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RESEARCH ARTICLE

Wijsman Lacunary $\mathcal{I}$-Invariant Convergence of Sequences of Sets

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Abstract In this paper, we study the concepts of Wijsman lacunary $\mathcal{I}$-invariant convergence $\left(\mathcal{I}_{\sigma\theta}^{W}\right)$, Wijsman lacunary $\mathcal{I}^{*}$-invariant convergence $\left(\mathcal{I}_{\sigma\theta}^{*W}\right)$, Wijsman p-strongly lacunary invariant convergence $\left([WN_{\sigma\theta}]_p\right)$ of sequences of sets and investigate the relationships among Wijsman lacunary invariant convergence, $[WN_{\sigma\theta}]_p$, $\mathcal{I}_{\sigma\theta}^{W}$ and $\mathcal{I}_{\sigma\theta}^{*W}$. Also, we introduce the concepts of $\mathcal{I}_{\sigma\theta}^{W}$-Cauchy sequence and $\mathcal{I}_{\sigma\theta}^{*W}$-Cauchy sequence of sets.

Keywords Lacunary sequence · Invariant convergence · $\mathcal{I}$-convergence · Wijsman convergence · Cauchy sequence · Sequences of sets

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1 Introduction and Background

The concept of convergence of a sequence of real numbers $\mathbb{R}$ has been extended to statistical convergence independently by Fast [1], Schoenberg [2] and studied by many authors. Nuray and Ruckle [3] introduced it with another name: generalized statistical convergence. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [4] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of positive integers $\mathbb{N}$.

Nuray and Rhoades [5] extended convergence of set sequences to statistical convergence and gave some basic theorems. Ulusu and Nuray [6] defined the Wijsman lacunary statistical convergence of set sequences and considered its relation with Wiijsman statistical convergence defined by Nuray and Rhoades. Kişi and Nuray [7] introduced a new convergence notion for sequence of sets called Wijsman $\mathcal{I}$-convergence. The concept of convergence of sequence of numbers has been extended by several authors to convergence of set sequences [8–17].

Several authors including Raimi [18], Schaefer [19], Mursaleen [20], Pancaroğlu and Nuray [21] have studied invariant convergent sequences. Nuray et al. [22] defined the concepts of σ-uniform density of any subsets A of $\mathbb{N}$, $\mathcal{I}_{\sigma}$-convergence and investigated relationships between $\mathcal{I}_{\sigma}$-convergence and invariant convergence also $\mathcal{I}_{\sigma}$-convergence and $[V_{\sigma}]_p$-convergence. The concept of strongly σ-convergence was defined by Mursaleen [23]. Savaş and Nuray [24] introduced the concepts of σ-statistical convergence and lacunary σ-statistical convergence and gave some inclusion relations. The concept of strong σ-convergence was generalized by Savaş [25]. Recently, Nuray and Ulusu [26] investigated lacunary $\mathcal{I}$-invariant convergence and lacunary $\mathcal{I}$-invariant Cauchy sequence of real numbers. Pancaroğlu Akın et al. [27] studied Wijsman $\mathcal{I}$-invariant convergence of sequences of sets.

2 Definitions and Notations

Now, we recall the ideal convergence, invariant convergence, sequence of sets and basic definitions and concepts (see [4, 5, 13, 14, 18, 19, 21, 22, 26–30]).

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A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if(i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called nontrivial if $\mathbb{N} \notin \mathcal{I}$ and nontrivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is called a filter if and only if(i) $\emptyset \notin \mathcal{F}$, (ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, (iii) For each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$$

is a filter on X, called the filter associated with $\mathcal{I}$.

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if and only if

1. $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
2. $\phi(e) = 1$, where $e = (1, 1, 1, \ldots)$, and
3. $\phi\left(x_{\sigma(n)}\right) = \phi(x_n)$ for all $x \in \ell_\infty$.

The mappings σ are one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. In the case σ is translation mappings $\sigma(n) = n + 1$, the σ-mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences [31].

It has been shown [32] that

$$V_\sigma = \left\{ x = (x_n) \in \ell_\infty : \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L \right\}.$$

uniformly in n.

A bounded sequence $x = (x_k)$ is said to be strongly σ-convergent to L if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \left| x_{\sigma^k(m)} - L \right| = 0,$$

uniformly in m and in this case, it is denoted by $x_k \to L[V_\sigma]$. By $[V_\sigma]$, we denote the set of all strongly σ-convergent sequences.

