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Research article

Characterizations for totally geodesic submanifolds of a K-paracontact manifold

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Abstract: The aim of the present paper is to study pseudoparallel invariant submanifolds of a K-paracontact metric manifold. We consider pseudoparallel, Ricci-generalized pseudoparallel and 2-Ricci generalized pseudo parallel invariant submanifolds of a K-paracontact manifold and we obtain new results. We think contributes to providing some new and interesting results in the area of geometric structures on manifolds geometry.

Keywords: K-paracontact manifold; pseudoparallel; Ricci-generalized pseudoparallel; 2-pseudoparallel submanifolds
Mathematics Subject Classification: 53C15, 53C44, 53D10

1. Introduction

The geometry of almost paracontact manifolds is a natural extension of the almost paraHermitian geometry. The study of almost paracontact metric manifolds started with [6]. A systematic study of almost paracontact metric manifolds was considered by Zamkovoy [7]. Almost paracontact metric manifolds have been extensively studied under several points of view in [6–12]. Also, invariant and anti-invariant submanifolds have been studied under several points of view in [13–16]. Similarly, in [17–23], Pishcoran, Mishra and other researchers have provided us with many studies that will shed light on these issues.

Many geometers studied paracontact metric manifolds and researched some important properties of these manifolds. The geometry of paracontact metric manifolds can be related to the theory of Legendre foliations. In [8], the authors introduced the class of paracontact metric manifolds for which the characteristic vector field ξ belongs to the (κ,μ)-nullity condition for some real constants κ and μ. Such manifolds are also known as (κ,μ)-paracontact metric manifolds.

Also, invariant submanifolds are used to discuss the properties of non-linear antronomous systems. Also, totally geodesic submanifolds play an important role in the relativity theory even though they are the simplest submanifolds.

Pseudoparallel submanifolds have been studied intensively by many geometers [1–5].

In this article motivated by the above studies, the pseudoparallel submanifolds of the $K-$paracontact metric manifold, which have not been tried until now, have been studied. Also, we obtain some necessary and sufficient conditions that an invariant submanifold to be pseudoparallel, generalized Ricci-pseudoparallel, 2-pseudoparallel, and 2-Ricci-generalized pseudoparallel under some conditions.

A $(2n+1)$-dimensional smooth manifold $\widetilde{M}^{2n+1}$ has an almost paracontact structure (φ, ξ, η) if it admits a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η satisfying the following conditions;

$$\varphi^2 = I - \eta \otimes \xi, \;\; \eta(\xi) = 1, \;\; \varphi\xi = 0, \;\; \eta \circ \varphi = 0. \tag{1}$$

If an almost paracontact manifold is endowed with a semi-Riemannian metric tensor g such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2}$$

for all vector fields X, Y on $\widetilde{M}^{2n+1}$, then $\widetilde{M}^{2n+1}(\varphi, \xi, \eta, g)$ is said to be almost paracontact metric manifold. The fundamental 2-form Φ of an almost paracontact metric manifold $\widetilde{M}^{2n+1}(\varphi, \xi, \eta, g)$ is defined by $\Phi(X, Y) = g(X, \varphi Y)$. If $d\eta = \Phi$, then almost paracontact metric manifold $\widetilde{M}^{2n+1}(\varphi, \xi, \eta, g)$ is called paracontact metric manifold. In addition, if ξ is a Killing vector field (equivalently $\ell_\xi = 0$, where ℓ denote the Lie-derivative), then $\widetilde{M}^{2n+1}(\varphi, \xi, \eta, g)$ is called a K-paracontact manifold. In a K-paracontact manifold, we have the following formulas.

$$(\widetilde{\nabla}_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X, \tag{3}$$

$$\widetilde{\nabla}_X \xi = -\varphi X, \tag{4}$$

$$\widetilde{R}(X, \xi)\xi = -X + \eta(X)\xi, \tag{5}$$

$$S(X, \xi) = -2n\eta(X), \tag{6}$$

for any vector fields X, Y on $\widetilde{M}^{2n+1}$, where $\widetilde{\nabla}$ is the Levi-Civita connection, $\widetilde{R}$ and S denote the Riemannian curvature tensor and Ricci tensor of $\widetilde{M}^{2n+1}$, respectively.

