



Wijsman asymptotic lacunary \mathcal{I}_2 -invariant equivalence for double set sequences

Uğur Ulusu¹ · Erdinç Dündar² · Nimet Pancaroğlu Akın²

Accepted: 23 August 2021 / Published online: 1 September 2021
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2021

Abstract

In this study, for double set sequences, we present the notions of Wijsman asymptotic lacunary invariant equivalence, Wijsman asymptotic lacunary \mathcal{I}_2 -invariant equivalence and Wijsman asymptotic lacunary \mathcal{I}_2^* -invariant equivalence. Also, we examine the relations between these notions and Wijsman asymptotic lacunary invariant statistical equivalence studied in this field before.

Keywords Wijsman convergence · Asymptotic equivalence · Double lacunary sequence · \mathcal{I} -convergence · Invariant convergence · Double set sequences

1 Introduction

The notion of convergence for double sequences was firstly introduced by Pringsheim Pringsheim (1900). Then, this notion was extended to the notion of lacunary statistical convergence by Patterson and Savaş Patterson and Savaş (2005) and the notion of \mathcal{I} -convergence by Das et al. Das et al. (2008).

The notion of asymptotic equivalence for double sequences was introduced by Patterson Patterson (2002). Then, this notion was extended to the notion of asymptotic double lacunary statistical equivalence by Esi Esi (2009) and the notion of asymptotic \mathcal{I} -equivalence by Hazarika and Kumar Hazarika and Kumar (2013).

Over the years, many authors have studied on the notions of various convergence for set sequences. One of them, discussed in this study, is the notion of convergence in the Wijsman sense Baronti and Papini (1986), Beer (1985, 1994), Wijsman (1964, 1966). Using the notions of lacunary statistical convergence, invariant mean and \mathcal{I} -convergence, the notion of convergence in the Wijsman sense was extended to new convergence notions for double set sequences by some authors Nuray et al. (2014, 2016), Tortop and Dündar (2018).

The notions of asymptotic equivalence in the Wijsman sense for double set sequences were firstly introduced by Nuray et al. Nuray et al. (2016) and studied by many authors. In this paper, using invariant mean, we study new asymptotic equivalence notions for double set sequences.

More study on the notions of convergence or asymptotic equivalence for real sequences or set sequences can be found in Çakan et al. (2006), Dündar and Altay (2014), Et and Şengül (2016), Fridy and Orhan (1993), Hazarika et al. (2013), Hazarika (2015), Kara et al. (2017), Kişi and Nuray (2013a, b), Kumar (2007), Marouf (1993), Mursaleen and Edely (2009), Nuray and Rhoades (2012), Pancaroğlu and Nuray (2013a, b), Pancaroğlu et al. (2013); Pancaroğlu Akın et al. (2020), Patterson and Savaş (2006), Raj and Anand (2018), Raj et al. (2018), Raj and Jamwal (2019), Savaş (1990), Savaş and Nuray (1993), Savaş and Patterson (2006), Savaş and Patterson (2009), Sever et al. (2014), Şengül (2018), Şengül et al. (2019), Şengül et al. (2020), Ulusu and Nuray (2012, 2013a, b), Ulusu and Savaş (2014), Ulusu and Dündar (2016, 2019).

2 Definitions and notations

The fundamental definitions and notations required for this study are following (see, Baronti and Papini (1986), Beer (1985), Das et al. (2008), Dündar et al. (2020), Kostyrko et al. (2000), Mursaleen (1979, 1983), Nuray et al. (2014,

✉ Uğur Ulusu
ugurulusu@cumhuriyet.edu.tr

¹ Sivas Cumhuriyet University, 58140 Sivas, Turkey

² Afyon Kocatepe University, 03200 Afyonkarahisar, Turkey

2016), Patterson (2002), Patterson and Savaş (2005), Savaş (1989), Ulusu et al. (2018), [55]).

Let σ be a mapping such that $\sigma : \mathbb{N}^+ \rightarrow \mathbb{N}^+$ (the set of positive integers). A continuous linear functional ψ on ℓ_∞ is said to be an invariant mean or a σ -mean if it satisfies the following conditions:

1. $\psi(x_n) \geq 0$, when the sequence (x_n) has $x_n \geq 0$ for all n ,
2. $\psi(e) = 1$, where $e = (1, 1, 1, \dots)$ and
3. $\psi(x_{\sigma(n)}) = \psi(x_n)$ for all $(x_n) \in \ell_\infty$.

The mappings σ are assumed to be one to one and such that $\sigma^m(n) \neq n$ for all $m, n \in \mathbb{N}^+$, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n . Thus, ψ extends the limit functional on c , in the sense that $\psi(x_n) = \lim x_n$ for all $(x_n) \in c$.

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is said to be an ideal if it satisfies the following conditions:

- (i) $\emptyset \in \mathcal{I}$,
- (ii) $E, F \in \mathcal{I} \Rightarrow E \cup F \in \mathcal{I}$,
- (iii) $(E \in \mathcal{I}) \wedge (F \subseteq E) \Rightarrow F \in \mathcal{I}$.

An ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is said to be nontrivial if $\mathbb{N} \notin \mathcal{I}$ and a nontrivial ideal is said to be admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

A nontrivial ideal $\mathcal{I}_2 \subseteq 2^{\mathbb{N} \times \mathbb{N}}$ is said to be strongly admissible if $\{n\} \times \mathbb{N}$ and $\mathbb{N} \times \{n\}$ belong to \mathcal{I}_2 for each $n \in \mathbb{N}$. Obviously, a strongly admissible ideal is admissible.

Throughout the study, $\mathcal{I}_2 \subseteq 2^{\mathbb{N} \times \mathbb{N}}$ will be considered as a strongly admissible ideal.

Let $\mathcal{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \notin A)\}$. Then, \mathcal{I}_2^0 is a strongly admissible ideal and clearly an ideal $\mathcal{I}_2 \subseteq 2^{\mathbb{N} \times \mathbb{N}}$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

An admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ satisfies the property (AP2) if every countable family of mutually disjoint sets $\{E_1, E_2, \dots\}$ belonging to \mathcal{I}_2 , there exists a countable family of sets $\{F_1, F_2, \dots\}$ such that $E_i \Delta F_i \in \mathcal{I}_2^0$, i.e., $E_i \Delta F_i$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $i \in \mathbb{N}$ and $F = \bigcup_{i=1}^\infty F_i \in \mathcal{I}_2$ (hence, $F_i \in \mathcal{I}_2$ for each $i \in \mathbb{N}$).

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is said to be a filter if it satisfies the following conditions:

- (i) $\emptyset \notin \mathcal{F}$,
- (ii) $E, F \in \mathcal{F} \Rightarrow E \cap F \in \mathcal{F}$,
- (iii) $(E \in \mathcal{F}) \wedge (F \supseteq E) \Rightarrow F \in \mathcal{F}$.

For any ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$, there is a filter $\mathcal{F}(\mathcal{I})$ corresponding with \mathcal{I} such that

$$\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : (\exists E \in \mathcal{I})(M = \mathbb{N} \setminus E)\}.$$

A double sequence $\theta_2 = \{(k_r, j_u)\}$ is said to be double lacunary sequence if there exist two increasing sequences of integers (k_r) and (j_u) such that

$$k_0 = 0, h_r = k_r - k_{r-1} \rightarrow \infty \text{ and } j_0 = 0, \bar{h}_u = j_u - j_{u-1} \rightarrow \infty \text{ as } r, u \rightarrow \infty.$$

For any double lacunary sequence $\theta_2 = \{(k_r, j_u)\}$, the following notations are used in general:

$$k_{ru} = k_r j_u, h_{ru} = h_r \bar{h}_u, I_{ru} = \{(k, j) : k_{r-1} < k \leq k_r \text{ and } j_{u-1} < j \leq j_u\}.$$

Throughout the study, $\theta_2 = \{(k_r, j_u)\}$ will be considered as a double lacunary sequence.

Let $\theta = \{(k_r, j_u)\}$ be a double lacunary sequence, $A \subseteq \mathbb{N} \times \mathbb{N}$ and

$$s_{ru} := \min_{m,n} |A \cap \{(\sigma^k(m), \sigma^j(n)) : (k, j) \in I_{ru}\}|$$

and

$$S_{ru} := \max_{m,n} |A \cap \{(\sigma^k(m), \sigma^j(n)) : (k, j) \in I_{ru}\}|.$$

If the following limits exist

$$\underline{V}_2^\theta(A) := \lim_{r,u \rightarrow \infty} \frac{s_{ru}}{h_{ru}} \text{ and } \overline{V}_2^\theta(A) := \lim_{r,u \rightarrow \infty} \frac{S_{ru}}{h_{ru}},$$

then they are said to be a lower lacunary σ -uniform density and an upper lacunary σ -uniform density of the set A , respectively. If $\underline{V}_2^\theta(A) = \overline{V}_2^\theta(A)$, then $V_2^\theta(A) = \underline{V}_2^\theta(A) = \overline{V}_2^\theta(A)$ is said to be lacunary σ -uniform density of the set A .

The class of all $A \subseteq \mathbb{N} \times \mathbb{N}$ with $V_2^\theta(A) = 0$ is denoted by $\mathcal{I}_2^{\sigma\theta}$. Obviously $\mathcal{I}_2^{\sigma\theta}$ is strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Two nonnegative double sequences (x_{kj}) and (y_{kj}) are said to be asymptotically equivalent if

$$\lim_{k,j \rightarrow \infty} \frac{x_{kj}}{y_{kj}} = 1.$$

It is denoted by $x_{kj} \sim y_{kj}$.

For a nonempty set X , let a function $f : \mathbb{N} \rightarrow 2^X$ (the power set of X) is defined by $f(k) = U_k \in 2^X$ for each $k \in \mathbb{N}$. Then, the sequence $\{U_k\} = (U_1, U_2, \dots)$, which are the codomain elements of f , is said to be set sequences.

For a metric space (X, ρ) , $\mu(x, U)$ denote the distance from x to U where

$$\mu(x, U) = \inf_{u \in U} \rho(x, u).$$

for any $x \in X$ and any nonempty set $U \subseteq X$.

Throughout the study, (X, ρ) will be considered as a metric space and U, U_{kj}, V_{kj} as any nonempty closed subsets of X .

