Fundamental Journal of Mathematics and Applications, 4 (2) (2021) 45-48
Research Article

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.org.tr/en/pub/fujma
ISSN: 2645-8845
doi: https://doi.org/10.33401/fujma.842104

$\mathfrak{I}$-Limit and $\mathfrak{I}$-Cluster Points for Functions Defined on Amenable Semigroups

Uğur Ulusu[1*], Fatih Nuray[2] and Erdinç Dündar[2]

[1]*Sivas Cumhuriyet University, 58140 Sivas, Turkey*
[2]*Afyon Kocatepe University, 03200 Afyonkarahisar, Turkey*
[*]*Corresponding author*

Article Info

Keywords: *Amenable semigroups, Folner sequence, $\mathfrak{I}$-cluster points, $\mathfrak{I}$-convergence, $\mathfrak{I}$-limit points*
2010 AMS: *40A05, 40A35, 43A07*
Received: *18 December 2020*
Accepted: *25 February 2021*
Available online: *05 March 2021*

Abstract

In this paper firstly, for functions defined on discrete countable amenable semigroups (DCASG), the notions of $\mathfrak{I}$-limit and $\mathfrak{I}$-cluster points are introduced. Then, for the functions, the notions of $\mathfrak{I}$-limit superior and inferior are examined.

1. Preliminaries

The notion of $\mathfrak{I}$-convergence, based on the structure of the ideal $\mathfrak{I}$ of subset of the set of natural numbers $\mathbb{N}$, was introduced and studied by Kostyrko et al. [1, 2]. After than, regarding this notion, Demirci [3] examined the notions of $\mathfrak{I}$-limit superior and inferior.

One of first studies on amenable semigroups (ASG) is made by Day [4]. Then, Douglass [5] and Mah [6] studied the notions of summability in ASG. The notion of arithmetic mean was extended to ASG by Douglas [5] and Douglas obtained a characterization for the notion of almost convergence in ASG. Recently, Nuray and Rhoades [7] introduced the notions of convergence and statistical convergence in ASG.

The aim of this paper is to introduce some new notions for functions defined on DCASG and to examine some properties of them. Our new notions yield the notions in [2, 3] when the ASG is the additive positive integers.

Now, for better understanding our study, we recall the basic notations (see, [1, 2, 7, 8, 9]).

Let $\mathscr{G}$ be a DCASG with identity in which both left and right cancelation laws hold and $r(\mathscr{G})$ denote the space of real functions on $\mathscr{G}$.

If $\mathscr{G}$ is a countable amenable group, then there exists a sequence $\{\lambda_i\}$ of finite subsets of $\mathscr{G}$ such that

i. $\mathscr{G} = \bigcup_{i=1}^{\infty} \lambda_i$,

ii. $\lambda_i \subset \lambda_{i+1} \quad (i = 1, 2, \ldots)$,

iii. $\lim\limits_{i\to\infty} \frac{|\lambda_i \vartheta \cap \lambda_i|}{|\lambda_i|} = 1, \quad \lim\limits_{i\to\infty} \frac{|\vartheta \lambda_i \cap \lambda_i|}{|\lambda_i|} = 1, \quad$ for all $\vartheta \in \mathscr{G}$ (see, [9]).

If a sequence of finite subsets of $\mathscr{G}$ satisfy $(i)-(iii)$, then this sequence is called a Folner sequence of $\mathscr{G}$.

A familiar Folner sequence giving rise to the classical Cesàro method of summability is the sequence

$$\lambda_i = \{0, 1, 2, \ldots, i-1\}.$$

Email addresses and ORCID numbers: **ugurulusu@cumhuriyet.edu.tr, 0000-0001-7658-6114 (U. Ulusu), fnuray@aku.edu.tr, 0000-0003-0160-4001 (F. Nuray), edundar@aku.edu.tr, 0000-0002-0545-7486 (E. Dündar)**

Let $\mathscr{G}$ be a DCASG with identity in which both left and right cancelation laws hold. For any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$, a function $h \in r(\mathscr{G})$ is called convergent to $l \in \mathbb{R}$ if every $\xi > 0$ there exists a $s_0 \in \mathbb{N}$ such that

$$|h(\vartheta) - l| < \xi,$$

for all $n > s_0$ and $\vartheta \in \mathscr{G} \setminus \lambda_n$.
Let $Y \neq \emptyset$. A family of sets $\mathfrak{I} \subseteq 2^Y$ (the power set of Y) is called an ideal if and only if

i. $\emptyset \in \mathfrak{I}$,

ii. $U \cup V \in \mathfrak{I}$ for $U, V \in \mathfrak{I}$,

iii. $V \in \mathfrak{I}$ for $U \in \mathfrak{I}$ and $V \subseteq U$.