Now, let M be an immersed submanifold of a paracontact metric manifold $\widetilde{M}^{2n+1}$. By $\Gamma(TM)$ and $\Gamma(T^\perp M)$, we denote the tangent and normal subspaces of M in $\widetilde{M}$. Then the Gauss and Weingarten formulae are, respectively, given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{7}$$

and

$$\widetilde{\nabla}_X V = -A_V X + \nabla^\perp_X V, \tag{8}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, where ∇ and $\nabla^\perp$ are the connections on M and $\Gamma(T^\perp M)$ and σ and A are called the second fundamental form and shape operator of M, respectively. They are related by

$$g(A_V X, Y) = g(\sigma(X, Y), V). \tag{9}$$

The covariant derivative of σ is defined by

$$(\widetilde{\nabla}_X\sigma)(Y,Z) = \nabla^{\perp}_X\sigma(Y,Z) - \sigma(\nabla_X Y,Z) - \sigma(Y,\nabla_X Z), \tag{10}$$

for all $X, Y, Z \in \Gamma(TM)$. If $\widetilde{\nabla}\sigma = 0$, then the submanifold M is said to be its second fundamental form is parallel.

By R, we denote the Riemannian curvature tensor of M, we have the following Gauss equation

$$\begin{aligned}\widetilde{R}(X,Y)Z = \quad & R(X,Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X \\ & +(\widetilde{\nabla}_X\sigma)(Y,Z) - (\widetilde{\nabla}_Y\sigma)(X,Z),\end{aligned} \tag{11}$$

For a $(0,k)$-type tensor field T, $k \geq 1$ and a $(0,2)$-type tensor field A on a Riemannian manifold (M,g), $Q(A,T)$-tensor field is defined by

$$\begin{aligned}Q(A,T)(X_1,X_2,...,X_k;X,Y) = \quad & -T((X\Lambda_A Y)X_1,X_2,...,X_k)... \\ & -T(X_1,X_2,...X_{k-1},(X\Lambda_A Y)X_k),\end{aligned} \tag{12}$$

for all $X_1, X_2, ..., X_k, X, Y \in \Gamma(TM)$, where

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y. \tag{13}$$

Definition 1. *A submanifold of a Riemannian manifold (M,g) is said to be pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel and 2-Ricci-generalized pseudoparallel if*

$$\begin{aligned}&\widetilde{R}\cdot\sigma \;\; and \;\; Q(g,\sigma)\\ &\widetilde{R}\cdot\widetilde{\nabla}\sigma \;\; and \;\; Q(g,\widetilde{\nabla}\sigma)\\ &\widetilde{R}\cdot\sigma \;\; and \;\; Q(S,\sigma)\\ &\widetilde{R}\cdot\widetilde{\nabla}\sigma \;\; and \;\; Q(S,\widetilde{\nabla}\sigma)\end{aligned}$$

are linearly dependent, respectively.

Equivalently, these can be expressed by the following equations;

$$\widetilde{R}\cdot\sigma = L_1 Q(g,\sigma), \tag{14}$$

$$\widetilde{R}\cdot\widetilde{\nabla}\sigma = L_2 Q(g,\widetilde{\nabla}\sigma), \tag{15}$$

$$\widetilde{R}\cdot\sigma = L_3 Q(S,\sigma), \tag{16}$$

$$\widetilde{R}\cdot\widetilde{\nabla}\sigma = L_4 Q(S,\widetilde{\nabla}\sigma), \tag{17}$$

where L_1, L_2, L_3 and L_4 are, respectively, functions defined on
$M_1 = \{x \in M : \sigma(x) \neq g(x)\}$, $M_2 = \{x \in M : \widetilde{\nabla}\sigma(x) \neq g(x)\}$,
$M_3 = \{x \in M : S(x) \neq \sigma(x)\}$ and $M_4 = \{x \in M : S(x) \neq \widetilde{\nabla}\sigma(x)\}$.

Particularly, if $L_1 = 0$, then the submanifold is said to be semiparallel, if $L_2 = 0$, the submanifold is said to be 2-semiparallel.

