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EIGENVALUES OF THE DIRAC OPERATOR WITH NONLOCAL BOUNDARY CONDITIONS ON TIME SCALES

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Abstract. In this paper, a Dirac system under some integral type nonlocal boundary conditions on a disctere set is considered. A formulation for the number of eigenvalues of the problem is obtained and a method for calculation of eigenvalues of the problem according to known coefficients is given.

1. Introduction

Nonlocal boundary conditions appear when data can not be measured directly at the boundary. These kinds of conditions arise in various applied problems of biology, biotechnology, physics and etc. Two types of nonlocal boundary conditions come to the fore. One class is called integral type conditions, and the other is the Bitsadze-Samarskii-type conditions.

Bitsadze and Samarskii are considered the originators of such conditions. Nonlocal boundary conditions of the Bitsadze-Samarskii type were first applied to elliptic equations by them [**6**]. Some important results on the properties of eigenvalues and eigenfunctions of nonlocal boundary value problems for Sturm-Liouville type operators have been in various publications (see, for example, [**2**], [**14**], [**17**], and the references therein).

Some spectral problems for the Dirac operator with nonlocal boundary conditions on a continuous interval are studied in [**7**], [**11**], [**12**], [**13**], [**18**], [**19**], [**21**], and the references therein. Moreover, in [**1**], [**8**], and [**9**], the Dirac operator with non-separated boundary conditions is investigated. In [**20**], nonlocal conditions together with the Dirac system are considered. Dirac operator with some integral

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type nonlocal boundary conditions is studied in [**16**]. We can also refer to [**3**], [**4**], [**5**], [**10**], and [**15**], for studies that include the discrete Dirac system.

We consider the Dirac operator under some integral type nonlocal boundary conditions on a discrete set and obtain the formulation of the eigenvalues-number of the problem. Moreover, we give a method to calculate the eigenvalues of the problem when the coefficients are given.

Let us consider the following boundary value problem on the set
$\mathbb{T} = \{0, 1, 2, ..., n, n+1\}$

$$L[Y(x)] := B\Delta(Y) + Q(x)Y = \lambda Y, \quad x \in \mathbb{T}\backslash\{n+1\} \tag{1.1}$$

$$U(y) := y_1(0)\sin\alpha + y_2(0)\cos\alpha - N_f(Y) = 0, \tag{1.2}$$

$$V(y) := y_1(n+1)\sin\beta + y_2(n+1)\cos\beta - N_g(Y) = 0 \tag{1.3}$$

where $Q(x) = \begin{pmatrix} p(x) & 0 \\ 0 & q(x) \end{pmatrix}$ is a matrix-function defined on $\mathbb{T}$, $Y = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$ is an unknown vector-valued function,

$$\Delta(Y) = \begin{pmatrix} y_1(x+1) - y_1(x) \\ y_2(x+1) - y_2(x) \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and α and β are real numbers, λ is the spectral parameter,

$$N_f(Y) = \sum_{k=0}^{k=n} [f_1(k)y_1(k) + f_2(k)y_2(k)], \tag{1.4}$$

$$N_g(Y) = \sum_{k=0}^{k=n} [g_1(k)y_1(k) + g_2(k)y_2(k)]. \tag{1.5}$$

2. The number of eigenvalues

THEOREM 2.1. *If* $[f_1(n)\sin\beta + f_2(n)\cos\beta] \neq 0$ *, the problem (1.1)-(1.3) has exactly* $(2n+1)$ *eigenvalues*

PROOF. Let $S(x,\lambda) = \begin{pmatrix} S_1(x,\lambda) \\ S_2(x,\lambda) \end{pmatrix}$ and $C(x,\lambda) = \begin{pmatrix} C_1(x,\lambda) \\ C_2(x,\lambda) \end{pmatrix}$ be the solutions of (1.1) under the initial conditions

$$S_1(0,\lambda) = 0,\ S_2(0,\lambda) = -1, \tag{2.1}$$

$$C_1(0,\lambda) = 1,\ C_2(0,\lambda) = 0, \tag{2.2}$$

respectively. It is obvious that $S(x,\lambda)$ and $C(x,\lambda)$ satisfy the following relations.

$$\begin{cases} C_1(x+1) = C_1(x) + (q(x)-\lambda)C_2(x), \\ C_2(x+1) = C_2(x) + (\lambda - p(x))C_1(x), \\ S_1(x+1) = S_1(x) + (q(x)-\lambda)S_2(x), \\ S_2(x+1) = S_2(x) + (\lambda - p(x))S_1(x). \end{cases} \tag{2.3}$$