An ideal $\mathfrak{I} \subseteq 2^Y$ is called non-trivial if $Y \notin \mathfrak{I}$. A non-trivial ideal $\mathfrak{I} \subseteq 2^Y$ is called admissible if

$$\mathfrak{I} \supset \{\{y\} : y \in Y\}.$$

All ideals in this paper are assumed to be admissible in $\mathbb{N}$.
Let $\mathscr{G}$ be a DCASG with identity in which both left and right cancelation laws hold. For any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$, a function $h \in r(\mathscr{G})$ is called $\mathfrak{I}$-convergent to $l \in \mathbb{R}$ if every $\xi > 0$

$$\left\{\vartheta \in \mathscr{G} : |h(\vartheta) - l| \geq \xi\right\} \in \mathfrak{I}.$$

2. Main results

In this section firstly, for functions defined on DCASG, the notions of $\mathfrak{I}$-limit and $\mathfrak{I}$-cluster points are introduced.

Definition 2.1. *For any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$, a number $l \in \mathbb{R}$ is called a $\mathfrak{I}$-limit point of a function $h \in r(\mathscr{G})$ if there exists a set $F \subset \mathscr{G}$ ($F \notin \mathfrak{I}$) such that*

$$\lim h(\vartheta) = l \quad (\vartheta \in F \setminus \lambda_i).$$

Definition 2.2. *For any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$, a number $c \in \mathbb{R}$ is called an $\mathfrak{I}$-cluster point of a function $h \in r(\mathscr{G})$ if every $\xi > 0$*

$$\left\{\vartheta \in \mathscr{G} : |h(\vartheta) - c| < \xi\right\} \notin \mathfrak{I}.$$

For any function $h \in r(\mathscr{G})$, let $\mathfrak{I}^h_\Lambda(\mathscr{G})$ and $\mathfrak{I}^h_\Gamma(\mathscr{G})$ denote the set of all $\mathfrak{I}$-limit and $\mathfrak{I}$-cluster points of the function h, respectively.

Theorem 2.3. *For each function $h \in r(\mathscr{G})$,*

$$\mathfrak{I}^h_\Lambda(\mathscr{G}) \subseteq \mathfrak{I}^h_\Gamma(\mathscr{G}).$$

Proof. Let $l \in \mathfrak{I}^h_\Lambda(\mathscr{G})$. Then, there exists a set $F \notin \mathfrak{I}$ such that

$$\lim h(\vartheta) = l \quad (\vartheta \in F \setminus \lambda_i).$$

Hence, for every $\delta > 0$ there exists a $s_0 = s_0(\delta) \in \mathbb{N}$ such that for $\vartheta \in F \setminus \lambda_i$ we have

$$|h(\vartheta) - l| < \delta,$$

for all $i > s_0$. Therefore,

$$\left\{\vartheta \in \mathscr{G} : |h(\vartheta) - l| < \delta\right\} \supset F \setminus \lambda_i$$

and so

$$\left\{\vartheta \in \mathscr{G} : |h(\vartheta) - l| < \delta\right\} \notin \mathfrak{I},$$

which means that $l \in \mathfrak{I}^h_\Gamma(\mathscr{G})$. □

Theorem 2.4. *For each function $h \in r(\mathscr{G})$, the set $\mathfrak{I}^h_\Lambda(G)$ is a closed set in $\mathbb{R}$.*

Proof. Let $l \in \overline{\mathfrak{I}_{\Lambda}^{h}(\mathscr{G})}$ and $\xi > 0$. Then, there exists

$$l_0 \in \mathfrak{I}_{\Lambda}^{h}(\mathscr{G}) \cap B(l, \xi).$$

Choose $\delta \geq 0$ such that

$$B(l_0, \delta) \subset B(l, \xi).$$

Obviously, we have

$$\left\{\vartheta \in \mathscr{G} : |l - h(\vartheta)| < \xi\right\} \supset \left\{\vartheta \in \mathscr{G} : |l_0 - h(\vartheta)| < \delta\right\}.$$

Therefore,

$$\left\{\vartheta \in \mathscr{G} : |l - h(\vartheta)| < \xi\right\} \notin \mathfrak{I}$$

and so $l \in \mathfrak{I}_{\Lambda}^{h}(\mathscr{G})$. □

Now secondly, for functions defined on DCASG, the notions of $\mathfrak{I}$-limit superior and inferior are examined.
For a function $h \in r(\mathscr{G})$, we define the following sets:

$$A_h := \left\{a \in \mathbb{R} : \{\vartheta \in \mathscr{G} : h(\vartheta) < a\} \notin \mathfrak{I}\right\},$$

similarly

$$B_h := \left\{b \in \mathbb{R} : \{\vartheta \in \mathscr{G} : h(\vartheta) > b\} \notin \mathfrak{I}\right\}$$

for any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$.