2. Invariant pseudoparallel submanifolds of K-paracontact manifold

Now, we will investigate the above cases for the invariant submanifold M of a K-paracontact manifold $\widetilde{M}^{2n+1}(\varphi, \xi, \eta, g)$.

Now, let M be an immersed submanifold of a K-paracontact metric manifold $\widetilde{M}^{2n+1}(\varphi, \xi, g, \eta)$. If $\varphi(T_xM) \subseteq T_xM$, for each point $x \in M$, then M is said to be an invariant submanifold. We note that all of the properties of an invariant submanifold are inherited by the ambient manifold.

In the rest of this paper, we will assume that M is an invariant submanifold of a K-paracontact manifold $\widetilde{M}^{2n+1}(\varphi, \xi, \eta, g)$. Thus by using (3) and (7), we have

$$\sigma(X, \xi) = 0, \quad \sigma(\varphi X, Y) = \sigma(X, \varphi Y) = \varphi\sigma(X, Y), \tag{18}$$

for all $X, Y \in \Gamma(TM)$.

Lemma 1. *Let M be an invariant submanifold of a K-paracontact manifold $\widetilde{M}$. The second fundamental form σ of M is parallel if and only if M is totally geodesic.*

Proof. Let us assume that σ is parallel. Then

$$(\widetilde{\nabla}_X\sigma)(Y, Z) = \nabla^{\perp}_X\sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) = 0,$$

for all $X, Y, Z \in \Gamma(TM)$. Here, taking $Z = \xi$, by virtue of (4) and (18), we obtain

$$-\sigma(\nabla_X\xi, Y) = -\sigma(-\varphi X, Y) = \varphi\sigma(X, Y) = 0.$$

This proves our assertion. The converse is obvious. □

Lemma 1 is important for later theorems.

Theorem 1. *Let M be an invariant pseudoparallel submanifold of a K-paracontact manifold $\widetilde{M}^{2n+1}(\varphi, \xi, \eta, g)$. Then M is either totally geodesic or $L_1 = -1$.*

Proof. Let M be pseudoparallel, then from (14) we have

$$(\widetilde{R}(X, Y) \cdot \sigma)(U, V) = L_1 Q(g, \sigma)(U, V; X, Y),$$

for all $X, Y, U, V \in \Gamma(TM)$. This leads to

$$\begin{aligned} &R^{\perp}(X, Y)\sigma(U, V) - \sigma(R(X, Y)U, V) - \sigma(U, R(X, Y)V) \\ &= -L_1\{\sigma((X \wedge_g Y)U, V) + \sigma(U, (X \wedge_g Y)V)\} \\ &= -L_1\{\sigma(g(Y, U)X - g(X, U)Y, V) \\ &+\sigma(U, g(Y, V)X - g(X, V)Y)\} \end{aligned} \tag{19}$$

for all $X, Y, U, V \in \Gamma(TM)$. Taking $V = \xi$ in (19) and by using (18), we obtain

$$\sigma(R(X, Y)\xi, U) = L_1\{\eta(Y)\sigma(X, U) - \eta(X)\sigma(U, Y)\}.$$

Again taking $Y = \xi$ and making use of (5), we conclude that

$$
\begin{aligned}
L_1\sigma(X, U) &= \sigma(R(X, \xi)\xi, U) \\
&= \sigma(-X + \eta(X)\xi, U) = -\sigma(X, U).
\end{aligned}
$$

This completes the proof. □

From the Theorem 1, we have the following corollary.

Corollary 1. *Let M be an invariant pseudoparallel submanifold of a K-paracontact manifold $\widetilde{M}^{2n+1}(\varphi, \xi, \eta, g)$. Then M is semiparallel if and only if M is totally geodesic.*

Theorem 2. *Let M be an invariant 2-pseudoparallel submanifold of a K-paracontact metric manifold $\widetilde{M}^{2n+1}(\varphi, \xi, \eta, g)$. Then M is either totally geodesic or $L_2 = -1$.*

