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Inverse nodal problem for Dirac operator with integral type nonlocal boundary conditions

A. Sinan Ozkan[1] | **İbrahim Adalar**[2]

[1]Department of Mathematics, Faculty of Science, Sivas Cumhuriyet University, Sivas, Turkey

[2]Zara Veysel Dursun Colleges of Applied Sciences, Sivas Cumhuriyet University, Sivas, Turkey

Correspondence
A. Sinan Ozkan, Department of Mathematics, Faculty of Science, Sivas Cumhuriyet University 58140 Sivas, Sivas, Turkey.
Email: sozkan@cumhuriyet.edu.tr

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In this paper, Dirac operator with some integral type nonlocal boundary conditions is studied. We show that the coefficients of the problem can be uniquely determined by a dense set of nodal points. Moreover, we give an algorithm for the reconstruction of some coefficients of the operator.

KEYWORDS

Dirac operator, inverse nodal problem, nonlocal boundary condition

MSC CLASSIFICATION

34A55, 34L05, 34K29, 34K10

1 | INTRODUCTION

The inverse nodal problem was posed and solved firstly by McLaughlin for a Sturm–Liouville operator.[1] McLaughlin showed that the potential of a Sturm–Liouville problem can be determined by a given dense subset of nodal points. Later on, Hald and McLaughlin give some numerical schemes for the reconstruction of the potential.[2] X.F. Yang gave an algorithm for the solution of the inverse nodal Sturm–Liouville problem.[3] His method is the first of the algorithms still in use today. Inverse nodal problems for various Sturm–Liouville operators have been studied in the papers ([4–23]).

Nonlocal boundary conditions appear when we cannot measure data directly atthe boundar y. These kinds of conditions arise in various some applied problems of biology, biotechnology, physics, and so on. Two types of nonlocal boundary conditions come to the fore. One class of them is called integral type conditions, and the other is the Bitsadze–Samarskii-type conditions. Some inverse problems for a class of Sturm–Liouville operators with nonlocal boundary conditions are investigated in earlier studies.[24,25] In particular, inverse nodal problems for this-type operators with different nonlocal integral boundary conditions are studied in previous research[26–29]. The inverse nodal problems for Dirac operators with local and separated boundary conditions are studied in earlier studies[30–36]. In their works, it is shown that the zeros of the first components of the eigenfunctions determine the coefficients of operator. The uniqueness of the recovery of the Dirac operator with nonseparated boundary conditions is investigated in earlier works[37–39]. In Yang and Yurko,[40] nonlocal conditions together with the Dirac system are considered. They give some uniqueness theorems according to the classical spectral data. As far as we know, inverse nodal problem for the Dirac system with nonlocal boundary conditions has not been considered before. In the present paper, we consider Dirac operator under some integral type nonlocal boundary conditions and obtain the uniqueness of coefficients of the problem according to a set of nodal points. Moreover, we give an algorithm for the reconstruction of these coefficients. We consider the boundary value problem L generated by the system of Dirac differential equations system:

$$\ell\,[Y(x)] := BY'(x) + Q(x)Y(x) = \lambda Y(x), x \in (0, \pi), \tag{1}$$

subject to the boundary conditions

$$U(y) := y_1(0)\sin\alpha + y_2(0)\cos\alpha - I_f(Y) = 0, \tag{2}$$

$$V(y) := y_1(\pi)\sin\beta + y_2(\pi)\cos\beta - I_g(Y) = 0, \tag{3}$$

where $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $Q(x) = \begin{pmatrix} V(x)+m & 0 \\ 0 & V(x)-m \end{pmatrix}$, $Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $I_f(Y) = \int_0^\pi (f_1(x)y_1(x) + f_2(x)y_2(x))\,dx$, $I_g(Y) = \int_0^\pi (g_1(x)y_1(x) + g_2(x)y_2(x))\,dx$, α, β, and m are real constants, and λ is the spectral parameter. We assume that $V(x)$, $f_i(x)$ and $g_i(x)$ are real-valued functions in the class of $W_2^1(0,\pi)$ for $i = 1,2$ and put $F_0 := f_1(0)\sin\alpha + f_2(0)\cos\alpha$, $F_\pi := f_1(\pi)\sin\beta + f_2(\pi)\cos\beta$, $G_0 := g_1(0)\sin\alpha + g_2(0)\cos\alpha$ and $G_\pi := g_1(\pi)\sin\beta + g_2(\pi)\cos\beta$.

