [1–3]). Taking into consideration that $L_p \subset L_1$ for every $p \in [1, \infty]$, in this paper for definition of the distance between the subsets of the spaces L_p, $p \in [1, \infty]$, the metric of the space L_1 is chosen. Using the introduced metric, the right upper semicontinuity at $p = 1$ and continuity at $p = \infty$ of the closed balls of the spaces L_p, $p \in [1, \infty]$, are established.

Let (Ω, Σ, μ) be a positive and finite measure space, $(\mathcal{X}, \|\cdot\|)$ be a separable Banach space, $L_p(\Omega, \Sigma, \mu; \mathcal{X})$ stand for the space of all (equivalence classes of) μ-measurable functions $x(\cdot) : \Omega \to \mathcal{X}$ such that $\|x(\cdot)\|_p < \infty$, where $p \in [1, \infty)$,

$$\|x(\cdot)\|_p = \Big(\int_\Omega \|x(s)\|^p \mu(ds) \Big)^{\frac{1}{p}},$$

integration is understood in the sense of Bochner, $\|\cdot\|$ denotes the norm of the space $\mathcal{X}$. By $L_\infty(\Omega, \Sigma, \mu; \mathcal{X})$ we denote the space of all (equivalence classes of) μ-measurable functions $v(\cdot) : \Omega \to \mathcal{X}$ such that $\|v(\cdot)\|_\infty < \infty$, where $\|v(\cdot)\|_\infty = \inf\{\rho > 0 : \|v(s)\| \le \rho$ for μ-a.a. $s \in \Omega\}$.

For given $p \in [1, \infty]$ and $r > 0$ we denote

$$B_{\Omega,\mathcal{X},p}(r) = \{x(\cdot) \in L_p(\Omega, \Sigma, \mu; \mathcal{X}) : \|x(\cdot)\|_p \le r\}. \tag{1.1}$$

Let $G \subset L_{p_1}(\Omega, \Sigma, \mu; \mathcal{X})$ and $Q \subset L_{p_2}(\Omega, \Sigma, \mu; \mathcal{X})$ be bounded sets, where $p_1 \in [1, \infty]$, $p_2 \in [1, \infty]$. The Hausdorff distance between the sets G and Q is denoted by symbol $\mathcal{H}_1(G, Q)$ and defined as

$$\mathcal{H}_1(G, Q) = \max\Big\{ \sup_{y(\cdot)\in G} d_1(y(\cdot), Q), \sup_{w(\cdot)\in Q} d_1(w(\cdot), G)\Big\}, \tag{1.2}$$

where $d_1(y(\cdot), Q) = \inf\{\|y(\cdot) - w(\cdot)\|_1 : w(\cdot) \in Q\}$.

***Corresponding author: Nesir Huseyin,** Department of Mathematics and Science Education, Sivas Cumhuriyet University, Sivas, Türkiye, e-mail: nhuseyin@cumhuriyet.edu.tr. https://orcid.org/0000-0001-7652-1505
Anar Huseyin, Department of Mathematics and Science Education, Sivas Cumhuriyet University, Sivas, Türkiye, e-mail: ahuseyin@cumhuriyet.edu.tr. https://orcid.org/0000-0002-3911-2304