Proof. Let M be 2-pseudoparallel. Then from (15), we have

$$
(\widetilde{R}(X, Y) \cdot \widetilde{\nabla}\sigma)(U, V, Z) = L_2 Q(g, \widetilde{\nabla}\sigma)(U, V, Z; X, Y),
$$

for all $X, Y, U, V, Z \in \Gamma(TM)$. This means that

$$
\begin{aligned}
&R^{\perp}(X, Y)(\widetilde{\nabla}_U\sigma)(V, Z) - (\widetilde{\nabla}_{R(X,Y)U}\sigma)(V, Z) \\
&-(\widetilde{\nabla}_U\sigma)(R(X, Y)V, Z) - (\widetilde{\nabla}_U\sigma)(V, R(X, Y)Z) \\
&= -L_2\{(\widetilde{\nabla}_{(X\wedge_g Y)U}\sigma)(V, Z) \\
&+(\widetilde{\nabla}_U\sigma)((X \wedge_g Y)V, Z) \\
&+(\widetilde{\nabla}_U\sigma)(V, (X \wedge_g Y)Z)\},
\end{aligned}
$$

that is,In the last equality, taking $X = Z = \xi$, we obtain

$$
\begin{aligned}
&R^{\perp}(\xi, Y)(\widetilde{\nabla}_U\sigma)(V, \xi) - (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(V, \xi) \\
&-(\widetilde{\nabla}_U\sigma)(R(\xi, Y)V, \xi) - (\widetilde{\nabla}_U\sigma)(V, R(\xi, Y)\xi) \\
&= -L_2\{g(Y, U)(\widetilde{\nabla}_\xi\sigma)(V, \xi) \qquad (20)\\
&-\eta(U)(\widetilde{\nabla}_Y\sigma)(V, \xi) + (\widetilde{\nabla}_U\sigma)(g(Y, V)\xi \\
&-\eta(V)Y, \xi) + (\widetilde{\nabla}_U\sigma)(V, \eta(Y)\xi - Y)\}.
\end{aligned}
$$

Now, let's calculate each of these expressions. From (4), (10) and (18), we obtain

$$
\begin{aligned}
R^{\perp}(\xi, Y)(\widetilde{\nabla}_U\sigma)(V,\xi) &= R^{\perp}(\xi, Y)\{\nabla^{\perp}_U\sigma(V,\xi) - \sigma(\nabla_U V,\xi) - \sigma(V,\nabla_U\xi)\} \\
&= R^{\perp}(\xi, Y)\{-\sigma(V,\nabla_U\xi)\} \\
&= -R^{\perp}(\xi, Y)\sigma(V,-\varphi U) \\
&= R^{\perp}(\xi, Y)\sigma(V,\varphi U) = R^{\perp}(\xi, Y)\varphi\sigma(V,U)
\end{aligned} \tag{21}
$$

