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Inverse nodal problems for Sturm-Liouville equation with nonlocal boundary conditions ☆

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ABSTRACT

In this paper, a Sturm–Liouville problem with some nonlocal boundary conditions of the Bitsadze-Samarskii type is studied. We show that the coefficients of the problem can be uniquely determined by a dense set of nodal points. Moreover, we give an algorithm for the reconstruction of the potential function and some other coefficients in the boundary conditions.

1. Introduction

The inverse nodal problem for a Sturm-Liouville operator consists in reconstructing the operator from zeros of its eigenfunctions, namely nodal points. This problem was studied firstly by McLaughlin in 1988 [16]. She showed that the potential of a Sturm-Liouville problem with Dirichlet boundary conditions can be determined by a given dense subset of nodal points. Immediately after, Hald and McLaughlin gave some numerical schemes for the reconstruction of the potential [9]. In 1997, X.F. Yang gave a solution algorithm of an inverse nodal problem for the Sturm-Liouville operator with separated boundary conditions [31]. Inverse nodal problems for Sturm-Liouville operators with the classical boundary conditions have been studied in the papers ([1], [4–8], [10], [12–15], [21,22,24], [28–30], [32,33]).

Nonlocal boundary conditions appear when we cannot measure data directly at the boundary. This kind conditions arise in various some applied problems of biology, biotechnology, physics and etc. As it is known there are two kinds of nonlocal boundary conditions. One class of them is called integral type conditions, and the other is the Bitsadze-Samarskii-type conditions. Bitsadze and Samarskii are considered the originators

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of such conditions. Nonlocal boundary conditions of the Bitsadze-Samarskii type were first applied to elliptic equations by them [3]. Some important results on the properties of eigenvalues and eigenfunctions of nonlocal boundary value problems for Sturm-Liouville type operators have been published in various publications (see, for example, [20,23] and the references therein).

Some inverse problems for Sturm-Liouville and Dirac operators with nonlocal boundary conditions are investigated in [2,17,18]. In the literature, there are only a few studies about inverse nodal problems with nonlocal boundary conditions. Moreover all of them include integral type conditions. Inverse nodal problems for this-type operators with different nonlocal integral boundary are studied in ([11,19,25,26]). Especially, C.F. Yang et al. solved inverse-nodal Sturm-Liouville problems with nonlocal integral-type boundary conditions at only one or both end-points (see [11] and [26]).

In the present paper, we consider Sturm-Liouville problems under some the Bitsadze-Samarskii type nonlocal boundary conditions and obtain the uniqueness of coefficients of the problem according to a set of nodal points. Moreover, we give an algorithm for the reconstruction of these coefficients.

Let us consider the following boundary value problem $L = L(q, h, H, \gamma_0, \gamma_1, \xi_0, \xi_1)$:

$$\ell y := -y'' + q(x)y = \lambda y, \quad x \in \Omega = (0,1) \tag{1}$$

$$U(y) := y'(0) + hy(0) - \gamma_0 y(\xi_0) = 0, \tag{2}$$

$$V(y) := y'(1) + Hy(1) - \gamma_1 y(\xi_1) = 0, \tag{3}$$

where $q(x)$ is a real valued continuously differentiable function, h, $H \in \mathbb{R} \cup \{\infty\}$, $\gamma_i \neq 0$ are real numbers for $i = 0, 1$, ξ_i are rational numbers in $(0,1)$ for $i = 0, 1$, and λ is the spectral parameter.

(2) and (3) are nonlocal conditions of a Bitsadze-Samarskii type. It is clear that if $\xi_0 = 0$ and $\xi_1 = 1$, (2) and (3) are not other than the classical separated boundary conditions. On the other hand, while $\xi_0 = 1$ and $\xi_1 = 0$ (2) and (3) turn into non-separated conditions. Inverse nodal problems for this type of boundary conditions are studied by C.F. Yang [27]. Therefore, we focus on the case $\xi_i \in (0,1)$ in our investigation. In fact, since ξ_0 and ξ_1 are arbitrary rational numbers, the problem we consider involves a relatively large class of nonlocal boundary conditions.

The main goal of this paper is to solve inverse nodal problems for (1)-(3) in each of the following cases

$$\text{i) } h, \quad H \in \mathbb{R},$$
$$\text{ii) } h = \infty, \quad H \in \mathbb{R},$$
$$\text{iii) } h \in \mathbb{R}, \quad H = \infty.$$

We note that if $h = \infty$, $H \in \mathbb{R}$, and $h \in \mathbb{R}$, $H = \infty$ the boundary conditions can be written as

$$y(0) = 0,$$
$$y'(1) + Hy(1) = \gamma_1 y(\xi_1),$$

and

$$y'(0) + hy(0) = \gamma_0 y(\xi_0),$$
$$y(1) = 0,$$

respectively.