In the case $\sigma(n) = n + 1$, the space $[V_\sigma]$ is the set of strongly almost convergent sequences $[\hat{c}]$.

Nuray et al. [22] introduced the concepts of σ-uniform density and $\mathcal{I}_\sigma$-convergence.

Let $A \subseteq \mathbb{N}$ and $s_n = \min_m \left| A \cap \left\{ \sigma(m), \sigma^2(m), \ldots, \sigma^n(m) \right\} \right|$ and $S_n = \max_m \left| A \cap \left\{ \sigma(m), \sigma^2(m), \ldots, \sigma^n(m) \right\} \right|$.

If the limits $\underline{V}(A) = \lim_{n \to \infty} \frac{s_n}{n}$, $\bar{V}(A) = \lim_{n \to \infty} \frac{S_n}{n}$ exists then, they are called a lower and an upper σ-uniform density of the set A, respectively. If $\underline{V}(A) = \bar{V}(A)$, then $V(A) = \underline{V}(A) = \bar{V}(A)$ is called the σ-uniform density of A.

Denote by $\mathcal{I}_\sigma$ the class of all $A \subseteq \mathbb{N}$ with $V(A) = 0$.

Throughout the paper, let (X, ρ) be a metric space, $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal and A, A_k be any non-empty closed subsets of X.

For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

A sequence $\{A_k\}$ is bounded if $\sup_k d(x, A_k) < \infty$, for each $x \in X$. L_∞ denotes the set of bounded sequences of sets.

A sequence $\{A_k\}$ is said to be Wijsman invariant statistical convergent to A, if for every $\varepsilon > 0$ and for each $x \in X$

$$\lim_{n \to \infty} \frac{1}{n} \left| \left\{ 0 \leq k \leq n : \left| d\left(x, A_{\sigma^k(m)}\right) - d(x, A) \right| \geq \varepsilon \right\} \right| = 0,$$

uniformly in m. In this case, we write $A_k \to A(WS_\sigma)$ and the set of all Wijsman invariant statistical convergent sequences of sets is denoted by WS_σ.

A sequence $\{A_k\}$ is said to be Wijsman $\mathcal{I}$-invariant convergent to A or $\mathcal{I}_\sigma^W$-convergent to A if for every $\varepsilon > 0$

$$A(\varepsilon, x) = \{k : |d(x, A_k) - d(x, A)| \geq \varepsilon\} \in \mathcal{I}_\sigma,$$

that is, $V(A(\varepsilon, x)) = 0$. In this case, we write $A_k \to A\left(\mathcal{I}_\sigma^W\right)$ and the set of all Wijsman $\mathcal{I}$-invariant convergent sequences of sets is denoted by $\mathcal{I}_\sigma^W$.

Let (X, ρ) be a separable metric space. The sequence $\{A_k\}$ is Wijsman $\mathcal{I}^*$-invariant convergent or $\mathcal{I}_\sigma^{*W}$-convergent to A if there exists a set $M = \{m_1 < \cdots < m_k < \cdots\} \in \mathcal{F}(\mathcal{I}_\sigma)$ such that for each $x \in X$,

$$\lim_{k \to \infty} d(x, A_{m_k}) = d(x, A).$$

A sequence $\{A_k\}$ is said to be Wijsman $\mathcal{I}$-invariant Cauchy sequence or $\mathcal{I}_\sigma^W$-Cauchy sequence if for every $\varepsilon > 0$ and for each $x \in X$, there exists a number $N = N(\varepsilon, x) \in \mathbb{N}$ such that

$$A(\varepsilon, x) = \{k : |d(x, A_k) - d(x, A_N)| \geq \varepsilon\} \in \mathcal{I}_\sigma,$$

that is, $V(A(\varepsilon, x)) = 0$.

A sequence $\{A_k\}$ is said to be Wijsman $\mathcal{I}^*$-invariant Cauchy sequence or $\mathcal{I}_\sigma^{*W}$-Cauchy sequence if there exists a set $M = \{m_1 < \ldots < m_k < \ldots\} \in \mathcal{F}(\mathcal{I}_\sigma)$ such that

$$\lim_{k,p \to \infty} \left| d(x, A_{m_k}) - d\left(x, A_{m_p}\right) \right| = 0,$$

for each $x \in X$.