Moreover, taking into account (4) and (18), we have

$$
\begin{aligned}
(\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(V,\xi) &= \nabla^{\perp}_{R(\xi,Y)U}\sigma(V,\xi) - \sigma(\nabla_{R(\xi,Y)U}V,\xi) \\
&-\sigma(\nabla_{R(\xi,Y)U}\xi, V) \\
&= -\sigma(-\varphi R(\xi,Y)U, V) \\
&= \varphi\sigma(R(\xi,Y)U, V).
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)(R(\xi,Y)V,\xi) &= \nabla^{\perp}_U\sigma(R(\xi,Y)V,\xi) - \sigma(\nabla_U R(\xi,Y)V,\xi) \\
&-\sigma(R(\xi,Y)V,\nabla_U\xi) \\
&= -\sigma(-\varphi U, R(\xi,Y)V) \\
&= \varphi\sigma(U, R(\xi,Y)V).
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)(V, R(\xi,Y)\xi) &= (\widetilde{\nabla}_U\sigma)(V, Y - \eta(Y)\xi) \\
&= (\widetilde{\nabla}_U\sigma)(V,Y) - (\widetilde{\nabla}_U\sigma)(V,\eta(Y)\xi) \\
&= (\widetilde{\nabla}_U\sigma)(V,Y) - \nabla^{\perp}_U\sigma(V,\eta(Y)\xi) \\
&+\sigma(\nabla_U V,\eta(Y)\xi) + \sigma(V,\nabla_U\eta(Y)\xi) \\
&= (\widetilde{\nabla}_U\sigma)(V,Y) + \sigma(V, U\eta(Y)\xi + \eta(Y)\nabla_U\xi) \\
&= (\widetilde{\nabla}_U\sigma)(V,Y) - \eta(Y)\varphi\sigma(V,U).
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
(\widetilde{\nabla}_{(\xi\wedge_g Y)U}\sigma)(V,\xi) &= \nabla^{\perp}_{(\xi\wedge_g Y)U}\sigma(V,\xi) - \sigma(\nabla_{(\xi\wedge_g Y)U}V,\xi) \\
&-\sigma(V,\nabla_{(\xi\wedge_g Y)U}\xi) \\
&= -\sigma(V,\nabla_{g(Y,U)\xi-\eta(U)Y}\xi) \\
&= -\sigma(V,-\varphi(g(Y,U)\xi-\eta(U)Y)) \\
&= -\sigma(V,\eta(U)\varphi Y) = -\eta(U)\varphi\sigma(V,Y).
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)((\xi\wedge_g Y)V,\xi) &= \nabla^{\perp}_U\sigma((\xi\wedge_g Y)V,\xi) - \sigma(\nabla_U(\xi\wedge_g Y)V,\xi) \\
&-\sigma((\xi\wedge_g Y)V,\nabla_U\xi) \\
&= -\sigma(g(Y,V)\xi-\eta(V)Y,-\varphi U) \\
&= -\eta(V)\varphi\sigma(Y,U).
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)(V,(\xi\wedge_g Y)\xi) &= (\widetilde{\nabla}_U\sigma)(V,\eta(Y)\xi - Y) \\
&= (\widetilde{\nabla}_U\sigma)(V,\eta(Y)\xi) - (\widetilde{\nabla}_U\sigma)(V,Y) \\
&= \nabla^{\perp}_U\sigma(V,\eta(Y)\xi) - \sigma(\nabla_U V,\eta(Y)\xi) \\
&-\sigma(V,\nabla_U\eta(Y)\xi) - (\widetilde{\nabla}_U\sigma)(V,Y) \\
&= -\sigma(V,U\eta(Y)\xi + \eta(Y)\nabla_U\xi) - (\widetilde{\nabla}_U\sigma)(V,Y) \\
&= -\eta(Y)\sigma(V,-\varphi U) - (\widetilde{\nabla}_U\sigma)(V,Y) \\
&= \eta(Y)\varphi\sigma(V,U) - (\widetilde{\nabla}_U\sigma)(V,Y).
\end{aligned} \tag{27}
$$

Consequently, if we put (21), (22), (23), (24), (25), (26) and (27) in (20), we reach at

$$
\begin{aligned}
&R^{\perp}(\xi,Y)\varphi\sigma(U,V) - \varphi\sigma(R(\xi,Y)U,V) \\
&-\varphi\sigma(U,R(\xi,Y)V) - (\widetilde{\nabla}_U\sigma)(V,Y) + \eta(Y)\varphi\sigma(V,U) \\
&= -L_2\{-\eta(U)\varphi\sigma(V,Y) - \eta(V)\varphi\sigma(Y,U) \\
&+\eta(Y)\varphi\sigma(V,U) - (\widetilde{\nabla}_U\sigma)(V,Y)\}.
\end{aligned} \tag{28}
$$

If ξ is taken of V at (28), considering (18) and (5), we get

$$\begin{aligned} &-\varphi\sigma(U, R(\xi, Y)\xi) - (\widetilde{\nabla}_U\sigma)(\xi, Y) \\ &= L_2\{\varphi\sigma(U, Y) + (\widetilde{\nabla}_U\sigma)(\xi, Y)\}, \end{aligned} \tag{29}$$

where

$$\begin{aligned} (\widetilde{\nabla}_U\sigma)(\xi, Y) &= \nabla^{\perp}_U\sigma(Y, \xi) - \sigma(\nabla_U Y, \xi) - \sigma(Y, \nabla_U\xi) \\ &= -\sigma(-\varphi U, Y) \\ &= \sigma(\varphi U, Y) = \varphi\sigma(U, Y). \end{aligned} \tag{30}$$

From (29) and (30), we conclude that

$$L_2\{\varphi\sigma(U, Y)\} = -\varphi\sigma(U, Y)$$

which proves our assertions. □

From Theorem 2, we have the following corollary.