From (2.1), (2.2) and (2.3) it can be obtained that

$$C_1(k,\lambda) = \begin{cases} (-1)^{\frac{k-1}{2}} k\lambda^{k-1} + \left[\lambda^{k-2}\right], & k \text{ is odd}, \\ (-1)^{\frac{k}{2}} \lambda^{k} + \left[\lambda^{k-1}\right], & k \text{ is even}, \end{cases} \tag{2.4}$$

$$C_2(k,\lambda) = \begin{cases} (-1)^{\frac{k-1}{2}} \lambda^{k} + \left[\lambda^{k-1}\right], & k \text{ is odd}, \\ (-1)^{\frac{k+2}{2}} k\lambda^{k-1} + \left[\lambda^{k-2}\right], & k \text{ is even}, \end{cases} \tag{2.5}$$

$$S_1(k,\lambda) = \begin{cases} (-1)^{\frac{k-1}{2}} \lambda^{k} + \left[\lambda^{k-1}\right], & k \text{ is odd}, \\ (-1)^{\frac{k+2}{2}} k\lambda^{k-1} + \left[\lambda^{k-2}\right], & k \text{ is even}, \end{cases} \tag{2.6}$$

$$S_2(k,\lambda) = \begin{cases} (-1)^{\frac{k+1}{2}} k\lambda^{k-1} + \left[\lambda^{k-2}\right], & k \text{ is odd}, \\ (-1)^{\frac{k+2}{2}} \lambda^{k} + \left[\lambda^{k-1}\right], & k \text{ is even}, \end{cases} \tag{2.7}$$

for $k \in \{2,3,...,n+1\}$, where the term $\left[\lambda^{j}\right]$ denotes a polynomial whose degree is j.
On the other hand, the characteristic function of the problem (1.1)-(1.3) is

$$D(\lambda) = \det \begin{pmatrix} U(C) & U(S) \\ V(C) & V(S) \end{pmatrix}. \tag{2.8}$$

Using (2.4)-(2.7), the following equalities can be obtained.

$$\begin{aligned} U(C) &= C_1(0,\lambda)\sin\alpha + C_2(0,\lambda)\cos\alpha - N_f(C) \\ &= \sin\alpha - N_f(C) \\ &= \sin\alpha - \sum_{k=0}^{k=n} \left[f_1(k)C_1(k,\lambda) + f_2(k)C_2(k,\lambda)\right] \\ &= \begin{cases} (-1)^{\frac{n+1}{2}} f_2(n)\lambda^{n} + \left[\lambda^{n-1}\right], \text{ n is odd}, \\ (-1)^{\frac{n+2}{2}} f_1(n)\lambda^{n} + \left[\lambda^{n-1}\right], \text{ n is even}, \end{cases} \end{aligned}$$

$$\begin{aligned} U(S) &= S_1(0,\lambda)\sin\alpha + S_2(0,\lambda)\cos\alpha - N_f(S) \\ &= -\cos\alpha - N_f(S) \\ &= -\cos\alpha - \sum_{k=0}^{k=n}\left[f_1(k)S_1(k,\lambda) + f_2(k)S_2(k,\lambda)\right] \\ &= \begin{cases} (-1)^{\frac{n+1}{2}} f_1(n)\lambda^n + \left[\lambda^{n-1}\right], \text{ n is odd,} \\ (-1)^{\frac{n}{2}} f_2(n)\lambda^n + \left[\lambda^{n-1}\right], \text{ n is even,} \end{cases} \end{aligned}$$

$$\begin{aligned} V(C) &= C_1(n+1,\lambda)\sin\beta + C_2(n+1,\lambda)\cos\beta - N_g(C) \\ &= C_1(n+1,\lambda)\sin\beta + C_2(n+1,\lambda)\cos\beta - \sum_{k=0}^{k=n}\left[g_1(k)C_1(k,\lambda) + g_2(k)C_2(k,\lambda)\right] \\ &= \begin{cases} (-1)^{\frac{n+1}{2}} \lambda^{n+1}\sin\beta + \left[\lambda^{n}\right] \text{ , n is odd,} \\ (-1)^{\frac{n}{2}} \lambda^{n+1}\cos\beta + \left[\lambda^{n}\right] \text{ , n is even,} \end{cases} \end{aligned}$$

and

$$\begin{aligned} V(S) &= S_1(n+1,\lambda)\sin\beta + S_2(n+1,\lambda)\cos\beta - N_g(S) \\ &= S_1(n+1,\lambda)\sin\beta + S_2(n+1,\lambda)\cos\beta - \sum_{k=0}^{k=n}\left[g_1(k)S_1(k,\lambda) + g_2(k)S_2(k,\lambda)\right] \\ &= \begin{cases} (-1)^{\frac{n+3}{2}} \lambda^{n+1}\cos\beta + \left[\lambda^{n}\right] \text{ , n is odd,} \\ (-1)^{\frac{n}{2}} \lambda^{n+1}\sin\beta + \left[\lambda^{n}\right] \text{ , n is even.} \end{cases} \end{aligned}$$