The double sequence $\{U_{kj}\}$ is said to be Wijsman convergent to U if each $x \in X$,

$$\lim_{k,j \rightarrow \infty} \mu(x, U_{kj}) = \mu(x, U).$$

The double sequence $\{U_{kj}\}$ is said to be Wijsman lacunary invariant convergent to U if each $x \in X$,

$$\lim_{r,u \rightarrow \infty} \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} \mu(x, U_{\sigma^k(m)\sigma^j(n)}) = \mu(x, U)$$

uniformly in $m, n \in \mathbb{N}$.

The double sequence $\{U_{kj}\}$ is said to be Wijsman strongly lacunary invariant convergent to U if each $x \in X$,

$$\lim_{r,u \rightarrow \infty} \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} |\mu(x, U_{\sigma^k(m)\sigma^j(n)}) - \mu(x, U)| = 0$$

uniformly in m and n .

Let $0 < p < \infty$. The double sequence $\{U_{kj}\}$ is said to be Wijsman strongly p -lacunary invariant convergent to U if each $x \in X$,

$$\lim_{r,u \rightarrow \infty} \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} |\mu(x, U_{\sigma^k(m)\sigma^j(n)}) - \mu(x, U)|^p = 0$$

uniformly in m and n .

The double sequence $\{U_{kj}\}$ is said to be Wijsman lacunary \mathcal{I}_2 -invariant convergent or $\mathcal{I}_2^{\sigma\theta}$ -convergent to U if every $\varepsilon > 0$ and each $x \in X$, the set

$$A^2(\varepsilon) := \{(k, j) \in I_{ru} : |\mu(x, U_{kj}) - \mu(x, U)| \geq \varepsilon\}$$

belongs to $\mathcal{I}_2^{\sigma\theta}$, i.e., $V_2^\theta(A^2(\varepsilon)) = 0$. It is denoted by $U_{kj} \xrightarrow{W(\mathcal{I}_2^{\sigma\theta})} U$.

The term $\mu_x\left(\frac{U_{kj}}{V_{kj}}\right)$ is defined as follows:

$$\mu_x\left(\frac{U_{kj}}{V_{kj}}\right) = \begin{cases} \frac{\mu(x, U_{kj})}{\mu(x, V_{kj})}, & x \notin U_{kj} \cup V_{kj} \\ L, & x \in U_{kj} \cup V_{kj}. \end{cases}$$

The double sequences $\{U_{kj}\}$ and $\{V_{kj}\}$ are said to be Wijsman asymptotically equivalent of multiple L if each $x \in X$,

$$\lim_{k,j \rightarrow \infty} \mu_x\left(\frac{U_{kj}}{V_{kj}}\right) = L.$$

It is denoted by $U_{kj} \overset{L}{\sim} V_{kj}$ and simply is said to be Wijsman asymptotically equivalent if $L = 1$.

As an example, consider the following sequences of circles in the (x, y) -plane:

$$U_{kj} = \{(x, y) : x^2 + y^2 + 2kx + 2jy = 0\}$$

and

$$V_{kj} = \{(x, y) : x^2 + y^2 - 2kx - 2jy = 0\}.$$

Then, the double sequences $\{U_{kj}\}$ and $\{V_{kj}\}$ are Wijsman asymptotically equivalent, i.e., $U_{kj} \sim V_{kj}$.

The double sequences $\{U_{kj}\}$ and $\{V_{kj}\}$ are Wijsman asymptotically lacunary invariant statistical equivalent of multiple L if every $\varepsilon > 0$ and each $x \in X$,

$$\lim_{r,u \rightarrow \infty} \frac{1}{h_{ru}} \left\{ (k, j) \in I_{ru} : \left| \mu_x\left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}}\right) - L \right| \geq \varepsilon \right\} = 0$$

uniformly in m and n and it is denoted by $U_{kj} \overset{W(S_{2(L)}^{\sigma\theta})}{\sim} V_{kj}$, and simply is said to be Wijsman asymptotically lacunary invariant statistical equivalent if $L = 1$.

3 Main results

In this section, for double set sequences, we present the notions of Wijsman asymptotic lacunary invariant equivalence $(W(N_{2(L)}^{\sigma\theta}), W[N_{2(L)}^{\sigma\theta}], W[N_{2(L)}^{\sigma\theta}]^p)$, Wijsman asymptotic lacunary \mathcal{I}_2 -invariant equivalence and Wijsman asymptotic lacunary \mathcal{I}_2^* -invariant equivalence. Also, we examine the relations between these notions and Wijsman asymptotic lacunary invariant statistical equivalence studied in this field before.

Definition 1 Two double set sequences $\{U_{kj}\}$ and $\{V_{kj}\}$ are Wijsman asymptotically lacunary invariant equivalent of multiple L if each $x \in X$,

$$\lim_{r,u \rightarrow \infty} \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} \mu_x\left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}}\right) = L$$

uniformly in m and n . In this case, we write $U_{kj} \overset{W(N_{2(L)}^{\sigma\theta})}{\sim} V_{kj}$ and simply say Wijsman asymptotically lacunary invariant equivalent if $L = 1$.