Corollary 2. *Let M be an invariant pseudoparallel submanifold of a K-paracontact manifold $\widetilde{M}^{2n+1}(\varphi, \xi, \eta, g)$. Then M is 2-semiparallel if and only if M is totally geodesic.*

Theorem 3. *Let M be an invariant Ricci-generalized pseudoparallel submanifold of a K-paracontact manifold $\widetilde{M}^{2n+1}(\varphi, \xi, \eta, g)$. Then M is either totally geodesic or $L_3 = \frac{1}{2n}$.*

Proof. If M is Ricci-generalized pseudoparallel, then from (12) and (16), we have

$$\begin{aligned} (\widetilde{R}(X, Y) \cdot \sigma)(U, V) &= L_3 Q(S, \sigma)(U, V; X, Y) \\ &= -L_3 \{\sigma((X \wedge_S Y)U, V) + \sigma(U, (X \wedge_S Y)V)\}, \end{aligned}$$

for all $X, Y, U, V \in \Gamma(TM)$. This means that

$$\begin{aligned} &R^{\perp}(X, Y)\sigma(U, V) - \sigma(R(X, Y)U, V) - \sigma(U, R(X, Y)V) \\ &= -L_3\{\sigma(S(Y, U)X - S(X, U)Y, V) \\ &+\sigma(S(V, Y)X - S(X, V)Y, U)\}. \end{aligned}$$

Here taking $X = V = \xi$ and by using (18), we reach at

$$\begin{aligned} &R^{\perp}(\xi, Y)\sigma(U, \xi) - \sigma(R(\xi, Y)U, \xi) - \sigma(U, R(\xi, Y)\xi) \\ &= -L_3\{\sigma(S(Y, U)\xi - S(\xi, U)Y, \xi) \\ &+\sigma(S(\xi, Y)\xi - S(\xi, \xi)Y, U)\}. \end{aligned} \tag{31}$$

By using (6) and (18), we can infer

$$L_3S(\xi,\xi)\sigma(U,Y) = -\sigma(R(\xi,Y)\xi,U)$$

$$-2nL_3\sigma(U,Y) = -\sigma(Y-\eta(Y)\xi,U) = -\sigma(Y,U).$$

This proves our assertion. □

Theorem 4. *Let M be an invariant 2-Ricci-generalized pseudoparallel submanifold of a K-paracontact manifold $\widetilde{M}^{2n+1}(\varphi,\xi,\eta,g)$. Then M is either totally geodesic or $L_4 = \frac{1}{2n}$.*

Proof. Let us assume that M is 2-Ricci-generalized pseudoparallel submanifold. Then from (17), we have

$$(\widetilde{R}(X,Y)\cdot\widetilde{\nabla}\sigma)(U,V,Z) = L_4 Q(S,\widetilde{\nabla}\sigma)(U,V,Z;X,Y),$$

for all $X,Y,U,V,Z \in \Gamma(TM)$. This implies that

$$R^{\perp}(X,Y)(\widetilde{\nabla}_U\sigma)(V,Z) - (\widetilde{\nabla}_{R(X,Y)U}\sigma)(V,Z)$$

$$-(\widetilde{\nabla}_U\sigma)(R(X,Y)V,Z) - (\widetilde{\nabla}_U\sigma)(V,R(X,Y)Z)$$

$$= -L_4\{(\widetilde{\nabla}_{(X\wedge_S Y)U}\sigma)(V,Z)$$

$$+(\widetilde{\nabla}_U\sigma)((X\wedge_S Y)V,Z) + (\widetilde{\nabla}_U\sigma)(V,(X\wedge_S Y)Z)\}.$$

Here taking $X = V = \xi$, we have

$$\begin{aligned} &R^{\perp}(\xi,Y)(\widetilde{\nabla}_U\sigma)(\xi,Z) - (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(\xi,Z)\\ &-(\widetilde{\nabla}_U\sigma)(R(\xi,Y)\xi,Z) - (\widetilde{\nabla}_U\sigma)(\xi,R(\xi,Y)Z)\\ &= -L_4\{(\widetilde{\nabla}_{(\xi\wedge_S Y)U}\sigma)(\xi,Z)\\ &+(\widetilde{\nabla}_U\sigma)((\xi\wedge_S Y)\xi,Z) + (\widetilde{\nabla}_U\sigma)(\xi,(\xi\wedge_S Y)Z)\}. \end{aligned} \tag{32}$$