2 | SPECTRAL PROPERTIES OF THE PROBLEM

Let $S(x,\lambda) = \begin{pmatrix} S_1(x,\lambda) \\ S_2(x,\lambda) \end{pmatrix}$ and $C(x,\lambda) = \begin{pmatrix} C_1(x,\lambda) \\ C_2(x,\lambda) \end{pmatrix}$ be the solutions of (1) under the initial conditions

$$S_1(0,\lambda) = 0,\ S_2(0,\lambda) = -1,$$
$$C_1(0,\lambda) = 1,\ C_2(0,\lambda) = 0,$$

respectively. $C(x,\lambda)$ and $S(x,\lambda)$ are entire functions of λ for each fixed x and the following asymptotic relations hold for $|\lambda| \to \infty$ (see Keskin and Ozkan[30]).

$$C_1(x,\lambda) = \cos(\lambda x - \omega(x)) + \frac{m^2 x}{2\lambda}\sin(\lambda x - \omega(x)) + o\left(\frac{e^{|\tau|x}}{\lambda}\right), \tag{4}$$

$$C_2(x,\lambda) = \sin(\lambda x - \omega(x)) - \frac{m}{\lambda}\sin(\lambda x - \omega(x)) - \frac{m^2 x}{2\lambda}\cos(\lambda x - \omega(x)) + o\left(\frac{e^{|\tau|x}}{\lambda}\right), \tag{5}$$

$$S_1(x,\lambda) = \sin(\lambda x - \omega(x)) + \frac{m}{\lambda}\sin(\lambda x - \omega(x)) - \frac{m^2 x}{2\lambda}\cos(\lambda x - \omega(x)) + o\left(\frac{e^{|\tau|x}}{\lambda}\right), \tag{6}$$

$$S_2(x,\lambda) = -\cos(\lambda x - \omega(x)) - \frac{m^2 x}{2\lambda}\sin(\lambda x - \omega(x)) + o\left(\frac{e^{|\tau|x}}{\lambda}\right), \tag{7}$$

where $\omega(x) = \int_0^x V(t)dt$ and $\tau = \operatorname{Im}\lambda$. The characteristic function of problem (1)–(3) is

$$\Delta(\lambda) = \det\begin{pmatrix} U(C) & U(S) \\ V(C) & V(S) \end{pmatrix} \tag{8}$$

and the zeros of $\Delta(\lambda)$ coincide with the eigenvalues of problems (1)–(3). Clearly, $\Delta(\lambda)$ is entire function and so the problem has a discrete spectrum. Using the asymptotic formulas (4)–(7) in (8), one can easily obtain

$$\begin{aligned}
\Delta(\lambda) = {} & \sin(\lambda\pi - \omega(\pi) + \beta - \alpha) - \frac{m^2\pi}{2\lambda}\cos(\lambda\pi - \omega(\pi) + \beta - \alpha) \\
& - \frac{1}{\lambda}(f_1(\pi)\sin\beta + f_2(\pi)\cos\beta - f_2(0)\cos(\lambda\pi - \omega(\pi) + \beta)) \\
& - \frac{1}{\lambda}(g_1(\pi)\sin(\lambda\pi - \omega(\pi) - \alpha) - g_2(\pi)\cos(\lambda\pi - \omega(\pi) - \alpha)) \\
& + \frac{1}{\lambda}(f_1(0)\sin(\lambda\pi - \omega(\pi) + \beta) - g_1(0)\sin\alpha - g_2(0)\cos\alpha) \\
& - \frac{m}{\lambda}\sin(\lambda\pi - \omega(\pi) + \beta - \alpha)\cos(\beta + \alpha)\cos(\beta - \alpha) \\
& + \frac{m}{\lambda}\cos(\lambda\pi - \omega(\pi) + \beta - \alpha)\cos(\beta + \alpha)\sin(\beta - \alpha) + o\left(\frac{e^{|\tau|\pi}}{\lambda}\right)
\end{aligned}$$