Definition 2 Two double set sequences $\{U_{kj}\}$ and $\{V_{kj}\}$ are Wijsman asymptotically lacunary \mathcal{I}_2 -invariant equivalent of multiple L if every $\varepsilon > 0$ and each $x \in X$, the set

$$\tilde{A}(\varepsilon, x) := \left\{ (k, j) \in I_{ru} : \left| \mu_x \left(\frac{U_{kj}}{V_{kj}} \right) - L \right| \geq \varepsilon \right\}$$

belongs to $\mathcal{I}_2^{\sigma\theta}$, i.e., $V_2^\theta(\tilde{A}(\varepsilon, x)) = 0$. In this case, we write $U_{kj} \stackrel{W(\mathcal{I}_2^{\sigma\theta})}{\sim} V_{kj}$ and simply say Wijsman asymptotically lacunary \mathcal{I}_2 -invariant equivalent if $L = 1$.

Theorem 1 *If $\mu_x(U_{kj}) = \mathcal{O}(\mu_x(V_{kj}))$, then*

$$U_{kj} \stackrel{W(\mathcal{I}_2^{\sigma\theta})}{\sim} V_{kj} \text{ implies } U_{kj} \stackrel{W(N_{2(L)}^{\sigma\theta})}{\sim} V_{kj}.$$

Proof Let $m, n \in \mathbb{N}$ be arbitrary and $\varepsilon > 0$ is given. Also, we assume that $U_{kj} \stackrel{W(\mathcal{I}_2^{\sigma\theta})}{\sim} V_{kj}$. Now, we calculate

$$\mathcal{S}(m, n) := \left| \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} \mu_x \left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}} \right) - L \right|.$$

For every $m, n = 1, 2, \dots$ and each $x \in X$, we have

$$\mathcal{S}(m, n) \leq \mathcal{S}_1(m, n) + \mathcal{S}_2(m, n),$$

where

$$\mathcal{S}_1(m, n) := \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} \left| \mu_x \left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}} \right) - L \right| \geq \varepsilon$$

and

$$\mathcal{S}_2(m, n) := \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} \left| \mu_x \left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}} \right) - L \right| < \varepsilon$$

For every $m, n = 1, 2, \dots$ and each $x \in X$, it is obvious that $\mathcal{S}_2(m, n) < \varepsilon$. Since $\mu_x(U_{kj}) = \mathcal{O}(\mu_x(V_{kj}))$, there exists a $\lambda > 0$ such that

$$\left| \mu_x \left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}} \right) - L \right| \leq \lambda,$$

for each $x \in X$ and $((k, j) \in I_{ru}, m, n = 1, 2, \dots)$, so we have

$$\begin{aligned} \mathcal{S}_1(m, n) &\leq \frac{\lambda}{h_{ru}} \left| \left\{ (k, j) \in I_{ru} : \left| \mu_x \left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}} \right) - L \right| \geq \varepsilon \right\} \right| \\ &\leq \lambda \frac{\max_{m,n} \left\{ \left| \left\{ (k, j) \in I_{ru} : \left| \mu_x \left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}} \right) - L \right| \geq \varepsilon \right\} \right| \right\}}{h_{ru}} \\ &= \lambda \frac{S_{ru}}{h_{ru}}. \end{aligned}$$

Hence, due to our assumption, $U_{kj} \stackrel{W(N_{2(L)}^{\sigma\theta})}{\sim} V_{kj}$. □

Definition 3 Two double set sequences $\{U_{kj}\}$ and $\{V_{kj}\}$ are Wijsman asymptotically strongly lacunary invariant equivalent of multiple L if each $x \in X$,

$$\lim_{r,u \rightarrow \infty} \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} \left| \mu_x \left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}} \right) - L \right| = 0$$

uniformly in m and n . In this case, we write $U_{kj} \stackrel{W[N_{2(L)}^{\sigma\theta}]}{\sim} V_{kj}$ and simply say Wijsman asymptotically strongly lacunary invariant equivalent if $L = 1$.

Definition 4 Let $0 < p < \infty$. Two double set sequences $\{U_{kj}\}$ and $\{V_{kj}\}$ are Wijsman asymptotically strongly p -lacunary invariant equivalent of multiple L if each $x \in X$,

$$\lim_{r,u \rightarrow \infty} \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} \left| \mu_x \left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}} \right) - L \right|^p = 0$$

uniformly in m and n . In this case, we write $U_{kj} \stackrel{W[N_{2(L)}^{\sigma\theta}]^p}{\sim} V_{kj}$ and simply say Wijsman asymptotically strongly p -lacunary invariant equivalent if $L = 1$.

Theorem 2 *If $U_{kj} \stackrel{W[N_{2(L)}^{\sigma\theta}]^p}{\sim} V_{kj}$, then $U_{kj} \stackrel{W(\mathcal{I}_2^{\sigma\theta})}{\sim} V_{kj}$.*