Now, let's calculate each of these expressions. Also taking into account of (4) and (18), we arrive at

$$\begin{aligned} R^{\perp}(\xi,Y)(\widetilde{\nabla}_U\sigma)(\xi,Z) &= R^{\perp}(\xi,Y)\{\nabla^{\perp}_U\sigma(\xi,Z)\\ &-\sigma(\nabla_U Z,\xi) - \sigma(Z,\nabla_U\xi)\}\\ &= -R^{\perp}(\xi,Y)\sigma(-\varphi U,Z)\\ &= R^{\perp}(\xi,Y)\sigma(\varphi U,Z) = R^{\perp}(\xi,Y)\varphi\sigma(U,Z). \end{aligned} \tag{33}$$

On the other hand, by using (4) and (18), we have

$$
\begin{aligned}
(\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(\xi,Z) &= \nabla^{\perp}_{R(\xi,Y)U}\sigma(\xi,Z)-\sigma(\nabla_{R(\xi,Y)U}\xi,Z)\\
&-\sigma(\xi,\nabla_{R(\xi,Y)U}Z)\\
&=-\sigma(-\varphi R(\xi,Y)U,Z)\\
&=\varphi\sigma(R(\xi,Y)U,Z)
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
(\widetilde{\nabla}_{U}\sigma)(R(\xi,Y)\xi,Z) &= (\widetilde{\nabla}_{U}\sigma)(Y-\eta(Y)\xi,Z)\\
&=(\widetilde{\nabla}_{U}\sigma)(Y,Z)-(\widetilde{\nabla}_{U}\sigma)(\eta(Y)\xi,Z)\\
&=(\widetilde{\nabla}_{U}\sigma)(Y,Z)-\nabla^{\perp}_{U}\sigma(\eta(Y)\xi,Z)\\
&+\sigma(\nabla_{U}\eta(Y)\xi,Z)+\sigma(\eta(Y)\xi,\nabla_{U}Z)\\
&=(\widetilde{\nabla}_{U}\sigma)(Y,Z)+\sigma(U\eta(Y)\xi+\eta(Y)\nabla_{U}\xi,Z)\\
&=(\widetilde{\nabla}_{U}\sigma)(Y,Z)+\sigma(-\varphi U,Z)\eta(Y)\\
&=(\widetilde{\nabla}_{U}\sigma)(Y,Z)-\eta(Y)\varphi\sigma(U,Z).
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
(\widetilde{\nabla}_{U}\sigma)(\xi,R(\xi,Y)Z) &= \nabla^{\perp}_{U}\sigma(\xi,R(\xi,Y)Z)-\sigma(\nabla_{U}\xi,R(\xi,Y)Z)\\
&-\sigma(\xi,\nabla_{U}R(\xi,Y)Z)\\
&=-\sigma(-\varphi U,R(\xi,Y)Z)=\varphi\sigma(U,R(\xi,Y)Z).
\end{aligned}
\tag{36}
$$

Now, let's calculate the left side of (32). Making use of (4), (6) and (18), we have