A sequence $\{A_k\}$ is said to be Wijsman p-strongly invariant convergent to A, if for each $x \in X$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|d\left(x,A_{\sigma^k(m)}\right)-d(x,A)\right|^p=0,$$

uniformly in m, where $0<p<\infty$. In this case, we write $A_k\to A[WV_\sigma]_p$ and the set of all Wijsman p-strongly invariant convergent sequences of sets is denoted by $[WV_\sigma]_p$.

By a lacunary sequence we mean an increasing integer sequence $\theta=\{k_r\}$ such that $k_0=0$ and $h_r=k_r-k_{r-1}\to\infty$ as $r\to\infty$. The intervals determined by θ are denoted by $I_r=(k_{r-1},k_r]$.

The concept of lacunary strong σ-convergence was introduced by Savaş [32] as:

$$L_\theta=\left\{x=(x_k):\lim_r\frac{1}{h_r}\sum_{k\in I_r}\left|x_{\sigma^k(m)}-L\right|=0\right\},$$

uniformly in m.

Pancaroğlu and Nuray [21] defined the concept of lacunary invariant summability and the space $[V_{\sigma\theta}]_q$ as follows:

A sequence $x=(x_k)$ is said to be lacunary invariant summable to L if

$$\lim_r\frac{1}{h_r}\sum_{m\in I_r}x_{\sigma^m(n)}=L,$$

uniformly in $n=1,2,\ldots$.

A sequence $x=(x_k)$ is said to be strongly lacunary q-invariant convergent $(0<q<\infty)$ to L if

$$\lim_r\frac{1}{h_r}\sum_{m\in I_r}\left|x_{\sigma^m(n)}-L\right|^q=0,$$

uniformly in $n=1,2,\ldots$ and we write $x_k\to L(\left[V_{\sigma\theta}\right]_q)$.

Let $\theta=\{k_r\}$ be a lacunary sequence, $A\subseteq\mathbb{N}$ and $s_r=\min_n\{|A\cap\{\sigma^m(n):m\in I_r\}|\}$ and $S_r=\max_n\{|A\cap\{\sigma^m(n):m\in I_r\}|\}$.

If the limits $\underline{V}_\theta(A)=\lim_{r\to\infty}\frac{s_r}{h_r}$ and $\overline{V_\theta}(A)=\lim_{r\to\infty}\frac{S_r}{h_r}$ exist then, they are called a lower lacunary σ-uniform (lower $\sigma\theta$-uniform) density and an upper lacunary σ-uniform (upper $\sigma\theta$-uniform) density of the set A, respectively. If $\underline{V}_\theta(A)=\overline{V_\theta}(A)$, then $V_\theta(A)=\underline{V}_\theta(A)=\overline{V_\theta}(A)$ is called the lacunary σ-uniform density or $\sigma\theta$-uniform density of A.

Denoted by $\mathcal{I}_{\sigma\theta}$ the class of all $A\subseteq\mathbb{N}$ with $V_\theta(A)=0$.

A sequence (x_k) is said to be lacunary $\mathcal{I}_\sigma$-convergent to L or $\mathcal{I}_{\sigma\theta}$-convergent to L if for every $\varepsilon>0$, the set $A_\varepsilon=\{k:|x_k-L|\geq\varepsilon\}$ belongs to $\mathcal{I}_{\sigma\theta}$, i.e., $V_\theta(A_\varepsilon)=0$. In this case, we write $\mathcal{I}_{\sigma\theta}-\lim x_k=L$.

The set of all $\mathcal{I}_{\sigma\theta}$-convergent sequences is denoted by $\Im_{\sigma\theta}$.

A sequence $x=(x_k)$ is said to be $\mathcal{I}^*_{\sigma\theta}$-convergent to the number L if there exists a set $M=\{m_1<m_2<\cdots\}\in\mathcal{F}(\mathcal{I}_{\sigma\theta})$ such that $\lim_{k\to\infty}x_{m_k}=L$. In this case, we write $\mathcal{I}^*_{\sigma\theta}-\lim x_k=L$.

A sequence (x_k) is said to be lacunary $\mathcal{I}_\sigma$-Cauchy sequence or $\mathcal{I}_{\sigma\theta}$-Cauchy sequence if for every $\varepsilon>0$, there exists a number $N=N(\varepsilon)\in\mathbb{N}$ such that the set $A(\varepsilon)=\{k:|x_k-x_N|\geq\varepsilon\}$ belongs to $\mathcal{I}_{\sigma\theta}$, i.e., $V_\theta(A(\varepsilon))=0$.