Proof Let $0 < p < \infty$ and $\varepsilon > 0$ is given. Also, we assume that $U_{kj} \stackrel{W[N_{2(L)}^{\sigma\theta}]^p}{\sim} V_{kj}$. For every $m, n = 1, 2, \dots$ and each $x \in X$, we have

$$\begin{aligned} &\sum_{(k,j) \in I_{ru}} \left| \mu_x \left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}} \right) - L \right|^p \\ &\geq \sum_{(k,j) \in I_{ru}} \left| \mu_x \left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}} \right) - L \right|^p \\ &\quad \left| \mu_x \left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}} \right) - L \right| \geq \varepsilon \\ &\geq \varepsilon^p \left| \left\{ (k, j) \in I_{ru} : \left| \mu_x \left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}} \right) - L \right| \geq \varepsilon \right\} \right| \\ &\geq \varepsilon^p \max_{m,n} \left\{ \left| \left\{ (k, j) \in I_{ru} : \left| \mu_x \left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}} \right) - L \right| \geq \varepsilon \right\} \right| \right\} \\ &\geq \varepsilon \left\{ \left\{ \right\} \right\} \end{aligned}$$

and so

$$\begin{aligned} & \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} \left| \mu_x \left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}} \right) - L \right|^p \\ & \geq \varepsilon^p \frac{\max_{m,n} \left\{ \left| \left\{ (k,j) \in I_{ru} : \left| \mu_x \left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}} \right) - L \right| \geq \varepsilon \right\} \right| \right\}}{h_{ru}} \\ & = \varepsilon^p \frac{S_{ru}}{h_{ru}}. \end{aligned}$$

Hence, due to our assumption, $U_{kj} \overset{W(\mathcal{I}_2^{\sigma\theta})}{\sim} V_{kj}$. □

Theorem 3 If $\mu_x(U_{kj}) = \mathcal{O}(\mu_x(V_{kj}))$, then

$$U_{kj} \overset{W(\mathcal{I}_2^{\sigma\theta})}{\sim} V_{kj} \text{ implies } U_{kj} \overset{W[N_{2(L)}^{\sigma\theta}]^p}{\sim} V_{kj}.$$

Proof Let $\mu_x(U_{kj}) = \mathcal{O}(\mu_x(V_{kj}))$ and $\varepsilon > 0$ is given.

Also, we assume that $U_{kj} \overset{W(\mathcal{I}_2^{\sigma\theta})}{\sim} V_{kj}$. Since $\mu_x(U_{kj}) = \mathcal{O}(\mu_x(V_{kj}))$, there exists a $\lambda > 0$ such that

$$\left| \mu_x \left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}} \right) - L \right| \leq \lambda,$$

for each $x \in X$ and $((k, j) \in I_{ru}, m, n = 1, 2, \dots)$, so we have

$$\begin{aligned} & \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} \left| \mu_x \left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}} \right) - L \right|^p \\ & = \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} \left| \mu_x \left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}} \right) - L \right|^p \\ & \quad \left| \mu_x \left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}} \right) - L \right| \geq \varepsilon \\ & \quad + \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} \left| \mu_x \left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}} \right) - L \right|^p \\ & \quad \left| \mu_x \left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}} \right) - L \right| < \varepsilon \\ & \leq \lambda \frac{\max_{m,n} \left\{ \left| \left\{ (k,j) \in I_{ru} : \left| \mu_x \left(\frac{U_{\sigma^k(m)\sigma^j(n)}}{V_{\sigma^k(m)\sigma^j(n)}} \right) - L \right| \geq \varepsilon \right\} \right| \right\}}{h_{ru}} \\ & \quad + \varepsilon^p \\ & \leq \lambda \frac{S_{ru}}{h_{ru}} + \varepsilon^p. \end{aligned}$$

Hence, due to our assumption, $U_{kj} \overset{W[N_{2(L)}^{\sigma\theta}]^p}{\sim} V_{kj}$. □

Theorem 4 If $\mu_x(U_{kj}) = \mathcal{O}(\mu_x(V_{kj}))$, then

$$U_{kj} \overset{W(\mathcal{I}_2^{\sigma\theta})}{\sim} V_{kj} \Leftrightarrow U_{kj} \overset{W[N_{2(L)}^{\sigma\theta}]^p}{\sim} V_{kj}.$$

Proof The proof is obvious from Theorem 2 and Theorem 3. □

Now, without proof, we shall present a theorem that gives a relationship between the notions of Wijsman asymptotic lacunary \mathcal{I}_2 -invariant equivalence of multiple L and Wijsman asymptotic lacunary invariant statistical equivalence of multiple L .

Theorem 5 For any double set sequences $\{U_{kj}\}$ and $\{V_{kj}\}$,

$$U_{kj} \overset{W(\mathcal{I}_2^{\sigma\theta})}{\sim} V_{kj} \Leftrightarrow U_{kj} \overset{W(S_{2(L)}^{\sigma\theta})}{\sim} V_{kj}.$$

Definition 5 Two double sequences $\{U_{kj}\}$ and $\{V_{kj}\}$ are Wijsman asymptotically lacunary \mathcal{I}_2^* -invariant equivalent of multiple L if and only if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2^{\sigma\theta})$ ($\mathbb{N} \times \mathbb{N} \setminus M_2 = H \in \mathcal{I}_2^{\sigma\theta}$) such that each $x \in X$,

$$\lim_{\substack{k,j \rightarrow \infty \\ (k,j) \in M_2}} \mu_x \left(\frac{U_{kj}}{V_{kj}} \right) = L.$$

In this case, we write $U_{kj} \overset{W(\mathcal{I}_2^{*\sigma\theta})}{\sim} V_{kj}$ and simply say Wijsman asymptotically lacunary \mathcal{I}_2^* -invariant equivalent if $L = 1$.

Theorem 6 If $U_{kj} \overset{W(\mathcal{I}_2^{*\sigma\theta})}{\sim} V_{kj}$, then $U_{kj} \overset{W(\mathcal{I}_2^{\sigma\theta})}{\sim} V_{kj}$.