$$
\begin{aligned}
(\widetilde{\nabla}_{(\xi\wedge_S Y)U}\sigma)(\xi,Z) &= \nabla^{\perp}_{(\xi\wedge_S Y)U}\sigma(\xi,Z)-\sigma(\nabla_{(\xi\wedge_S Y)U}\xi,Z)\\
&-\sigma(\xi,\nabla_{(\xi\wedge_S Y)U}Z)\\
&=-\sigma(-\varphi(S(Y,U)\xi-S(\xi,U)Y),Z)\\
&=-\sigma(\varphi S(U,\xi)Y,Z)=2n\eta(U)\varphi\sigma(Y,Z).
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)((\xi\wedge_S Y)\xi, Z) &= (\widetilde{\nabla}_U\sigma)(S(Y,\xi)\xi - S(\xi,\xi)Y, Z)\\
&= (\widetilde{\nabla}_U\sigma)(2nY - 2n\eta(Y)\xi, Z)\\
&= 2n(\widetilde{\nabla}_U\sigma)(Y - \eta(Y)\xi, Z)\\
&= 2n\{(\widetilde{\nabla}_U\sigma)(Y,Z) - (\widetilde{\nabla}_U\sigma)(\eta(Y)\xi, Z)\}\\
&= 2n\{(\widetilde{\nabla}_U\sigma)(Y,Z) - \nabla^\perp_U\sigma(\eta(Y)\xi, Z) \qquad (38)\\
&+\sigma(\nabla_U\eta(Y)\xi, Z) + \sigma(\eta(Y)\xi, \nabla_U Z)\}\\
&= 2n\{(\widetilde{\nabla}_U\sigma)(Y,Z) + \sigma(U\eta(Y)\xi + \eta(Y)\nabla_U\xi, Z)\}\\
&= 2n\{(\widetilde{\nabla}_U\sigma)(Y,Z) + \eta(Y)\sigma(-\varphi U, Z)\}\\
&= 2n\{(\widetilde{\nabla}_U\sigma)(Y,Z) - \eta(Y)\varphi\sigma(U,Z)\}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
(\widetilde{\nabla}_U\sigma)(\xi, (\xi\wedge_S Y)Z) &= (\widetilde{\nabla}_U\sigma)(\xi, S(Y,Z)\xi - S(\xi,Z)Y)\\
&= (\widetilde{\nabla}_U\sigma)(\xi, S(Y,Z)\xi) + 2n(\widetilde{\nabla}_U\sigma)(\xi, \eta(Z)Y)\\
&= \nabla^\perp_U\sigma(\xi, S(Y,Z)\xi) - \sigma(\nabla_U\xi, S(Y,Z)\xi)\\
&-\sigma(\xi, \nabla_U S(Y,Z)\xi) + 2n\{\nabla^\perp_U\sigma(\xi, \eta(Z)Y) \qquad (39)\\
&-\sigma(\nabla_U\xi, \eta(Z)Y) - \sigma(\xi, \nabla_U\eta(Z)Y)\}\\
&= 2n\{-\sigma(-\varphi U, \eta(Z)Y)\}\\
&= 2n\eta(Z)\varphi\sigma(Y,U).
\end{aligned}
$$

By substituting (33), (34), (35), (36), (37), (38) and (39) into (32) we reach at

$$
\begin{aligned}
&R^\perp(\xi,Y)\varphi\sigma(U,Z) - \varphi\sigma(R(\xi,Y)U,Z) - (\widetilde{\nabla}_U\sigma)(Y,Z)\\
&+\eta(Y)\varphi\sigma(U,Z) - \varphi\sigma(U,R(\xi,Y)Z)\\
&= -2nL_4\{\eta(U)\varphi\sigma(Y,Z)\\
&+(\widetilde{\nabla}_U\sigma)(Y,Z) - \eta(Y)\varphi\sigma(U,Z) + \eta(Z)\varphi\sigma(U,Y)\}.
\end{aligned}
\qquad (40)
$$

Here if taking $Z = \xi$, then (40) reduce

$$\begin{aligned} 2nL_4\{(\widetilde{\nabla}_U\sigma)(Y,\xi) + \varphi\sigma(U,Y)\} &= (\widetilde{\nabla}_U\sigma)(Y,\xi) \\ &\quad +\varphi\sigma(U,R(\xi,Y)\xi). \end{aligned}$$

From (30), we conclude that

$$(2nL_4 - 1)\sigma(U,Y) = 0$$

which proves our assertion. □

3. Conclusions

For many years, many studies have been done on the geometry of paracontact metric manifolds. This study has been prepared to contribute to making more detailed studies on K-paracontact metric manifolds. In the introduction section, a summary of the literature, basic definitions and theorems are given for a better understanding of the subject. In the following sections, Invariant pseudoparallel submanifolds of K-paracontact manifold are examined in detail. As a result, this study has been presented to the literature as a resource that will be used by every scientist who will study paracontact metric manifolds.

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Conflict of interest

Authors have declared that no competing interests exist.

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