A sequence $x=(x_k)$ is said to be $\mathcal{I}^*_{\sigma\theta}$-Cauchy sequences if there exists a set $M=\{m_1<m_2<\cdots<m_k<\cdots\}\in\mathcal{F}(\mathcal{I}_{\sigma\theta})$ such that $lim_{k,p\to\infty}\left|x_{m_k}-x_{m_p}\right|=0$.

A sequence $\{A_k\}$ is said to be Wijsman lacunary invariant convergent to A, for each $x\in X$

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}d\left(x,A_{\sigma^k(m)}\right)=d(x,A),$$

uniformly in m.

An admissible ideal $\mathcal{I}\subset 2^{\mathbb{N}}$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{E_1,E_2,\ldots\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{F_1,F_2,\ldots\}$ such that $E_j\Delta F_j$ is a finite set for $j\in\mathbb{N}$ and $F=\bigcup_{j=1}^{\infty}F_j\in\mathcal{I}$.

Several authors studied on ideal convergence and lacunary sequence (see [33–43]).

3 Main Results

In this section, we study the concepts of Wijsman Lacunary $\mathcal{I}$-invariant convergence $(\mathcal{I}^W_{\sigma\theta})$, Wijsman Lacunary $\mathcal{I}^*$-invariant convergence $(\mathcal{I}^{*W}_{\sigma\theta})$, Wijsman p-strongly Lacunary invariant convergence $\left([WN_{\sigma\theta}]_p\right)$ and investigate the relationships among Wijsman Lacunary invariant convergence, $[WN_{\sigma\theta}]_p$, $\mathcal{I}^W_{\sigma\theta}$ and $\mathcal{I}^{*W}_{\sigma\theta}$.

Definition 3.1 *Asequence $\{A_k\}$ is said to be Wijsman lacunary $\mathcal{I}$-invariant convergent or $\mathcal{I}^W_{\sigma\theta}$-convergent to A if for every $\varepsilon>0$ and for each $x\in X$, the set*

$$A(\varepsilon,x)=\{k:|d(x,A_k)-d(x,A)|\geq\varepsilon\}$$

belongs to $\mathcal{I}_{\sigma\theta}$, that is, $V_\theta(A(\varepsilon,x))=0$. In this case, we write $A_k\to A\left(\mathcal{I}^W_{\sigma\theta}\right)$ and the set of all Wijsman lacunary $\mathcal{I}$-invariant convergent sequences of sets is denoted by $\mathcal{I}^W_{\sigma\theta}$.

Theorem 3.1 *Let $\{A_k\}$ is bounded sequence. If $\{A_k\}$ is $\mathcal{I}^W_{\sigma\theta}$-convergent to A, then $\{A_k\}$ is Wijsman lacunary invariant convergent to A.*

Proof Let θ be a lacunary sequence, $m\in\mathbb{N}$ be arbitrary and $\varepsilon>0$. For each $x\in X$, we calculate

$$t(m,r,x):=\left|\frac{d\left(x,A_{\sigma(m)}\right)+d\left(x,A_{\sigma^2(m)}\right)+\ldots+d\left(x,A_{\sigma^n(m)}\right)}{h_r}-d(x,A)\right|.$$

Then, for each $x\in X$ we have

$$t(m,r,x)\leq t^1(m,r,x)+t^2(m,r,x),$$

where

$$t^1(m,r,x) := \frac{1}{h_r} \sum_{\substack{j \in I_r \\ |d(x,A_{\sigma^j(m)}) - d(x,A)| \geq \varepsilon}} |d(x,A_{\sigma^j(m)}) - d(x,A)|$$

and

$$t^2(m,r,x) := \frac{1}{h_r} \sum_{\substack{j \in I_r \\ |d(x,A_{\sigma^j(m)}) - d(x,A)| < \varepsilon}} |d(x,A_{\sigma^j(m)}) - d(x,A)|.$$

Therefore, we have $t^2(m,r,x) < \varepsilon$ for each $x \in X$ and for every $m = 1,2,\ldots$. The boundedness of $\{A_k\}$ implies that there exist $T > 0$ such that for each $x \in X$

$$|d(x,A_{\sigma^j(m)}) - d(x,A)| \leq T, (j \in I_r; m = 1,2\ldots)$$

and this implies that

$$\begin{aligned} t^1(m,r,x) &\leq \frac{T}{h_r} |\{j \in I_r : |d(x,A_{\sigma^j(m)}) - d(x,A)| \geq \varepsilon\}| \\ &\leq T \frac{\max_m |\{j \in I_r : |d(x,A_{\sigma^j(m)}) - d(x,A)| \geq \varepsilon\}|}{h_r} \\ &= T\frac{S_r}{h_r}. \end{aligned}$$

Hence, $\{A_k\}$ is Wijsman lacunary invariant convergent to A.