for sufficiently large $|\lambda|$. Let $\{\lambda_n : n = 0, \pm 1, \pm 2, \ldots\}$ be the set of eigenvalues. Since $\Delta(\lambda) = \sin(\lambda\pi - \omega(\pi) + \beta - \alpha) + O\left(\frac{e^{|\tau|x}}{\lambda}\right)$, the numbers λ_n are real for $|n| \to \infty$ and if Rouché 's theorem is used, the eigenvalues satisfy the following asymptotic formula

$$\lambda_n = n + \frac{\omega(\pi) - \beta + \alpha}{\pi} + O\left(\frac{1}{n}\right), |n| \to \infty.$$

Moreover, we can write the following equation.

$$\left(1 + O\left(\frac{1}{\lambda_n}\right)\right) \tan(\lambda_n\pi - \omega(\pi) + \beta - \alpha) = \frac{1}{\lambda_n}\left[\frac{m^2\pi}{2} - F_0 - G_\pi + m\cos(\beta + \alpha)\sin(\alpha - \beta)\right] + \frac{F_\pi + G_0}{\lambda_n \cos(\lambda\pi - \omega(\pi) + \beta - \alpha)} + o\left(\frac{1}{\lambda_n}\right).$$

Since $\left(1 + O(\frac{1}{\lambda_n})\right)^{-1} = 1 + O(\frac{1}{\lambda_n})$, this implies that

$$\tan(\lambda_n\pi - \omega(\pi) + \beta - \alpha) = \frac{A_1}{\lambda_n} + \frac{A_2}{\lambda_n \cos(\lambda\pi - \omega(\pi) + \beta - \alpha)} + o\left(\frac{1}{\lambda_n}\right) \tag{9}$$

for sufficiently large $|n|$, where

$$A_1 = \frac{m^2\pi}{2} + m\cos(\beta + \alpha)\sin(\alpha - \beta) - (F_0 + G_\pi),$$

$$A_2 = F_\pi + G_0.$$

From (9), we can see that

$$\lambda_n\pi - \omega(\pi) + \beta - \alpha = n\pi + O\left(\frac{1}{n}\right), \quad |n| \to \infty.$$

Hence,

$$\frac{1}{\cos(\lambda_n\pi - \omega(\pi) + \beta - \alpha)} = \frac{1}{\cos\left(n\pi + O\left(\frac{1}{n}\right)\right)} = (-1)^n\left(1 + o\left(\frac{1}{n}\right)\right), \quad |n| \to \infty. \tag{10}$$

Using (9) and (10) together, we obtain the following lemma.

Lemma 1. *The following asymptotic relation is valid for* $|n| \to \infty$.

$$\lambda_n = n + \frac{1}{\pi}\int_0^\pi V(t)dt + \frac{\alpha - \beta}{\pi} + \frac{A_1}{n\pi} + \frac{(-1)^n A_2}{n\pi} + o\left(\frac{1}{n}\right). \tag{11}$$

3 | NODAL POINTS

Let $\varphi(x, \lambda_n) = \begin{pmatrix} \varphi_1(x, \lambda_n) \\ \varphi_2(x, \lambda_n) \end{pmatrix}$ be the eigenfunction of (1)–(3) corresponding to the eigenvalue λ_n. It is clear that

$$\varphi_i(x, \lambda_n) = U(S(x, \lambda_n))C_i(x, \lambda_n) - U(C(x, \lambda_n))S_i(x, \lambda_n), \; i = 1, 2. \tag{12}$$

