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# RECONSTRUCTION OF A STURM-LIOUVILLE OPERATOR WITH SOME NONLOCAL BOUNDARY CONDITIONS

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Abstract. In the present paper, a Sturm-Liouville equation and two nonlocal boundary conditions which are generalized versions of Bitsadze-Samarskii-type conditions are considered. The main goal of this paper is to obtain the uniqueness and the recovering algorithm of the operator via nodal points.

## 1. Introduction

Let us consider the following boundary value problem $L$ :

$$\ell y := -y'' + q(x)y = \lambda y, \quad x \in (0,1) \tag{1.1}$$

$$U(y) := y'(0) + hy(0) - \sum_{i=1}^{m_1} \alpha_i y(\xi_i) = 0, \tag{1.2}$$

$$V(y) := y'(1) + Hy(1) - \sum_{j=1}^{m_2} \beta_j y(\psi_j) = 0, \tag{1.3}$$

where $q(x)$ is a real valued continuously differentiable function; $h,\ H \in \mathbb{R}$ and $\alpha_i, \beta_j \in \mathbb{R} \setminus \{0\}$; $\xi_i$ and $\psi_i$ are rational numbers in $(0,1)$ for $i = 1, ..., m_1,\ j = 1, 2, ..., m_2$ and $\lambda$ is the spectral parameter.

Conditions like (1.2) and (1.3) are called nonlocal boundary conditions due to including some points except boundaries of the interval. Nonlocal boundary

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conditions encountered in some applied sciences appear when data on the problem can not be measured at the boundary [**2**], [**8**], [**12**], [**14**], [**19**] and [**22**]. Particularly, a condition which includes a linear form of only one inner point is known as the Bitsadze-Samarskii-type condition. This kind of condition was applied first to elliptic equations by Bitsadze and Samarskii [**3**]. Some spectral properties of a Sturm-Liouville operator with this type of conditions are given in [**1**], [**16**], [**17**] and the references therein. When it comes to inverse problems which consist of some uniqueness theorems or reconstruction procedures of coefficients of operators from some data, there are some difficulties to study with Bitsadze-Samarskii-type conditions. It can be mentioned only one publication including inverse problems for the Sturm-Liouville operator with B-S type conditions. In this paper, Ozkan and Adalar consider an inverse nodal Sturm-Liouville problem with B-S type conditions and obtain the uniqueness and a reconstruction algorithm for the solution of this problem [**13**]. Inverse nodal problems for the Sturm-Liouville operator with local boundary conditions have already been studied in a lot of articles since the first result about this topic was published by McLaughlin in 1988 (see [**4**], [**5**], [**6**], [**7**], [**9**], [**10**], [**11**], [**13**], [**15**], [**18**], [**20**], [**21**]).

In the present paper, we aim to adapt the results in [**13**] to more general nonlocal boundary conditions and to obtain an procedure for recovering the coefficients of the operator from nodal points.

## 2. Main results

Before providing the main result we need to give some lemmas. The proof of the following lemma is similar to that in [**13**], so we omit it here.

LEMMA 2.1. *Let $\{\lambda_n\}_{n\geqslant 0}$ be the set of eigenvalues and $\varphi(x,\lambda_n)$ be the eigenfunction corresponding to the eigenvalue $\lambda_n$.*
*i) For sufficiently large $n$, the numbers $\lambda_n$ are real and satisfy the following asymptotic relation.*

$$\sqrt{\lambda_n} = s_n = n\pi + \frac{Q(1)+H-h}{n\pi} - (-1)^n \frac{\varkappa_n}{n\pi} + o(\frac{1}{n}) \tag{2.1}$$

*where $Q(x) := \frac{1}{2}\int_0^x q(t)dt$ and $\varkappa_n := \sum\limits_{j=1}^{m_2} \beta_j \cos(n\pi\psi_j) - \sum\limits_{i=1}^{m_1} \alpha_i \cos(n\pi(1-\xi_i))$.*
*ii) The asymptotic formula*

$$\begin{aligned}\varphi(x,\lambda_n) = \cos s_n x + \tfrac{(Q(x)-h)}{s_n}\sin s_n x + \tfrac{1}{s_n}\sum_{i=1}^{m_1}\alpha_i \sin s_n(x-\xi_i) + \\ +O\left(\frac{1}{s_n^2}\exp|\tau|\, x\right)\end{aligned} \tag{2.2}$$

*holds for sufficiently large $n$.*

It is clear from (2.1) and (2.2) that $\varphi(x,\lambda_n)$ has exactly $n-1$ zeros, namely nodal points in $(0,1)$ for sufficiently large $n$. Let us determine the set of the nodal points by $X = \left\{x_n^j : n = 0,1,2,\ldots \text{ and } j = 1,2,\ldots,n-1\right\}$, and assume $\int_0^1 q(x)dx =$

0. (Otherwise, the term $q(x)-\int_0^1 q(x)dx$ is determined uniquely, instead of $q(x)$ in the main result).