2. Spectral properties of the problem

Let $S(x,\lambda)$ and $C(x,\lambda)$ be the solutions of (1) under the initial conditions

$$S(0,\lambda)=0, \quad S'(0,\lambda)=1,$$
$$C(0,\lambda)=1, \quad C'(0,\lambda)=0,$$

respectively. It can be calculated that $C(x,\lambda)$ and $S(x,\lambda)$ satisfy the following asymptotic relations for $|\lambda|\to\infty$ (see [11] and [34])

$$C(x,\lambda)=\cos kx+\frac{\sin kx}{k}Q(x)+\frac{\cos kx}{k^2}q_1(x)+O\left(\frac{1}{k^3}\exp|\tau|x\right), \tag{4}$$

$$S(x,\lambda)=\frac{\sin kx}{k}-\frac{\cos kx}{k^2}Q(x)+O\left(\frac{1}{k^3}\exp|\tau|x\right), \tag{5}$$

where $k=\sqrt{\lambda}$, $\tau=|\mathrm{Im}\,k|$, $Q(x)=\frac{1}{2}\int_0^x q(t)dt$ and $q_1(x)=\frac{q(x)-q(0)}{4}-\frac{1}{8}\left(\int_0^x q(t)dt\right)^2$.

The characteristic function of problem (1)-(3)

$$\Delta(\lambda)=\det\begin{pmatrix} U(C) & U(S) \\ V(C) & V(S)\end{pmatrix} \tag{6}$$

and the zeros of the function $\Delta(\lambda)$ coincide with the eigenvalues of the problem (1)-(3). Clearly, $\Delta(\lambda)$ is entire function and so the problem has a discrete spectrum.

Let $\{\lambda_n\}_{n\geq 0}$ be the set of eigenvalues and $\varphi(x,\lambda_n)$ be the eigenfunction corresponding to the eigenvalue λ_n. Some asymptotic formulas of λ_n and $\varphi(x,\lambda_n)$ are given in the following Lemmas.

Lemma 1. *The numbers $\{\lambda_n\}_{n\geq 0}$ are real for sufficiently large n and they satisfy the following asymptotic relation for $n\to\infty$:*

$$\sqrt{\lambda_n}=k_n=k_n^0+\frac{\kappa_n}{n\pi}+o(\frac{1}{n})$$

where $k_n^0=\begin{cases} n\pi, & \text{if } h, \quad H\in\mathbb{R}, \\ \left(n+\frac{1}{2}\right)\pi, & \text{if } h=\infty, \quad H\in\mathbb{R}, \\ & \text{or } H=\infty, \quad h\in\mathbb{R}, \end{cases}$ *and*

$$\kappa_n=\begin{cases} Q(1)+H-h-(-1)^n\left[\gamma_1\cos\left(n\pi\xi_1\right)-\gamma_0\cos\left(n\pi(1-\xi_0)\right)\right], & \text{if } h, \quad H\in\mathbb{R}, \\ H+Q(1)-(-1)^n\gamma_1\sin\left(\left(n+\frac{1}{2}\right)\pi\xi_1\right), & \text{if } h=\infty, \quad H\in\mathbb{R}, \\ Q(1)-h+\gamma_0\cos\left(\left(n+\frac{1}{2}\right)\pi\xi_0\right), & \text{if } h\in\mathbb{R}, \quad H=\infty. \end{cases}$$

Proof. We give the proof for the case: h, $H\in\mathbb{R}$; the other cases are similar. From (6), we have that

$$\begin{aligned}\Delta(\lambda)=\,&hS'(1,\lambda)-\gamma_0C(\xi_0,\lambda)S'(1,\lambda)+HhS(1,\lambda)-\gamma_0HC(\xi_0,\lambda)S(1,\lambda)\\ &-h\gamma_1S(\xi_1,\lambda)+\gamma_1\gamma_0C(\xi_0,\lambda)S(\xi_1,\lambda)-C'(1,\lambda)+\gamma_0C'(1,\lambda)S(\xi_0,\lambda)\\ &-HC(1,\lambda)+H\gamma_0S(\xi_0,\lambda)C(1,\lambda)+\gamma_1C(\xi_1,\lambda)-\gamma_1\gamma_0S(\xi_0,\lambda)C(\xi_1,\lambda).\end{aligned}$$