From (12) and Lemma 1, we can give the following asymptotic formula for sufficiently large $|n|$

$$\begin{aligned}
\varphi_1(x,\lambda_n) = & -\cos(\lambda_n x-\omega(x)-\alpha)-\frac{m^2 x}{2\lambda_n}\sin(\lambda_n x-\omega(x)-\alpha) \\
& -\frac{m\sin 2\alpha}{2\lambda_n}\sin(\lambda_n x-\omega(x)-\alpha)-\frac{m\sin^2\alpha}{\lambda_n}\cos(\lambda_n x-\omega(x)-\alpha) \\
& +\frac{(-1)^n f_1(\pi)}{\lambda_n}(\cos\beta\cos(\lambda_n x-\omega(x)-\alpha)-\sin\beta\sin(\lambda_n x-\omega(x)-\alpha)) \\
& -\frac{(-1)^n f_2(\pi)}{\lambda_n}(\cos\beta\sin(\lambda_n x-\omega(x)-\alpha)+\sin\beta\cos(\lambda_n x-\omega(x)-\alpha)) \\
& +\frac{f_1(0)}{\lambda_n}(\sin\alpha\sin(\lambda_n x-\omega(x)-\alpha)-\cos\alpha\cos(\lambda_n x-\omega(x)-\alpha)) \\
& +\frac{f_2(0)}{\lambda_n}(\cos\alpha\sin(\lambda_n x-\omega(x)-\alpha)+\sin\alpha\cos(\lambda_n x-\omega(x)-\alpha)) \\
& +o\left(\frac{e^{|\tau|x}}{\lambda}\right).
\end{aligned}$$

Lemma 2. *For sufficiently large positive n, $\varphi_1(x,\lambda_n)$ has exactly n zeros, namely, nodal points, $\left\{x_n^j : j=\overline{0,n-1}\right\}$ in the interval $(0,\pi)$:*

The numbers $\left\{x_n^j\right\}$ satisfy the following asymptotic formula:

$$\begin{aligned}
x_n^j = & \frac{(j+1/2)\pi}{n}+\frac{\omega(x_n^j)+\alpha}{n}-\frac{(j+1/2)}{n}\left(\frac{\omega(\pi)+\alpha-\beta}{n}\right) \\
& -\frac{\omega(\pi)+\alpha-\beta}{n^2\pi}\left(\omega(x_n^j)+\alpha\right)+\frac{1}{n^2}\left(\frac{m^2 x_n^j}{2}+\frac{m\sin 2\alpha}{2}+(-1)^n F_\pi-F_0\right) \\
& -\frac{(j+1/2)}{n}\left(\frac{A_1+(-1)^n A_2}{n^2}\right)+o\left(\frac{1}{n^2}\right)
\end{aligned} \tag{13}$$

for $n\to\infty$.

Proof. Let $\kappa_{n,j} := \lambda_n x_n^j-\omega(x_n^j)-\alpha$. From $\varphi_1(x_n^j,\lambda_n)=0$, we can write for $n\to\infty$

$$\begin{aligned}
0 = & -\cos\kappa_{n,j}-\frac{m\sin 2\alpha}{2\lambda_n}\sin\kappa_{n,j}-\frac{m\sin^2\alpha}{\lambda_n}\cos\kappa_{n,j}-\frac{m^2 x_n^j}{2\lambda_n}\sin\kappa_{n,j}+ \\
& +\frac{(-1)^n f_1(\pi)}{\lambda_n}\left(\cos\kappa_{n,j}\cos\beta-\sin\kappa_{n,j}\sin\beta\right)+ \\
& -\frac{(-1)^n f_2(\pi)}{\lambda_n}\left(\sin\kappa_{n,j}\cos\beta+\cos\kappa_{n,j}\sin\beta\right)+ \\
& +\frac{f_1(0)}{\lambda_n}\left(\sin\kappa_{n,j}\sin\alpha-\cos\kappa_{n,j}\cos\alpha\right)+ \\
& +\frac{f_2(0)}{\lambda_n}\left(\sin\kappa_{n,j}\cos\alpha+\cos\kappa_{n,j}\sin\alpha\right)+o\left(\frac{e^{|\tau|\pi}}{\lambda_n}\right).
\end{aligned}$$