LEMMA 2.2. *The numbers $x_n^j$ satisfy the following asymptotic formula for $n\to\infty$,*

$$\begin{aligned} x_n^j &= \frac{j+1/2}{n}+\frac{h-H+(-1)^n\varkappa_n}{n^2\pi^2}\frac{(j+1/2)}{n}+ \\ &+\frac{\left(Q(x_n^j)-h\right)}{n^2\pi^2}+\frac{1}{n^2\pi^2}\sum_{i=1}^{m_1}\alpha_i\cos(n\pi\xi_i)+o\left(\tfrac{1}{n^2}\right). \end{aligned} \tag{2.3}$$

PROOF. It follows from

$$0=\varphi(x_n^j,\lambda_n)=\cos s_n x_n^j+\frac{\left(Q(x_n^j)-h\right)}{s_n}\sin s_n x_n^j+\frac{1}{s_n}\sum_{i=1}^{m_1}\alpha_i\sin s_n\left(x-\xi_i\right)+o\left(\frac{1}{s_n}\right)$$

that

$$\tan\left(s_n x_n^j-\frac{\pi}{2}\right)=\frac{\left(Q(x_n^j)-h\right)}{s_n}+\frac{1}{s_n}\frac{\sum\limits_{i=1}^{m_1}\alpha_i\sin s_n\left(x-\xi_i\right)}{\sin s_n x_n^j}+o\left(\frac{1}{s_n}\right).$$

Thus we get using series expansion of arctangent at zero that

$$x_n^j=\frac{(j+1/2)\,\pi}{s_n}+\frac{\left(Q(x_n^j)-h\right)}{s_n^2}+\frac{1}{s_n^2}\frac{\sum\limits_{i=1}^{m_1}\alpha_i\sin s_n\left(x-\xi_i\right)}{\sin s_n x_n^j}+o\left(\frac{1}{s_n^2}\right).$$

This yields $s_n x_n^j=(j+1/2)\,\pi+O(\frac{1}{n})$, $n\to\infty$. Therefore it can be calculated that

$$\frac{\sin s_n\left(x_n^j-\xi_i\right)}{s_n^2\sin s_n x_n^j}=\frac{\cos\left(n\pi\xi_i\right)}{n^2\pi^2}+o\left(\frac{1}{n^2}\right).$$

On the other hand, by taking into account

$$\begin{aligned} \frac{1}{s_n} &= \frac{1}{n\pi}\left(1+\frac{w}{n^2\pi^2}+\frac{(-1)^n}{n^2\pi^2}\varkappa_n+o\left(\frac{1}{n^3}\right)\right) \\ \frac{1}{s_n^2} &= \frac{1}{n^2\pi^2}+o\left(\frac{1}{n^3}\right), \end{aligned}$$

it is concluded that,

$$\begin{aligned} x_n^j &= \frac{j+1/2}{n}+\frac{h-H+(-1)^n\varkappa_n}{n^2\pi^2}\frac{(j+1/2)}{n}+ \\ &+\frac{\left(Q(x_n^j)-h\right)}{n^2\pi^2}+\frac{1}{n^2\pi^2}\sum_{i=1}^{m_1}\alpha_i\cos(n\pi\xi_i)+o\left(\frac{1}{n^2}\right). \end{aligned}$$

□

From Lemma 2.2 there exists a dense subset $X_0$ of $X$.

In this section, we reach the main goal of this study. To do this, let $r_i$ and $\rho_j$ denote denominators of $\xi_i$ and $\psi_j$, respectively, and $\nu=2n\prod\limits_{i=1}^{m_1}r_i\prod\limits_{j=1}^{m_2}\rho_j$. Assume that a dense subset $X_0$ of $X$ is given. From Lemma 2.2, elements of $X_0$ satisfy (2.3). In addition to this, there exist some sequences in $X_0$ for each $x \in [0,1]$ such that the following limit is finite.