Using (4) and (5), we obtain the following asymptotic formula for $\Delta(\lambda)$ as $|k|\to\infty$:

$$\begin{aligned}\Delta(\lambda) &= \left[k\sin k - \cos kQ(1) + \frac{\sin k}{k}q_1(1)\right] + \gamma_0\left[C'(1,\lambda)S(\xi_0,\lambda) - C(\xi_0,\lambda)S'(1,\lambda)\right]\\ &+h\left[\cos k + \frac{\sin k}{k}Q(1)\right] + Hh\left[\frac{\sin k}{k} - \frac{\cos k}{k^2}Q(1)\right]\\ &-H\left[\cos k + \frac{\sin k}{k}Q(1) + \frac{\cos k}{k^2}q_1(x)\right] + H\gamma_0\left[S(\xi_0,\lambda)C(1,\lambda) - C(\xi_0,\lambda)S(1,\lambda)\right]\\ &+\gamma_1\gamma_0\left[C(\xi_0,\lambda)S(\xi_1,\lambda) - S(\xi_0,\lambda)C(\xi_1,\lambda)\right]\\ &+\gamma_1\left[\cos k\xi_1 + \frac{\sin k\xi_1}{k}Q(1) + \frac{\cos k\xi_1}{k^2}q_1(\xi_1)\right]\\ &-\gamma_1\left[h\frac{\sin k\xi_1}{k} - h\frac{\cos k\xi_1}{k^2}Q(\xi_1)\right] + O\left(\frac{1}{k^3}\exp|\tau|\right)\end{aligned}$$

and so

$$\Delta(\lambda) = k\sin k + w\cos k + \gamma_1\cos(k\xi_1) - \gamma_0\cos k(1-\xi_0) + O(\exp|\tau|), \tag{7}$$

where $w = h - H - Q(1)$. Let $G_n(\varepsilon) = \{k : |k - n\pi| \geq \varepsilon\}$ for $n = 1, 2, \ldots$. It follows from (7) that there exist some $M(\varepsilon) > 0$ such that $|\Delta(\lambda)| \geq M(\varepsilon)|k|\exp|\tau|$ for sufficiently large $|k|$ in $G_n(\varepsilon)$. Therefore λ_n must be a real number for sufficiently large n.

Moreover, if we apply Rouché theorem to $h_1(\lambda) = k\sin k$ and $h_2(\lambda) = w\cos k + \gamma_1\cos(k\xi_1) - \gamma_0\cos k(1-\xi_0) + O(\exp|\tau|)$ on $\partial G_n(\varepsilon)$ for sufficiently small ε, we can see that zeros of $\Delta(\lambda)$ satisfy

$$k_n = n\pi + \mu_n, \quad \mu_n = o(1), \quad n \to \infty.$$

It follows from (7) that

$$\sin(n\pi + \mu_n) + O(\frac{1}{n}) = 0.$$

Hence $\sin(\mu_n) = O(\frac{1}{n})$ and so $\mu_n = O(\frac{1}{n})$. Thus

$$k_n = n\pi + O(\frac{1}{n}), \quad n \to \infty. \tag{8}$$

Using (7) and (8) together, we get

$$\sin k_n + \frac{w}{n\pi}\cos k_n + \frac{\gamma_1}{n\pi}\cos(k_n\xi_1) - \frac{\gamma_0}{n\pi}\cos(k_n(1-\xi_0)) + o(\frac{1}{n}) = 0.$$

Therefore, we obtain

$$\tan k_n = -\frac{w}{n\pi} - \frac{\gamma_1}{n\pi}\frac{\cos(k_n\xi_1)}{\cos k_n} + \frac{\gamma_0}{n\pi}\frac{\cos(k_n(1-\xi_0))}{\cos k_n} + o(\frac{1}{n}). \tag{9}$$

On the other hand, we have

$$\frac{\cos(k_n\xi_1)}{n\pi\cos k_n} = (-1)^n\frac{\cos(n\pi\xi_1)}{n\pi} + o(\frac{1}{n}), \tag{10}$$

and

$$\frac{\cos(k_n(1-\xi_0))}{n\pi\cos k_n} = (-1)^n\frac{\cos(n\pi(1-\xi_0))}{n\pi} + o(\frac{1}{n}). \tag{11}$$

Using (10) and (11) in (9), we get

$$\tan k_n = -\frac{w}{n\pi} - \gamma_1(-1)^n \frac{\cos(n\pi\xi_1)}{n\pi} + \gamma_0(-1)^n \frac{\cos(n\pi(1-\xi_0))}{n\pi} + o(\frac{1}{n}).$$

Using Taylor's expansion of Arctangent, the proof can be concluded. □

It is clear that

$$\varphi(x,\lambda_n) = U(S(x,\lambda_n))C(x,\lambda_n) - U(C(x,\lambda_n))S(x,\lambda_n). \tag{12}$$

From (12) and Lemma 1, we can prove easily the following lemma:

Lemma 2. *The asymptotic formula*

$$\varphi(x,\lambda_n) = \begin{cases} \cos k_n x + \frac{(Q(x)-h)}{k_n}\sin k_n x + \frac{\gamma_0}{k_n}\sin k_n(x-\xi_0) + O\left(\frac{1}{k_n^2}\right), \ \textit{for } h, H \in \mathbb{R}, \\ \frac{\sin k_n x}{k_n} - \frac{\cos kx}{k_n^2}Q(x) + O\left(\frac{1}{k_n^3}\right), \ \textit{for } h=\infty,\ H\in\mathbb{R}, \\ \frac{\sin k_n(1-x)}{k_n} + \frac{\cos k_n(1-x)}{k_n^2}Q(x) + O\left(\frac{1}{k_n^3}\right), \ \textit{for } h\in\mathbb{R},\ H=\infty \end{cases} \tag{13}$$

is valid for sufficiently large n.

3. Inverse nodal problems: uniqueness and reconstruction

We can see from Lemma 2 that $\varphi(x,\lambda_n)$ has exactly $n-1$ nodal points in $(0,1)$. Let $X = \left\{x_n^j : n = 0,1,2,\ldots \text{ and } j = 1,2,\ldots,n-1\right\}$ be the set of nodal points. We assume that $\int_0^1 q(x)dx = 0$. Otherwise, the term $q(x) - \int_0^1 q(x)dx$ is determined uniquely, instead of $q(x)$.

Lemma 3. *The elements of X satisfy the following asymptotic formulas for sufficiently large n,*

$$x_n^j = \begin{cases} \frac{j+1/2}{n} + \frac{h-H+(-1)^n A_n}{n^2\pi^2}\frac{(j+1/2)}{n} + \frac{\left(Q(x_n^j)-h\right)}{n^2\pi^2} + \\ \qquad + \frac{\gamma_0}{n^2\pi^2}\cos(n\pi\xi_0) + o\left(\frac{1}{n^2}\right), & \textit{if } h,\ H\in\mathbb{R}, \\ \frac{j}{n+\frac{1}{2}} - \frac{H-(-1)^n\gamma_1\sin\left(\left(n+\frac{1}{2}\right)\pi\xi_1\right)}{\left(n+\frac{1}{2}\right)^2\pi^2}\frac{j}{\left(n+\frac{1}{2}\right)} + \\ \qquad + \frac{Q(x_n^j)}{\left(n+\frac{1}{2}\right)^2\pi^2} + o\left(\frac{1}{n^2}\right), & \textit{if } h=\infty,\ H\in\mathbb{R}, \\ \frac{j+\frac{1}{2}}{n+\frac{1}{2}} + \left[h-\gamma_0\cos\left(\left(n+\frac{1}{2}\right)\pi\xi_0\right)\right]\frac{j+\frac{1}{2}}{\left(n+\frac{1}{2}\right)^3\pi^2} - \frac{h-Q(x_n^j)}{\left(n+\frac{1}{2}\right)^2\pi^2} + \\ \qquad + \gamma_0\frac{\cos\left(\left(n+\frac{1}{2}\right)\pi\xi_0\right)}{\left(n+\frac{1}{2}\right)^2\pi^2} + o\left(\frac{1}{n^2}\right), & \textit{if } H=\infty,\ h\in\mathbb{R} \end{cases}$$

where $A_n = [\gamma_1\cos(n\pi\xi_1) - \gamma_0\cos(n\pi(1-\xi_0))]$.

Proof. As before, we consider only the first case. One can obtain similarly desired formulas for the other cases. Use the asymptotic formula (13) to get

$$0=\varphi(x_n^j,\lambda_n)=\cos k_n x_n^j+\frac{\left(Q(x_n^j)-h\right)}{k_n}\sin k_n x_n^j+\frac{\gamma_0}{k_n}\sin k_n\left(x_n^j-\xi_0\right)+o\left(\frac{1}{k_n}\right)$$

and so

$$\tan\left(k_n x_n^j-\frac{\pi}{2}\right)=\frac{\left(Q(x_n^j)-h\right)}{k_n}+\frac{\gamma_0}{k_n}\frac{\sin k_n\left(x_n^j-\xi_0\right)}{\sin k_n x_n^j}+o\left(\frac{1}{k_n}\right).$$

This yields

$$x_n^j=\frac{(j+1/2)\,\pi}{k_n}+\frac{\left(Q(x_n^j)-h\right)}{k_n^2}+\frac{\gamma_0}{k_n^2}\frac{\sin k_n\left(x_n^j-\xi_0\right)}{\sin k_n x_n^j}+o\left(\frac{1}{k_n^2}\right). \tag{14}$$