Definition 2.5. *For a function $h \in r(\mathscr{G})$, $\mathfrak{I}$-limit inferior is given by*

$$\mathfrak{I} - \liminf h = \begin{cases} \inf A_h & , \quad A_h \neq \emptyset \\ \infty & , \quad A_h = \emptyset \end{cases}$$

also, $\mathfrak{I}$-limit superior is given by

$$\mathfrak{I} - \limsup h = \begin{cases} \sup B_h & , \quad B_h \neq \emptyset \\ -\infty & , \quad B_h = \emptyset. \end{cases}$$

for any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$.

For any function $h \in r(\mathscr{G})$, it is easy to see that

$$\mathfrak{I} - \liminf h \leq \mathfrak{I} - \limsup h$$

for any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$.

Definition 2.6. *For any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$, a function $h \in r(\mathscr{G})$ is called $\mathfrak{I}$-bounded if there exists a M such that*

$$\left\{\vartheta \in \mathscr{G} : |h(\vartheta)| < M\right\} \in \mathfrak{I}.$$

Note that $\mathfrak{I}$-boundedness for a function $h \in r(\mathscr{G})$ implies that $\mathfrak{I} - \liminf h$ and $\mathfrak{I} - \limsup h$ are finite for any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$.
The following theorem can be proved by a simple least upper bound argument.

Theorem 2.7. *For any function $h \in r(\mathscr{G})$; if $\gamma = \mathfrak{I} - \liminf h$ is finite, then for every $\xi > 0$*

$$\left\{\vartheta \in \mathscr{G} : h(\vartheta) < \gamma + \xi\right\} \notin \mathfrak{I} \text{ and } \left\{\vartheta \in \mathscr{G} : h(\vartheta) < \gamma - \xi\right\} \in \mathfrak{I}, \tag{2.1}$$

for any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$.
Conversely if (2.1) holds for every $\xi > 0$, then

$$\mathfrak{I} - \liminf h = \gamma.$$

The dual statement for $\mathfrak{I} - \limsup h$ is as follows:

Theorem 2.8. *For any function $h \in r(\mathscr{G})$; if $\eta = \Im - \limsup h$ is finite, then for every $\xi > 0$*

$$\left\{\vartheta \in \mathscr{G} : h(\vartheta) > \eta - \xi\right\} \notin \Im \text{ and } \left\{\vartheta \in \mathscr{G} : h(\vartheta) > \eta + \xi\right\} \in \Im, \tag{2.2}$$

for any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$.
Conversely if (2.2) holds for every $\xi > 0$, then

$$\Im - \limsup h = \eta.$$

Theorem 2.9. *For any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$; $\Im - \liminf h = \Im - \limsup h$ if and only if the $\Im$-bounded function h is $\Im$-convergent.*

Proof. For any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$, let

$$\gamma = \Im - \liminf h \text{ and } \eta = \Im - \limsup h.$$

Firstly, we assume that $\Im - \lim h = l$ and $\xi > 0$. Then,

$$\left\{\vartheta \in \mathscr{G} : |h(\vartheta) - l| \geq \xi\right\} \in \Im$$

and so

$$\left\{\vartheta \in \mathscr{G} : h(\vartheta) > l + \xi\right\} \in \Im,$$

which implies that $\eta \leq l$. Also, we have

$$\left\{\vartheta : h(\vartheta) < l - \xi\right\} \in \Im,$$

which implies that $l \leq \gamma$. Therefore $\eta \leq \gamma$, which we combine with the fact that

$$\Im - \liminf h \leq \Im - \limsup h,$$

to conclude that $\gamma = \eta$.
Now, secondly, we assume that for any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$,

$$\Im - \liminf h = \Im - \limsup h.$$

If $\xi > 0$, then (2.1) and (2.2) imply

$$\left\{\vartheta \in \mathscr{G} : h(\vartheta) > l + \frac{\xi}{2}\right\} \in \Im \text{ and } \left\{\vartheta \in \mathscr{G} : h(\vartheta) < l - \frac{\xi}{2}\right\} \in \Im.$$

Hence, for any Folner sequence $\{\lambda_i\}$ of $\mathscr{G}$, we have

$$\Im - \lim h = l.$$

□

3. Conclusion

We investigated the notions of $\Im$-limit points and $\Im$-cluster points for functions defined on discrete countable amenable semigroups. These notions can also be studied for double sequences in the future.

Acknowledgement

This study is supported by TÜBİTAK (The Scientific and Technological Research Council of Turkey) with the project number 120F082.

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