Proof Let $U_{kj} \overset{W(\mathcal{I}_2^{*\sigma\theta})}{\sim} V_{kj}$ and $\varepsilon > 0$ is given. Then, there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2^{\sigma\theta})$ ($\mathbb{N} \times \mathbb{N} \setminus M_2 = H \in \mathcal{I}_2^{\sigma\theta}$) such that for each $x \in X$,

$$\lim_{\substack{k,j \rightarrow \infty \\ (k,j) \in M_2}} \mu_x \left(\frac{U_{kj}}{V_{kj}} \right) = L$$

and so there exist $k_0, j_0 \in \mathbb{N}$ such that

$$\left| \mu_x \left(\frac{U_{kj}}{V_{kj}} \right) - L \right| < \varepsilon,$$

for all $(k, j) \in M_2$ where $k \geq k_0, j \geq j_0$. Hence, for every $\varepsilon > 0$ and each $x \in X$ it is obvious that

$$\begin{aligned} \mathcal{S}(\varepsilon) & := \left\{ (k, j) \in I_{ru} : \left| \mu_x \left(\frac{U_{kj}}{V_{kj}} \right) - L \right| \geq \varepsilon \right\} \\ & \subset H \cup \left(M_2 \cap \left((\{1, 2, \dots, (k_0 - 1)\} \times \mathbb{N}) \right. \right. \\ & \quad \left. \left. \cup (\mathbb{N} \times \{1, 2, \dots, (j_0 - 1)\}) \right) \right). \end{aligned}$$

Since $\mathcal{I}_2^{\sigma\theta} \subset 2^{\mathbb{N} \times \mathbb{N}}$ is a strongly admissible ideal,

$$H \cup \left(M_2 \cap \left((\{1, 2, \dots, (k_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \dots, (k_0 - 1)\}) \right) \right) \in \mathcal{I}_2^{\sigma\theta}$$

and so we have $S(\varepsilon) \in \mathcal{I}_2^{\sigma\theta}$. Consequently, $U_{kj} \overset{W(\mathcal{I}_2^{\sigma\theta})}{\sim} V_{kj}$. □

If $\mathcal{I}_2^{\sigma\theta}$ has property (AP2), then the converse of Theorem 6 is hold.

Theorem 7 *If $\mathcal{I}_2^{\sigma\theta}$ has property (AP2), then*

$$U_{kj} \overset{W(\mathcal{I}_2^{\sigma\theta})}{\sim} V_{kj} \text{ implies } U_{kj} \overset{W(\mathcal{I}_2^{*\sigma\theta})}{\sim} V_{kj}.$$

Proof Let $\mathcal{I}_2^{\sigma\theta}$ satisfies condition (AP2) and $\varepsilon > 0$ is given.

Also, suppose that $U_{kj} \overset{W(\mathcal{I}_2^{\sigma\theta})}{\sim} V_{kj}$. Then, for every $\varepsilon > 0$ and each $x \in X$ we have

$$\left\{ (k, j) \in I_{ru} : \left| \mu_x \left(\frac{U_{kj}}{V_{kj}} \right) - L \right| \geq \varepsilon \right\} \in \mathcal{I}_2^{\sigma\theta}.$$

For every $x \in X$, denote E_1, \dots, E_n as follows

$$E_1 := \left\{ (k, j) \in I_{ru} : \left| \mu_x \left(\frac{U_{kj}}{V_{kj}} \right) - L \right| \geq 1 \right\}$$

and

$$E_n := \left\{ (k, j) \in I_{ru} : \frac{1}{n} \leq \left| \mu_x \left(\frac{U_{kj}}{V_{kj}} \right) - L \right| < \frac{1}{n-1} \right\},$$

where $n \geq 2$ ($n \in \mathbb{N}$). For each $x \in X$, note that $E_i \cap E_j = \emptyset$ ($i \neq j$) and $E_i \in \mathcal{I}_2^{\sigma\theta}$ (for each $i \in \mathbb{N}$). Since $\mathcal{I}_2^{\sigma\theta}$ satisfies condition (AP2), there exists a sequence of sets $\{F_n\}_{n \in \mathbb{N}}$ such that $E_i \Delta F_i$ is included in finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ (for each $i \in \mathbb{N}$) and $F = \left(\bigcup_{i=1}^{\infty} F_i \right) \in \mathcal{I}_2^{\sigma\theta}$.

Now, to complete the proof, it is enough to prove that for each $x \in X$

$$\lim_{\substack{k, j \rightarrow \infty \\ (k, j) \in M_2}} \mu_x \left(\frac{U_{kj}}{V_{kj}} \right) = L \tag{1}$$

where $M_2 = \mathbb{N} \times \mathbb{N} \setminus F$. Let $\gamma > 0$ is given. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \gamma$. Then, for each $x \in X$ we have

$$\left\{ (k, j) \in I_{ru} : \left| \mu_x \left(\frac{U_{kj}}{V_{kj}} \right) - L \right| \geq \gamma \right\} \subset \bigcup_{i=1}^n E_i.$$