Definition 3.2 *Let (X,ρ) be a separable metric space. A sequence $\{A_k\}$ is Wijsman lacunary $\mathcal{I}^*$-invariant convergent or $\mathcal{I}^{*W}_{\sigma\theta}$-convergent to A if there exists a set $M = \{m_1 < \cdots < m_k < \cdots\} \in F(\mathcal{I}_{\sigma\theta})$ such that for each $x \in X$,*

$$\lim_{k\to\infty} d(x,A_{m_k}) = d(x,A).$$

Theorem 3.2 *If a sequence $\{A_k\}$ is $\mathcal{I}^{*W}_{\sigma\theta}$-convergent to A, then $\{A_k\}$ is $\mathcal{I}^{W}_{\sigma\theta}$-convergent to A.*

Proof By assumption, there exists a set $H \in \mathcal{I}_{\sigma\theta}$ such that for $M = \mathbb{N}\backslash H = \{m_1 < \cdots < m_k < \cdots\}$ we have

$$\lim_{k\to\infty} d(x,A_{m_k}) = d(x,A), \tag{3.1}$$

for each $x \in X$. Let $\varepsilon > 0$. By Eq. 3.1, there exists $k_0 \in \mathbb{N}$ such that for each $x \in X$

$$|d(x,A_{m_k}) - d(x,A)| < \varepsilon,$$

for each $k > k_0$. Then, obviously

$$\{k \in \mathbb{N} : |d(x,A_k) - d(x,A)| \geq \varepsilon\} \subset H \cup \{m_1 < m_2 < \cdots < m_{k_0}\}. \tag{3.2}$$

Since $\mathcal{I}_{\sigma\theta}$ is admissible, the set on the right-hand side of Eq. (3.2) belongs to $\mathcal{I}_{\sigma\theta}$. So $\{A_k\}$ is $\mathcal{I}^{W}_{\sigma\theta}$-convergent to A.

Theorem 3.3 *Let $\mathcal{I}_{\sigma\theta} \subset 2^{\mathbb{N}}$ be an admissible ideal with the property (AP). If $\{A_k\}$ is $\mathcal{I}^{W}_{\sigma\theta}$- convergent to A, then $\{A_k\}$ is $\mathcal{I}^{*W}_{\sigma\theta}$-convergent to A.*

Proof Suppose that $\mathcal{I}_{\sigma\theta}$ satisfies condition (AP) and $\{A_k\}$ is $\mathcal{I}^{W}_{\sigma\theta}$-convergent to A. Then, for every $\varepsilon > 0$ and for each $x \in X$

$$\{k : |d(x,A_k) - d(x,A)| \geq \varepsilon\} \in \mathcal{I}_{\sigma\theta}.$$

Put $E_1 = \{k : |d(x,A_k) - d(x,A)| \geq 1\}$ and $E_n = \{k : \frac{1}{n} \leq |d(x,A_k) - d(x,A)| < \frac{1}{n-1}\}$, for $n \geq 2$ and for each $x \in X$. Obviously $E_i \cap E_j = \emptyset$, for $i \neq j$. By condition (AP) there exists a sequence of $\{F_n\}_{n\in\mathbb{N}}$ such that $E_j \Delta F_j$ are finite sets for $j \in \mathbb{N}$ and $F = \left(\bigcup_{j=1}^{\infty} F_j\right) \in \mathcal{I}_{\sigma\theta}$. It is sufficient to prove that for $M = \mathbb{N}\backslash F$ and for each $x \in X$, we have

$$\lim_{k\to\infty} d(x,A_k) = d(x,A), (k \in M). \tag{3.3}$$

Let $\lambda > 0$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n+1} < \lambda$. Then, for each $x \in X$

$$\{k : |d(x,A_k) - d(x,A)| \geq \lambda\} \subset \bigcup_{j=1}^{n+1} E_j.$$

Since $E_j \Delta F_j$, $j = 1,2,\ldots,n+1$ are finite sets, there exists $k_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{n+1} F_j\right) \cap \{k : k > k_0\} = \left(\bigcup_{j=1}^{n+1} E_j\right) \cap \{k : k > k_0\}. \tag{3.4}$$