It can be obtained that

$$\begin{aligned}
\tan\left(\kappa_{n,j}-\frac{\pi}{2}\right) & = \frac{\frac{m^2 x_n^j}{2\lambda_n}+\frac{m\sin 2\alpha}{2\lambda_n}+\frac{(-1)^n F_\pi-F_0}{\lambda_n}+o\left(\frac{1}{\lambda_n}\right)}{1+O(\frac{1}{\lambda_n})} \\
& = \frac{m^2 x_n^j}{2\lambda_n}+\frac{m\sin 2\alpha}{2\lambda_n}+\frac{(-1)^n F_\pi-F_0}{\lambda_n}+o\left(\frac{1}{\lambda_n}\right).
\end{aligned}$$

Taking into account Taylor's expansion formulas of the arctangent, we get

$$\kappa_{n,j} = \left(j + \frac{1}{2}\right)\pi + \frac{1}{\lambda_n}\left(\frac{m^2 x_n^j}{2} + \frac{m \sin 2\alpha}{2} + (-1)^n F_\pi - F_0\right) + o\left(\frac{1}{\lambda_n}\right).$$

It follows from the last equality that

$$x_n^j = \frac{\left(j + \frac{1}{2}\right)\pi + \omega(x_n^j) + \alpha}{\lambda_n} + \frac{1}{\lambda_n^2}\left(\frac{m^2 x_n^j}{2} + \frac{m \sin 2\alpha}{2} + (-1)^n F_\pi - F_0\right) + o\left(\frac{1}{\lambda_n^2}\right).$$

Finally, using the asymptotic formula

$$\lambda_n^{-1} = \frac{1}{n}\left\{1 - \frac{\omega(\pi) + \alpha - \beta}{n\pi} - \frac{A_1}{n^2\pi} - \frac{(-1)^n A_2}{n^2\pi} + o\left(\frac{1}{n^3}\right)\right\},$$

we can obtain our desired formula: (13). □

4 | INVERSE NODAL PROBLEM

Let X be the set of nodal points and $\omega(\pi) = 0$. For each fixed $x \in (0, \pi)$, we can choose a sequence $\left(x_n^{j(n)}\right) \subset X$ so that $x_n^{j(n)}$ converges to x. Then the following limits are exist and finite:

$$\lim_{n\to\infty} n\left(x_n^{j(n)} - \frac{\left(j(n) + \frac{1}{2}\right)\pi}{n}\right) = \psi_1(x), \tag{14}$$

where

$$\psi_1(x) = \omega(x) + \frac{(\beta - \alpha)x}{\pi} + \alpha$$

and

$$\lim_{n\to\infty} n^2\pi\left(x_n^{j(n)} - \frac{\left(j(n) + \frac{1}{2}\right)\pi + \left(\omega(x_n^{j(n)}) + \alpha\right)}{n} + \frac{(j(n) + 1/2)}{n}\left(\frac{\alpha - \beta}{n}\right)\right) = \psi_2(x), \tag{15}$$

where

$$\psi_2(x) = \begin{cases} \psi_2^+(x), & n \text{ is even} \\ \psi_2^-(x), & n \text{ is odd} \end{cases}$$

and

$$\psi_2^+(x) = (\beta - \alpha)(\omega(x) + \alpha) - x(A_1 + A_2) + \pi\left(\frac{m^2 x}{2} + \frac{m \sin 2\alpha}{2} + F_\pi - F_0\right),$$

$$\psi_2^-(x) = (\beta - \alpha)(\omega(x) + \alpha) - x(A_1 - A_2) + \pi\left(\frac{m^2 x}{2} + \frac{m \sin 2\alpha}{2} - F_\pi - F_0\right).$$

Therefore, proof of the following theorem is clear.