Since $E_i \Delta F_i$ ($i = 1, 2, \dots$) are included in finite union of rows and columns, there exists $n_0 \in \mathbb{N}$ such that for each $x \in X$

$$\begin{aligned} & \left(\bigcup_{i=1}^n E_i \right) \cap \{ (k, j) : k \geq n_0 \wedge j \geq n_0 \} \\ &= \left(\bigcup_{i=1}^n F_i \right) \cap \{ (k, j) : k \geq n_0 \wedge j \geq n_0 \}. \end{aligned} \tag{2}$$

If $k, j > n_0$ and $(k, j) \notin F$, then

$$(k, j) \notin \bigcup_{i=1}^n F_i$$

and by (2), for each $x \in X$

$$(k, j) \notin \bigcup_{i=1}^n E_i.$$

This implies that for each $x \in X$

$$\left| \mu_x \left(\frac{U_{kj}}{V_{kj}} \right) - L \right| < \frac{1}{n} < \gamma$$

and so (1) holds. Consequently, $U_{kj} \overset{W(\mathcal{I}_2^{*\sigma\theta})}{\sim} V_{kj}$. □

Theorem 8 *If $\mathcal{I}_2^{\sigma\theta}$ has property (AP2), then*

$$U_{kj} \overset{W(\mathcal{I}_2^{\sigma\theta})}{\sim} V_{kj} \Leftrightarrow U_{kj} \overset{W(\mathcal{I}_2^{*\sigma\theta})}{\sim} V_{kj}.$$

Proof This is an immediate consequence of Theorem 6 and Theorem 7. □

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participant or animals performed by the authors.

References

Baronti M, Papini P (1986) Convergence of sequences of sets. In: Methods of Functional Analysis in Approximation Theory (pp.133–155). Birkhäuser, Basel

Beer G (1985) On convergence of closed sets in a metric space and distance functions. Bull. Aust. Math. Soc. 31(3):421–432