Using $k_n x_n^j=(j+1/2)\,\pi+O(\frac{1}{n})$, $n\to\infty$ we can show

$$\frac{\sin k_n\left(x_n^j-\xi_0\right)}{k_n^2\sin k_n x_n^j}=\frac{\cos(n\pi\xi_0)}{n^2\pi^2}+o\left(\frac{1}{n^2}\right).$$

On the other hand, we have

$$\frac{1}{k_n}=\frac{1}{n\pi}\left(1+\frac{w}{n^2\pi^2}+\frac{(-1)^n}{n^2\pi^2}A_n+o\left(\frac{1}{n^3}\right)\right),$$
$$\frac{1}{k_n^2}=\frac{1}{n^2\pi^2}+o\left(\frac{1}{n^3}\right),$$

by using Lemma 1. Therefore, it is concluded that,

$$x_n^j=\frac{j+1/2}{n}+\frac{h-H+(-1)^nA_n}{n^2\pi^2}\frac{(j+1/2)}{n}+\frac{\left(Q(x_n^j)-h\right)}{n^2\pi^2}+\frac{\gamma_0}{n^2\pi^2}\cos(n\pi\xi_0)+o\left(\frac{1}{n^2}\right).\quad\square$$

According to Lemma 3 the existence of a dense subset X_0 of X is obvious.

3.1. The Case $h,\,H\in\mathbb{R}$

Consider the problem $\widetilde{L}=L\left(\widetilde{q},\widetilde{h},\widetilde{H},\widetilde{\gamma}_0,\widetilde{\gamma}_1,\xi_0,\xi_1\right)$ under the same assumptions with L. It is assumed in what follows that if a certain symbol s denotes an object related to the problem L then $\widetilde{s}$ denotes the corresponding object related to the problem $\widetilde{L}$.

The following theorem is the first of our main results in this article.

Theorem 1 *(Uniqueness). If $X_0=\widetilde{X}_0$ then $q(x)=\widetilde{q}(x)$ in $(0,1)$, $h=\widetilde{h}$, $H=\widetilde{H}$, $\gamma_0=\widetilde{\gamma}_0$ and $\gamma_1=\widetilde{\gamma}_1$. Thus, the potential $q(x)$ in $(0,1)$, the coefficients γ_0, γ_1, h and H are uniquely determined by X_0.*

Proof. Step 1. Put $\xi_0=\dfrac{p_0}{r_0}$ and $\xi_1=\dfrac{p_1}{r_1}$, where $p_i,r_i\in\mathbb{Z}$ for $i=0,1$. For each fixed $x\in[0,1]$, there exists a sequence $\left(x_n^j\right)$ converges to x. Clearly the subsequence $\left(x_m^j\right)$ converges also to x for $m=2r_0r_1n$. On the other hand, $\lim\limits_{m\to\infty}A_m=\gamma_1-\gamma_0$. Therefore we can see from Lemma 3 the following limit exists and given equality holds:

$$\lim_{m\to\infty}m^2\pi^2\left(x_m^j-\frac{j}{m}\right)=f(x)=(h-H+\gamma_1-\gamma_0)\,x+Q(x)-h+\gamma_0. \tag{15}$$

Direct calculations in (15) yield

$$\gamma_0 - h = f(0),$$
$$\gamma_1 - H = f(1),$$
$$q(x) = 2\left(f'(x) + f(0) - f(1)\right).$$

Since $X_0 = \widetilde{X}_0$ then $f(x) = \widetilde{f}(x)$ and so $q(x) = \widetilde{q}(x)$, a.e. in $(0,1)$.

Step 2. To show $\widetilde{h} = h$ and $\gamma_0 = \widetilde{\gamma}_0$ consider a sequence $\left\{x_n^j\right\} \subset X_0$ converges to ξ_0 and write the equation (1) for $\varphi(x,\lambda_n)$ and $\widetilde{\varphi}(x,\widetilde{\lambda}_n)$:

$$-\widetilde{\varphi}''\left(x,\widetilde{\lambda}_n\right) + q(x)\widetilde{\varphi}\left(x,\widetilde{\lambda}_n\right) = \widetilde{\lambda}_n\widetilde{\varphi}\left(x,\widetilde{\lambda}_n\right),$$
$$-\varphi''\left(x,\lambda_n\right) + q(x)\varphi\left(x,\lambda_n\right) = \lambda_n\varphi\left(x,\lambda_n\right).$$

If we apply the procedure:

(i): multiplied by $\varphi\left(x,\lambda_n\right)$ and $\widetilde{\varphi}\left(x,\widetilde{\lambda}_n\right)$, respectively; (ii): subtracted from each other and (iii): integrated over the interval $\left(\xi_0, x_n^j\right)$ the equality

$$\varphi'\left(\xi_0,\lambda_n\right)\widetilde{\varphi}\left(\xi_0,\widetilde{\lambda}_n\right) - \widetilde{\varphi}'\left(\xi_0,\widetilde{\lambda}_n\right)\varphi\left(\xi_0,\lambda_n\right) = \left(\widetilde{\lambda}_n - \lambda_n\right)\int_{\xi_0}^{x_n^j}\widetilde{\varphi}\left(x,\widetilde{\lambda}_n\right)\varphi\left(x,\lambda_n\right)dx$$

is obtained. From Lemma 1 the following estimate holds for sufficiently large n

$$\varphi'\left(\xi_0,\lambda_n\right)\widetilde{\varphi}\left(\xi_0,\widetilde{\lambda}_n\right) - \widetilde{\varphi}'\left(\xi_0,\widetilde{\lambda}_n\right)\varphi\left(\xi_0,\lambda_n\right) = o(1), \quad n \to \infty. \tag{16}$$

Using (16) and Lemma 2 we get

$$\left[\varphi'(\xi_0,\lambda_n) - \widetilde{\varphi}'\left(\xi_0,\widetilde{\lambda}_n\right)\right]\cos n\pi\xi_0 = o(1), \quad n \to \infty.$$

The last equality yields

$$\left(\widetilde{h} - h\right)\cos^2 n\pi\xi_0 + \left(\gamma_0 - \widetilde{\gamma}_0\right)\cos n\pi\xi_0 = o(1), \quad n \to \infty.$$

Therefore, we conclude that $\widetilde{h} = h$ and $\gamma_0 = \widetilde{\gamma}_0$.

Step 3. Finally let us prove $\gamma_1 = \widetilde{\gamma}_1$ and $H = \widetilde{H}$. Consider another sequence $\left\{x_n^j\right\} \subset X_0$ converges to ξ_1. If we apply above procedure but take the integral from ξ_1 to x_n^j, we get

$$\widetilde{\varphi}'\left(\xi_1,\widetilde{\lambda}_n\right)\varphi\left(\xi_1,\lambda_n\right) - \varphi'\left(\xi_1,\lambda_n\right)\widetilde{\varphi}\left(\xi_1,\widetilde{\lambda}_n\right) = o(1), \quad n \to \infty$$

instead of (16). From (3), we have

$$\widetilde{\varphi}'\left(\xi_1,\widetilde{\lambda}_n\right)\left[\frac{\varphi'\left(1,\lambda_n\right) + H\varphi\left(1,\lambda_n\right)}{\gamma_1}\right] - \varphi'\left(\xi_1,\lambda_n\right)\left[\frac{\widetilde{\varphi}\left(1,\widetilde{\lambda}_n\right) + \widetilde{H}\widetilde{\varphi}\left(1,\widetilde{\lambda}_n\right)}{\widetilde{\gamma}_1}\right] = o(1), \quad n \to \infty.$$

Using Lemma 1 and Lemma 2, it can be calculated that

$$\sin n\pi\xi_1\left[\left(\frac{H-h}{\gamma_1} - \frac{\widetilde{H}-h}{\widetilde{\gamma}_1}\right)\frac{(-1)^n}{n\pi} + \frac{\gamma_0}{n\pi}\left(\frac{1}{\gamma_1} - \frac{1}{\widetilde{\gamma}_1}\right)\cos n\pi\left(1-\xi_0\right)\right] = o(\frac{1}{n}), \quad n \to \infty.$$

This yields

$$\sin n\pi\xi_1\left[(-1)^n\left(\frac{H-h}{\gamma_1}-\frac{\widetilde{H}-h}{\widetilde{\gamma}_1}\right)+\gamma_0\left(\frac{1}{\gamma_1}-\frac{1}{\widetilde{\gamma}_1}\right)\cos n\pi\left(1-\xi_0\right)\right]=o(1).$$

Hence $\gamma_1=\widetilde{\gamma}_1$ and $H=\widetilde{H}$. This completes the proof. □

Corollary 1 *(Reconstruction algorithm). Let X_0, $\xi_0=\dfrac{p_0}{r_0}$ and $\xi_1=\dfrac{p_1}{r_1}$ be given. Then $q(x)$, γ_0-h and γ_1-H can be reconstructed by the following algorithm:*
i) Denote $m=2r_0r_1n$;
ii) Find $f(x)$ by (15);
iii) Find $q(x)$, γ_0-h and γ_1-H by the formulas

$$\begin{aligned} q(x) &= 2\left(f'(x)+f(0)-f(1)\right), \\ \gamma_0-h &= f(0), \\ \gamma_1-H &= f(1). \end{aligned}$$

Note that if one of the pairs (h,H) and (γ_0,γ_1) is given, we can find the other pair.