Theorem 1. *The given dense subset of nodal points X uniquely determines the potential $V(x)$ a.e. on $(0,\pi)$ and the coefficients α, β, F_π, and G_0 in the boundary conditions; m, $V(x)$, α, β, F_π, and G_0 can be reconstructed by the following formulae:*

Step-1: *For each fixed $x \in (0,\pi)$, choose a sequence $\left(x_n^{j(n)}\right) \subset X$ such that $\lim\limits_{n\to\infty} x_n^{j(n)} = x$;*

Step-2: *Find the function $\psi_1(x)$ and $\psi_2(x)$ from (14) and (15) and calculate*

$$\alpha = \psi_1(0)$$
$$\beta = \psi_1(\pi)$$
$$V(x) = \psi_1'(x) + \frac{\alpha - \beta}{\pi}$$
$$F_\pi = \frac{\psi_2^+(0) - \psi_2^-(0)}{2\pi}$$
$$G_0 = \frac{\psi_2^-(\pi) - \psi_2^+(\pi)}{2\pi}.$$

Additionally, if m is known, then F_0 and G_π can be found by the following formulas

$$F_0 = -\frac{\psi_2^+(0) + \psi_2^-(0)}{2} + \alpha(\beta - \alpha) + m\pi \frac{\sin 2\alpha}{2},$$
$$G_\pi = \frac{\psi_2^-(\pi) + \psi_2^+(\pi)}{2\pi} - \frac{\alpha(\beta - \alpha)}{\pi} - m\frac{\sin 2\beta}{2}.$$

Example 1. Let $\{x_n^j\} \subset X$ be a subset of nodal points of problems (1)–(3), and the sequence $\{x_n^j\}$ satisfies the following asymptotics:

$$x_n^j = \frac{(j+1/2)\pi}{n} + \frac{\sin\frac{(j+1/2)\pi}{n} + \frac{\pi}{6}}{n} + \frac{(j+1/2)\pi}{6n^2} + \frac{1}{6n^2}\left(\sin\frac{(j+1/2)\pi}{n} + \frac{\pi}{6}\right) + \frac{1}{n^2}\left(\frac{3}{2}\left(\frac{(j+1/2)\pi}{n} + \frac{1}{2}\right) + (-1)^n 2\pi\right) - \frac{(j+1/2)}{n}\left(\frac{\left(2-\sqrt{3}\right)\pi + (-1)^n 4\pi}{2n^2}\right) + o\left(\frac{1}{n^2}\right).$$

It can be calculated from (14) and (15) that

$$\psi_1(x) = \sin x + \frac{1}{6}(x+\pi),$$
$$\psi_2^+(x) = \frac{\pi}{6}\left(\sin x + \frac{\pi}{6}\right) - \frac{\pi}{2}\left(6-\sqrt{3}\right)x + \pi\left(\frac{3}{2}\left(x+\frac{1}{2}\right) + 2\pi\right),$$
$$\psi_2^-(x) = \frac{\pi}{6}\left(\sin x + \frac{\pi}{6}\right) + \frac{\pi}{2}\left(2+\sqrt{3}\right)x + \pi\left(\frac{3}{2}\left(x+\frac{1}{2}\right) - 2\pi\right).$$

Therefore, it is obtained by using the algorithm in Theorem 1,

$$\alpha = \frac{\pi}{6},$$
$$\beta = \frac{\pi}{3},$$
$$V(x) = \cos x,$$
$$F_\pi = 2\pi,$$
$$G_0 = 0.$$

If $m = \sqrt{3}$, then

$$F_0 = 0,$$
$$G_\pi = \frac{\left(\sqrt{3}+1\right)\pi}{2}.$$

5 | CONCLUSION AND RECOMMENDATION

Problem (1)–(3) have countably many eigenvalues that satisfy the asymptotic formula (11). For the sequence of nodal points of the problem, the asymptotic formula (13) holds. If $\omega(\pi) = 0$, a dense set of nodal points uniquely determines the coefficients $V(x)$, α, β and the terms F_π and G_0. Also, if a nodal points-sequence that satisfies the asymptotic expression (13) is given, all of these coefficients and terms can be reconstructed by using the steps in Theorem 1. Our main result includes the full algorithm for the solution of the inverse nodal problem, therefore it would be useful for some problems in the applied sciences. After this, inverse nodal problems for a Dirac system with some different conditions (e.g., parameter-dependent nonlocal boundary conditions or transmission conditions) can be investigated.

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CONFLICT OF INTEREST

This work does not have any conflict of interest.

ORCID

A. Sinan Ozkan https://orcid.org/0000-0002-9703-8982
İbrahim Adalar https://orcid.org/0000-0002-4224-0972

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