Beer G (1994) Wijsman convergence: a survey. Set-Valued Anal. 2(1–2):77–94

- Çakan C, Altay B, Mursaleen M (2006) The σ -convergence and σ -core of double sequences. *Appl. Math. Lett.* 19:1122–1128
- Das P, Kostyrko P, Wilczyński W, Malik P (2008) \mathcal{I} and \mathcal{I}^* -convergence of double sequences. *Math. Slovaca* 58(5):605–620
- Dündar E, Altay B (2014) \mathcal{I}_2 -convergence and \mathcal{I}_2 -Cauchy of double sequences. *Acta Math. Sci.* 34B(2):343–353
- Dündar E, Akın, Pancaroğlu, Wijsman N (2020) lacunary ideal invariant convergence of double sequences of sets. *Honam Math. J.*, 42(2), 345–358
- Esi A (2009) On asymptotically double lacunary statistically equivalent sequences. *Appl. Math. Lett.* 22:1781–1785
- Et M, Şengül H (2016) On (Δ^m, \mathcal{I}) -lacunary statistical convergence of order α . *J. Math. Anal.* 7(5):78–84
- Fridy JA, Orhan C (1993) Lacunary statistical convergence. *Pacific J. Math.* 160(1):43–51
- Hazarika B, Esi A, Braha NL (2013) On asymptotically Wijsman lacunary σ -statistical convergence of set sequences. *J. Math. Anal.* 4(3):33–46
- Hazarika B, Kumar V (2013) On asymptotically double lacunary statistical equivalent sequences in ideal context. *J. Ineq. Appl.*, 2013(543), 15 pages
- Hazarika B (2015) On asymptotically ideal equivalent sequences. *J. Egyptian Math. Soc.* 23(1):67–72
- Kara EE, Datan M, İlkan M (2017) On lacunary ideal convergence of some sequences. *New Trends Math. Sci.* 5(1):234–242
- Kişİ, Ö, Nuray F (2013) New convergence definitions for sequences of sets. *Abstr. Appl. Anal.*, 2013(Article ID 852796), 6 pages
- Kişİ, Ö, Nuray F (2013) $S_\lambda^I(\mathcal{I})$ -asymptotically statistical equivalence of sequence of sets. *ISRN Math. Anal.*, 2013(Article ID 602963), 6 pages
- Kostyrko P, Wilczyński W, Šalát T (2000) \mathcal{I} -convergence. *Real Anal. Exchange* 26(2):669–686
- Kumar V (2007) On \mathcal{I} and \mathcal{I}^* -convergence of double sequences. *Math. Commun.* 12:171–181
- Marouf M (1993) Asymptotic equivalence and summability. *Int. J. Math. Math. Sci.* 16(4):755–762
- Mursaleen M (1979) Invariant means and some matrix transformations. *Tamkang J. Math.* 10(2):183–188
- Mursaleen M (1983) Matrix transformation between some new sequence spaces. *Houston J. Math.* 9(4):505–509
- Mursaleen M, Edely OHH (2009) On the invariant mean and statistical convergence. *Appl. Math. Lett.* 22(11):1700–1704
- Nuray F, Rhoades BE (2012) Statistical convergence of sequences of sets. *Fasc. Math.* 49(2):87–99
- Nuray F, Dündar E, Ulusu U (2014) Wijsman \mathcal{I}_2 -convergence of double sequences of closed sets. *Pure Appl. Math. Lett.* 2:35–39
- Nuray F, Ulusu U, Dündar E (2016) Lacunary statistical convergence of double sequences of sets. *Soft Comput.* 20:2883–2888
- Nuray F, Patterson RF, Dündar E (2016) Asymptotically lacunary statistical equivalence of double sequences of sets. *Demonstratio Math.* 49(2):183–196
- Pancaroğlu N, Nuray F (2013) On invariant statistically convergence and lacunary invariant statistically convergence of sequences of sets. *Progress Appl. Math.* 5(2):23–29
- Pancaroğlu N, Nuray F (2013) Statistical lacunary invariant summability. *Theoretical Math. Appl.* 3(2):71–78
- Pancaroğlu N, Nuray F, Savaş E (2013) On asymptotically lacunary invariant statistical equivalent set sequences. *AIP Conf. Proc.* 1558(1):780–781
- Pancaroğlu Akın N, Dündar E, Ulusu U, lacunary, Wijsman (2020) \mathcal{I} -invariant convergence of sequences of sets. *Proc. Nat. Acad. Sci. India Sect. A.* <https://doi.org/10.1007/s40010-020-00694-w>
- Patterson RF (2002) Rates of convergence for double sequences. *South-east Asian Bull. Math.* 26(3):469–478
- Patterson RF, Savaş E (2005) Lacunary statistical convergence of double sequences. *Math. Commun.* 10:55–61
- Patterson RF, Savaş E (2006) On asymptotically lacunary statistically equivalent sequences. *Thai J. Math.* 4(2):267–272
- Pringsheim A (1900) Zur theorie der zweifach unendlichen Zahlenfolgen. *Math. Ann.* 53:289–321
- Raj K, Anand R (2018) Double difference spaces of almost null and almost convergent sequences for Orlicz function. *J. Comput. Anal. Appl.* 24(4):773–783
- Raj K, Choudhary A, Sharma C (2018) Almost strongly Orlicz double sequence spaces of regular matrices and their applications to statistical convergence. *Asian-Eur. J. Math.* 11(5):1850073
- Raj K, Jamwal S (2019) Difference sequence spaces for K-functions and its extension to double by a new matrix theoretic approach. *Ann. Acad. Rom. Sci. Ser. Math. Appl.* 11:5–23
- Savaş E (1989) Some sequence spaces involving invariant means. *Indian J. Math.* 31:1–8
- Savaş E (1990) On lacunary strong σ -convergence. *Indian J. Pure Appl. Math.* 21(4):359–365
- Savaş E, Nuray F (1993) On σ -statistically convergence and lacunary σ -statistically convergence. *Math. Slovaca.* 43(3):309–315
- Savaş E, Patterson RF (2006) σ -asymptotically lacunary statistical equivalent sequences. *Cent. Eur. J. Math.* 4(4):648–655
- Savaş E, Patterson RF (2009) Double σ -convergence lacunary statistical sequences. *J. Comput. Anal. Appl.* 11(4):610–615
- Sever Y, Ulusu U, Dündar E (2014) On strongly \mathcal{I} and \mathcal{I}^* -lacunary convergence of sequences of sets. *AIP Conf. Proc.* 1611(1):357–362
- Şengül H (2018) On Wijsman \mathcal{I} -lacunary statistical equivalence of order (η, μ) . *J. Ineq. Spec. Funct.* 9(2):92–101
- Şengül H, Et M, Işık M (2019) On \mathcal{I} -deferred statistical convergence of order α . *Filomat* 33(9):2833–2840
- Şengül, H., Çakallı, H. and Et, M. A variation on strongly ideal lacunary ward continuity. *Bol. Soc. Parana. Mat.* (3), 38(7) (2020), 99–108
- Tortop Ş, Dündar E (2018) Wijsman \mathcal{I}_2 -invariant convergence of double sequences of sets. *J. Ineq. Spec. Funct.* 9(4):90–100
- Ulusu U, Nuray F (2012) Lacunary statistical convergence of sequences of sets. *Progress Appl. Math.* 4(2):99–109
- Ulusu U, Nuray F (2013) On asymptotically lacunary statistical equivalent set sequences. *Journal of Math.*, 2013(Article ID 310438), 5 pages
- Ulusu U, Nuray F (2013) On strongly lacunary summability of sequences of sets. *J. Appl. Math. Bio.* 3(3):75–88
- Ulusu U, Savaş E (2014) An extension of asymptotically lacunary statistical equivalence set sequences. *J. Ineq. Appl.*, 2014(134), 8 pages
- Ulusu U, Dündar E (2016) Asymptotically \mathcal{I}_2 -lacunary statistical equivalence of double sequences of sets. *J. Ineq. Spec. Funct.* 7(2):44–56
- Ulusu U, Dündar E, Nuray F (2018) Lacunary \mathcal{I}_2 -invariant convergence and some properties. *Int. J. Anal. Appl.* 16(3):317–327
- Ulusu U, Dündar E (2019) Asymptotically lacunary \mathcal{I}_2 -invariant equivalence. *J. Intell. Fuzzy Systems* 36(1):467–472
- Ulusu U, Dündar E, Pancaroğlu Akın N. Lacunary invariant statistical equivalence for double set sequences. (in review)
- Wijsman RA (1964) Convergence of sequences of convex sets, cones and functions. *Bull. Amer. Math. Soc.* 70(1):186–188
- Wijsman RA (1966) Convergence of sequences of convex sets, cones and functions II. *Trans. Amer. Math. Soc.* 123(1):32–45