Example 1. Consider the nonlocal BVP

$$L:\begin{cases} \ell y:=-y''+q(x)y=\lambda y, \quad x\in(0,1) \\ U(y):=y'(0)+hy(0)=\gamma_0 y(\frac{2}{5}), \\ V(y):=y'(1)+Hy(1)=\gamma_1 y(\frac{6}{7}), \end{cases}$$

where $q(x)\in C^1[0,1]$, γ_0, γ_1, h, and $H\in\mathbb{R}$ are unknown coefficients. Let $X_0=\left\{x_n^j\right\}$ be the given subset of nodal points which satisfy the following asymptotics

$$\begin{aligned} x_n^j &= \frac{(j+1/2)}{n}+\frac{-1+(-1)^n\left[6\cos\left(\frac{6n\pi}{7}\right)-3\cos\left(\frac{3n\pi}{5}\right)\right]}{n^2\pi^2}\frac{(j+1/2)}{n}+\frac{\left(\sin\frac{(j+1/2)\pi}{n}-2\pi\right)}{2n^2\pi^3} \\ &\quad +\frac{3}{n^2\pi^2}\cos\left(2n\frac{\pi}{5}\right)+o\left(\frac{1}{n^2}\right). \end{aligned}$$

Let $m:=70n$. One can calculate that,

$$\lim_{m\to\infty} m^2\pi^2\left(x_m^j-\frac{j}{m}\right)=f(x)=2x+\left(\frac{\sin\pi x-2\pi}{2\pi}\right)+3.$$

According to Theorem 1, we find

$$q(x)=2\left(f'(x)+f(0)-f(1)\right)=\cos\pi x$$

and

$$\begin{aligned} \gamma_0-h &= f(0)=2, \\ \gamma_1-H &= f(1)=4. \end{aligned}$$

If the pair (h,H) is given as, for example, $h=1$ and $H=2$ then we find $\gamma_0=3$ and $\gamma_1=6$.

3.2. The Case $h = \infty$, $H \in \mathbb{R}$

In this subsection, we consider the equation (1) with one Dirichlet boundary condition

$$U(y) := y(0) = 0 \tag{17}$$

and with the nonlocal boundary condition (3).

Let X_0 be a dense nodal points-set. For each fixed x in $(0,1)$, it can be choosen a sequence $\left(x_n^j\right) \subset X_0$ which converges to x. Therefore we can show from Lemma 3 that the following limit exists and is finite for $m = 2r_1 n$:

$$\lim_{m\to\infty} \left(m + \frac{1}{2}\right)^2 \pi^2 \left(x_m^j - \frac{j}{m + \frac{1}{2}}\right) = g(x)$$
$$= (\gamma_1 \sin \frac{\pi}{2}\xi_1 - H)x + \frac{1}{2}\int_0^x q(t)dt.$$

Thus, we can prove the following theorem using methods similar to one in the proof of Theorem 1.

Theorem 2. *If $X_0 = \widetilde{X}_0$ then $q(x) = \widetilde{q}(x)$ a.e. in $(0,1)$, $H = \widetilde{H}$, and $\gamma_1 = \widetilde{\gamma}_1$. Moreover if X_0 and $\xi_1 = \dfrac{p_1}{r_1}$ is given, $q(x)$ and $\gamma_1 \sin \frac{\pi}{2}\xi_1 - H$ can be reconstructed by the following formulas:*

$$q(x) = 2\left(g'(x) - g(1)\right),$$
$$\gamma_1 \sin \frac{\pi}{2}\xi_1 - H = g(1).$$

Example 2. Consider the nonlocal BVP

$$L : \begin{cases} -y'' + q(x)y = \lambda y, \quad x \in (0,1) \\ y(0) = 0, \\ y'(1) + 2y(1) - \gamma_1 y(\frac{2}{5}) = 0, \end{cases}$$

where $q(x) \in C^1[0,1]$ and γ_1 are unknown real coefficients. Let $X_0 = \left\{x_n^j\right\}$ be the given subset of nodal points which satisfy the following asymptotics

$$x_n^j = \frac{j}{n+\frac{1}{2}} - \frac{2 - (-1)^n 3 \sin\left(\frac{2n+1}{5}\pi\right)}{\left(n+\frac{1}{2}\right)^2 \pi^2} \frac{j}{\left(n+\frac{1}{2}\right)} +$$
$$+ \frac{1}{2\left(n+\frac{1}{2}\right)^2 \pi^3}\left(\sin\frac{j\pi}{n+\frac{1}{2}} + \frac{j\pi}{n+\frac{1}{2}}\left(\frac{j}{2n+1} - \frac{1}{2}\right)\right) + o\left(\frac{1}{n^2}\right).$$