Hence, it is clear that

$$D(\lambda) = -\left[f_1(n)\sin\beta + f_2(n)\cos\beta\right]\lambda^{2n+1} + \left[\lambda^{2n}\right].$$

Since the eigenvalues of the problem are the zeros of $D(\lambda)$ which has degree of $2n+1$, the proof is obvious. □

COROLLARY 2.1. *The number of eigenvalues of the problem depends only on the number of elements of* $\mathbb{T}$, *and coefficients* f_1, f_2 *and* β *(it depends on neither* $Q(x)$ *nor the other coefficients.)*

3. A calculation of eigenvalues

If we write (1.1) for each element in $\mathbb{T}\backslash\{n+1\}$, and use the conditions (1.2) and (1.3), we get the following system of linear algebraic equations.

$$AV = 0$$

where $V = (y_1(0) \;\; y_1(1)...y_1(n+1) \;\; y_2(0) \;\; y_2(1)...y_2(n+1))^T$ and A is a block matrix such that

$$A = \begin{pmatrix} A_p & A_0 \\ -A_0 & A_q \\ B_1 & B_2 \end{pmatrix}_{(2n+4)\times(2n+4)}$$

where

$$A_p = \begin{pmatrix} \lambda - p(0) & 0 & ... & 0 & 0 \\ 0 & \lambda - p(1) & ... & 0 & 0 \\ . & . & & . & . \\ . & . & & . & . \\ . & . & & . & . \\ 0 & 0 & ... & \lambda - p(n) & 0 \end{pmatrix}_{(n+1)\times(n+2)},$$

$$A_q = \begin{pmatrix} \lambda - q(0) & 0 & ... & 0 & 0 \\ 0 & \lambda - q(1) & ... & 0 & 0 \\ . & . & & . & . \\ . & . & & . & . \\ . & . & & . & . \\ 0 & 0 & ... & \lambda - q(n) & 0 \end{pmatrix}_{(n+1)\times(n+2)},$$

$$A_0 = \begin{pmatrix} 1 & -1 & 0 & ... & 0 & 0 \\ 0 & 1 & -1 & ... & 0 & 0 \\ . & . & & & . & . \\ . & . & & & . & . \\ . & . & & & . & . \\ 0 & 0 & & ... & 1 & -1 \end{pmatrix}_{(n+1)\times(n+2)},$$

$$B_1 = \begin{pmatrix} \sin\alpha - f_1(0) & -f_1(1) & ... & -f_1(n) & 0 \\ -g_1(0) & -g_1(1) & ... & -g_1(n) & \sin\beta \end{pmatrix}_{2\times(n+2)}$$

and

$$B_2 = \begin{pmatrix} \cos\alpha - f_2(0) & -f_2(1) & ... & -f_2(n) & 0 \\ -g_2(0) & -g_2(1) & ... & -g_2(n) & \cos\beta \end{pmatrix}_{2\times(n+2)}.$$

It is clear that the characteristic function of the problem (1.1)-(1.3) is also $D(\lambda) = \det A$. If all the coefficients of the problem are given, we can calculate eigenvalues by this determinant.

EXAMPLE 3.1. *Let* $\mathbb{T} = \{0, 1, 2, 3\}$, $Q = 0$, $f_1(x) = -x$, $f_2(x) = -x^2$, $g_1(x) = g_2(x) = 0$, $\alpha = 0$, *and* $\beta = \pi/2$. *It is clear that*

$$D(\lambda)=\det\begin{pmatrix} \lambda & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & \lambda & 0 \\ 0 & 1 & 2 & 0 & 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}=2\lambda^5+3\lambda^4+6\lambda^3+5\lambda^2+$$

$4\lambda+6,$

and eigenvalues are approximately $\lambda_1=-1.1571;\quad \lambda_2=0.37758+0.98981i;\quad \lambda_3=0.37758-0.98981i;\quad \lambda_4=-0.54903-1.417\,i;\quad \lambda_5=-0.54903+1.4173i.$

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