If $k > k_0$ and $k \notin F$, then $k \notin \bigcup_{j=1}^{n+1} F_j$ and by Eq. (3.4) $k \notin \bigcup_{j=1}^{n+1} E_j$. But then

$$|d(x,A_k) - d(x,A)| < \frac{1}{n+1} < \lambda,$$

So Eq. (3.3) holds and $\{A_k\}$ is $\mathcal{I}^{*W}_{\sigma\theta}$-convergent to A.

Now, we define the concepts of Wijsman lacunary $\mathcal{I}$-invariant Cauchy sequence and Wijsman lacunary $\mathcal{I}^*$-invariant Cauchy sequence of sets.

Definition 3.3 *Asequence $\{A_k\}$ is said to be Wijsman lacunary $\mathcal{I}$-invariant Cauchy sequence or $\mathcal{I}^{W}_{\sigma\theta}$-Cauchy sequence if for every $\varepsilon > 0$ and for each $x \in X$, there exists a number $N = N(\varepsilon,x) \in \mathbb{N}$ such that*

$$A(\varepsilon,x) = \{k : |d(x,A_k) - d(x,A_N)| \geq \varepsilon\} \in \mathcal{I}_{\sigma\theta},$$

that is, $V_\theta(A(\varepsilon,x)) = 0$.

Definition 3.4 *Asequence $\{A_k\}$ is said to be Wijsman lacunary $\mathcal{I}^*$-invariant Cauchy sequence or $\mathcal{I}^{*W}_{\sigma\theta}$-Cauchy sequence if there exists a set $M = \{m_1 < \ldots < m_k < \ldots\} \in F(\mathcal{I}_{\sigma\theta})$ such that*

$$\lim_{k,p\to\infty} |d(x,A_{m_k}) - d(x,A_{m_p})| = 0,$$

for each $x \in X$.

We give following theorems which show relationships among $\mathcal{I}_{\sigma\theta}^{W}$-convergence, $\mathcal{I}_{\sigma\theta}^{W}$-Cauchy sequence and $\mathcal{I}_{\sigma\theta}^{*W}$-Cauchy sequence. The proof of them are similar to the proof of Theorems in [44, 45], so we omit them.

Theorem 3.4 *If a sequence $\{A_k\}$ is $\mathcal{I}_{\sigma\theta}^{W}$-convergent, then $\{A_k\}$ is $\mathcal{I}_{\sigma\theta}^{W}$-Cauchy sequence.*

Theorem 3.5 *If a sequence $\{A_k\}$ is $\mathcal{I}_{\sigma\theta}^{*W}$-Cauchy sequence, then $\{A_k\}$ is $\mathcal{I}_{\sigma\theta}^{W}$-Cauchy sequence.*

Theorem 3.6 *Let $\mathcal{I}_{\sigma\theta}$ has property (AP). Then the concepts $\mathcal{I}_{\sigma\theta}^{W}$-Cauchy sequence and $\mathcal{I}_{\sigma\theta}^{*W}$-Cauchy sequence coincides.*

Definition 3.5 *The sequence $\{A_k\}$ is said to be Wijsman p-strongly lacunary invariant convergent to A, if for each $x \in X$,*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left| d\left(x, A_{\sigma^k(m)}\right) - d(x,A) \right| = 0,$$

uniformly in m, where $0<p<\infty$. In this case, we write $A_k \to A[WN_{\sigma\theta}]_p$ and the set of all Wijsman p-strongly lacunary invariant convergent sequences of sets is denoted by $[WN_{\sigma\theta}]_p$.

