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Research Article

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A partial inverse problem for non-self-adjoint Sturm–Liouville operators with a constant delay

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Abstract: In this paper we study a partial inverse spectral problem for non-self-adjoint Sturm–Liouville operators with a constant delay and show that subspectra of two boundary value problems with one common boundary condition are sufficient to determine the complex potential. We developed the Horváth's method in [M. Horváth, On the inverse spectral theory of Schrödinger and Dirac operators, *Trans. Amer. Math. Soc.* **353** (2001), no. 10, 4155–4171] for the self-adjoint Sturm–Liouville operator without delay into the non-self-adjoint Sturm–Liouville differential operator with a constant delay.

Keywords: Inverse problem, non-self-adjoint Sturm–Liouville operators, constant delay, potential, eigenvalue

MSC 2020: 34A55, 34K29, 45J05

1 Introduction

In this paper we are concerned with a partial inverse spectral problem of non-self-adjoint Sturm–Liouville operators with a constant delay $L_\xi(q, a)$ which consists in

$$lu := -u''(x) + q(x)u(x - a) = \lambda u(x), \quad 0 < x < \pi, \tag{1.1}$$

$$u(0) = u^{(\xi)}(\pi) = 0, \quad \xi = 0, 1, \tag{1.2}$$

where λ is the spectral parameter, $q(x)$ is a complex function, $q \in L^2[a, \pi]$, $q(x) \equiv 0$, $x \in [0, a]$, and a is a constant in $(0, \pi)$.

It is well known that the differential equation with a constant delay has many applications in mathematics and naturel sciences (see [10]). We note that the area of inverse spectral problems for Sturm–Liouville differential operators with a constant delay remains incredibly active up to this day. Since we cannot possibly describe the recent developments in details in this paper, we refer, for instance, to [1, 2, 4, 5, 7, 10, 16, 19, 21, 24], and the references therein, which lead the interested reader into a variety of directions. Problem (1.1)–(1.2) for the case $a \in [\frac{\pi}{2}, \pi)$ is called the Sturm–Liouville differential operator with a large constant delay. Pikula [16] studied the inverse problem for this operator and reconstructed the potential $q(x)$ by two spectra $\sigma(L_0)$ and $\sigma(L_1)$ (see below), which was improved by Buterin and Yurko [5] by parts of two spectra. They presented necessary and sufficient conditions for the uniqueness theorem on the potential and a constructive procedure. For the case $a \in [0, \frac{\pi}{2})$, then problem (1.1)–(1.2) is non-linear, it is more difficult for us to study the inverse problem for problem (1.1)–(1.2). One had to divide into three cases $a \in [\frac{\pi}{2}, \pi)$, or $a \in [\frac{2\pi}{5}, \frac{\pi}{2})$, or $a \in (0, \frac{2\pi}{5})$ to deal with the inverse

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