To find $q(x)$ and γ_1 we take $m = 10n$ and calculate the following limit

$$\lim_{m\to\infty} \left(m + \frac{1}{2}\right)^2 \pi^2 \left(x_m^j - \frac{j}{\left(m + \frac{1}{2}\right)}\right) = g(x) = \left(3\sin\frac{\pi}{5} - 2\right)x + \frac{\sin \pi x}{2\pi} + \frac{x}{2}\left(\frac{x}{2} - \frac{1}{2}\right).$$

Thus, we find

$$q(x) = 2\left(g'(x) - g(1)\right) = \cos \pi x + x - \frac{1}{2},$$
$$\gamma_1 = \frac{g(1) + 2}{\sin\frac{\pi}{5}} = 3.$$

3.3. The Case $H=\infty$, $h\in\mathbb{R}$

In this subsection, we consider the equation (1) with nonlocal boundary condition (2) and one Dirichlet boundary condition

$$V(y):=y(1)=0. \tag{18}$$

Let $m:=2r_0n$. Here, r_0 denotes the denominators of ξ_0.

Let X_0 be a dense nodal points-set. For each fixed x in $(0,1)$, it can be choosen a sequence $\left(x_n^j\right)\subset X_0$ which converges to x. Therefore we can show from Lemma 3 that

$$\lim_{m\to\infty}\left(m+\frac{1}{2}\right)^2\pi^2\left(x_m^j-\frac{j+\frac{1}{2}}{m+\frac{1}{2}}\right)=\psi(x)$$

$$=(h-\gamma_0\cos\frac{\pi}{2}\xi_0)x-h+\gamma_0\cos\frac{\pi}{2}\xi_0+\frac{1}{2}\int_0^x q(t)dt.$$

Thus, we can give the following theorem.

Theorem 3. *If $X_0=\widetilde{X}_0$ then $q(x)=\widetilde{q}(x)$ a.e. in $(0,1)$, $h=\widetilde{h}$ and $\gamma_0=\widetilde{\gamma}_0$. Moreover if X_0 and $\xi_0=\dfrac{p_0}{r_0}$ are given, $q(x)$ and $\widetilde{\gamma}_0\cos\frac{\pi}{2}\xi_0-h$ can be reconstructed by the following formulae:*

$$q(x)=2\left(\psi'(x)+\psi(0)\right),$$
$$\gamma_0\cos\tfrac{\pi}{2}\xi_0-h=\psi(0).$$

Example 3. Consider the nonlocal BVP

$$L:\begin{cases}\ell y:=-y''+q(x)y=\lambda y, \quad x\in(0,1)\\ U(y):=y'(0)+y(0)=\gamma_0 y(\frac{2}{3}),\\ V(y):=y(1)=0,\end{cases}$$

where $q(x)\in C^1[0,1]$ and γ_0 are unknown coefficients. Let $X_0=\left\{x_n^j\right\}$ be the given subset of nodal points which satisfy the following asymptotics

$$x_n^j=\frac{j+\frac{1}{2}}{n+\frac{1}{2}}+\left[1-2\cos\left(\frac{2n+1}{3}\pi\right)\right]\frac{j+\frac{1}{2}}{\left(n+\frac{1}{2}\right)^3\pi^2}-\frac{1}{\left(n+\frac{1}{2}\right)^2\pi^2}$$
$$+\frac{2\cos\left(\frac{2n+1}{3}\pi\right)}{\left(n+\frac{1}{2}\right)^2\pi^2}-\frac{\cos\frac{(j+\frac{1}{2})\pi}{n+\frac{1}{2}}-1}{2\left(n+\frac{1}{2}\right)^2\pi^3}-\frac{j+\frac{1}{2}}{\left(n+\frac{1}{2}\right)^3\pi^3}+o\left(\frac{1}{n^2}\right).$$

Let $m:=6n$. One can calculate that,

$$\lim_{m\to\infty}\left(m+\frac{1}{2}\right)^2\pi^2\left(x_m^j-\frac{j}{m}\right)=\psi(x)=\left(1-2\cos\frac{\pi}{3}\right)x-1+2\cos\frac{\pi}{3}-\frac{\cos\pi x-1}{2\pi}-\frac{x}{\pi}.$$

According to Theorem 3, we find

$$q(x)=2\left(\psi'(x)+\psi(0)\right)=\sin\pi x-\frac{2}{\pi},$$
$$\gamma_0=2\psi(0)+2=2.$$

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