Theorem 3.7 *Let $\mathcal{I}_{\sigma\theta} \subset 2^{\mathbb{N}}$ be an admissible ideal and $0<p<\infty$.*

(i) *If $A_k \to A\left(\left[WN_{\sigma\theta}\right]_p\right)$, then $A_k \to A\left(\mathcal{I}_{\sigma\theta}^{W}\right)$.*
(ii) *If $\{A_k\} \in L_\infty$ and $A_k \to A\left(\mathcal{I}_{\sigma\theta}^{W}\right)$, then $A_k \to A\left(\left[WN_{\sigma\theta}\right]_p\right)$.*
(iii) *If $\{A_k\} \in L_\infty$, then $\{A_k\}$ is $\mathcal{I}_{\sigma\theta}^{W}$-convergent to A if and only if $A_k \to A\left(\left[WN_{\sigma\theta}\right]_p\right)$.*

Proof (i): If $A_k \to A\left(\left[WN_{\sigma\theta}\right]_p\right)$, then for $\varepsilon > 0$ and for each $x \in X$ we write

$$\sum_{j\in I_r} \left| d\left(x, A_{\sigma^j(m)}\right) - d(x,A) \right|^p \geq \sum_{\substack{j \in I_r \\ \left| d\left(x, A_{\sigma^j(m)}\right) - d(x,A) \right| \geq \varepsilon}} \left| d\left(x, A_{\sigma^j(m)}\right) - d(x,A) \right|^p$$
$$\geq \varepsilon^p \left| \left\{ j \in I_r : \left| d\left(x, A_{\sigma^j(m)}\right) - d(x,A) \right| \geq \varepsilon \right\} \right|$$
$$\geq \varepsilon^p \max_m \left| \left\{ j \in I_r : \left| d\left(x, A_{\sigma^j(m)}\right) - d(x,A) \right| \geq \varepsilon \right\} \right|$$

and so

$$\frac{1}{h_r} \sum_{j\in I_r} \left| d\left(x, A_{\sigma^j(m)}\right) - d(x,A) \right|^p \geq \varepsilon^p \frac{\max_m \left| \left\{ j \in I_r : \left| d\left(x, A_{\sigma^j(m)}\right) - d(x,A) \right| \geq \varepsilon \right\} \right|}{h_r}$$
$$= \varepsilon^p \frac{S_r}{r}$$

for every $m = 1, 2, \ldots$. This implies $\lim_{r\to\infty} \frac{S_r}{r} = 0$ and therefore, $\{A_k\}$ is $\left(\mathcal{I}_{\sigma\theta}^{W}\right)$-convergent to A.

(ii): Suppose that $\{A_k\} \in L_\infty$ and $A_k \to A\left(\mathcal{I}_{\sigma\theta}^{W}\right)$. Let $\varepsilon > 0$. By assumption we have $V_\theta(A(\varepsilon, x)) = 0$. Since $\{A_k\}$ is bounded, there exist $T > 0$ such that for each $x \in X$

for all j and m. Then, we have

$$\frac{1}{h_r} \sum_{j\in I_r} \left| d\left(x, A_{\sigma^j(m)}\right) - d(x,A) \right|^p$$
$$= \frac{1}{h_r} \sum_{\substack{j \in I_r \\ \left| d\left(x, A_{\sigma^j(m)}\right) - d(x,A) \right| \geq \varepsilon}} \left| d\left(x, A_{\sigma^j(m)}\right) - d(x,A) \right|^p$$
$$+ \frac{1}{h_r} \sum_{\substack{j \in I_r \\ \left| d\left(x, A_{\sigma^j(m)}\right) - d(x,A) \right| < \varepsilon}} \left| d\left(x, A_{\sigma^j(m)}\right) - d(x,A) \right|^p$$
$$\leq T \frac{\max_m \left| \left\{ j \in I_r : \left| d\left(x, A_{\sigma^j(m)}\right) - d(x,A) \right| \geq \varepsilon \right\} \right|}{h_r} + \varepsilon^p$$
$$\leq T \frac{S_r}{r} + \varepsilon^p,$$

for each $x \in X$. Hence, for each $x \in X$ we obtain

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{j\in I_r} \left| d\left(x, A_{\sigma^j(m)}\right) - d(x,A) \right|^p = 0,$$

uniformly in m.

(iii): This is immediate consequence of *(i)* and *(ii)*.

Now, we state a theorem that gives a relationship between $WS_{\sigma\theta}$ and $\mathcal{I}_{\sigma\theta}^{W}$.

Theorem 3.8 *Asequence $\{A_k\}$ is $WS_{\sigma\theta}$-convergent to A if and only if it is $\mathcal{I}_{\sigma\theta}^{W}$-convergent to A.*

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