(2.4)
$$\eta(x) := \lim_{\nu\to\infty} \nu^2\pi^2 \left( x_\nu^j - \frac{j}{\nu} \right) = \left( h - H + \sum_{j=1}^{m_2} \beta_j - \sum_{i=1}^{m_1} \alpha_i \right) x + Q(x) - h + \sum_{i=1}^{m_1} \alpha_i.$$

According to (2.4) the function $\eta$ is differentiable and

$$q(x) = 2\left(\eta'(x) + \eta(0) - \eta(1)\right) \tag{2.5}$$

is valid almost everywhere in $[0,1]$. On the other, hand it can be calculated easily that

$$h = \sum_{i=1}^{m_1} \alpha_i - \eta(0), \tag{2.6}$$

$$H = \sum_{j=1}^{m_2} \beta_j - \eta(1). \tag{2.7}$$

Consequently, $q(x)$, $h$, and $H$ can be uniquely determined when a dense subset $X_0$ of nodal points of the problem.

The following theorem is our main result whose proof is given just above.

THEOREM 2.1. *Assume that a dense subset, $X_0$, of nodal points satisfying (2.3) is given. In this case $q(x)$, $h$, and $H$ are uniquely determined by $X_0$. Moreover, $q(x)$, $h$, and $H$ can be reconstructed by the following algorithm:*

*i) Denote $\nu = 2n\prod_{i=1}^{m_1} r_i \prod_{j=1}^{m_1} \rho_j$, where $r_i$ and $\rho_j$ denote denominators of $\xi_i$ and $\psi_j$, respectively.*

*ii) Calculate $\eta(x)$ by (2.4);*

*iii) Establish $q(x)$, $h$, and $H$ by the formulas (2.5)-(2.7).*

*We must note that, if we suppose that the pairs $(h,H)$ is given and $\left(\sum_{i=1}^{m_1}\alpha_i, \sum_{j=1}^{m_2}\beta_j\right)$ is unknown at the beginning, we can find the latter by (2.6) and (2.7).*

EXAMPLE 2.1. *Example[section] Let us apply our solving method to the following problem*

$$L : \begin{cases} -y'' + q(x)y = \lambda y, \quad x \in (0,1) \\ y'(0) + hy(0) - y(\frac{1}{3}) - 2y(\frac{2}{3}) = 0, \\ y'(1) + Hy(1) - 3y(\frac{3}{4}) - 4y(\frac{2}{3}) - 5y(\frac{1}{2}) = 0, \end{cases}$$

*where $q(x) \in C^1[0,1]$. One can easily notice that it is taken in the problem that $\alpha_i = i$ for $i = 1,2$, $\beta_j = j+2$ for $j = 1,2,3$, and $\xi_1 = \frac{1}{3}$, $\xi_2 = \frac{2}{3}$, $\psi_1 = \frac{3}{4}$, $\psi_2 = \frac{2}{3}$, $\psi_3 = \frac{1}{2}$.*

*Let* $X_0 = \left\{x_n^j\right\}$ *be the given as*

$$\begin{aligned} x_n^j &= \frac{(j+1/2)}{n} + \frac{5+(-1)^n \varkappa_n}{n^2\pi^2}\frac{(j+1/2)}{n} \\ &+ \frac{1}{n^2\pi^2}\left[\frac{(j+1/2)}{4n}\left(\frac{(j+1/2)}{n}-1\right)-7\right] \\ &+ \frac{\cos\left(\frac{n\pi}{3}\right)+2\cos\left(\frac{2n\pi}{3}\right)}{n^2\pi^2} + o\left(\frac{1}{n^2}\right) \end{aligned}$$

*where* $\varkappa_n = 3\cos\frac{3n\pi}{4} + 3\cos\frac{2n\pi}{3} + 5\cos\frac{n\pi}{2} - 2\cos\frac{n\pi}{3}$.

*According to the algoritm in Theorem 2.1, we get* $\nu = 432n$ *and calculate the following limit*

$$\lim_{\nu\to\infty}\nu^2\pi^2\left(x_\nu^j - \frac{j}{\nu}\right) = \eta(x) = 14x + \frac{x}{4}(x-1) - 4$$

*Thus, we find*

$$h = \sum_{i=1}^{m_1}\alpha_i - \eta(0) = 7,$$

$$H = \sum_{j=1}^{m_2}\beta_j - \eta(1) = 2$$

*and*

$$q(x) = 2\left(\eta'(x) + \eta(0) - \eta(1)\right) = x - \frac{1}{2}.$$

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