Received: 21 August 2020 | Accepted: 11 September 2020

DOI: 10.1002/num.22559

RESEARCH ARTICLE

WILEY

Half-inverse problems for the quadratic pencil of the Sturm–Liouville equations with impulse

Rauf Amirov[1] | Abdullah Ergun[2] | Sevim Durak[1]

[1]Department of Mathematics, Faculty of Science and Arts, Cumhuriyet University, Sivas, Turkey

[2]Vocational School of Sivas, Cumhuriyet University, Sivas, Turkey

Correspondence

Abdullah Ergun, Vocational School of Sivas, Cumhuriyet University, 58140 Sivas, Turkey.

Email: aergun@cumhuriyet.edu.tr

Abstract

In this paper, we consider the inverse spectral problem for the impulsive Sturm–Liouville differential pencils on $[0, \pi]$ with the Robin boundary conditions and the jump conditions at the point $\frac{\pi}{2}$. We prove that two potentials functions on the whole interval and the parameters in the boundary and jump conditions can be determined from a set of eigenvalues for two cases: (i) the potentials given on $\left(0, \frac{\pi}{2}\right)$. and (ii) the potentials given on $\left(\frac{\pi}{2}, \pi\right)$, where $0 < \alpha < 1$, respectively. Inverse spectral problems, Sturm–Liouville operator, spectrum, uniqueness.

KEYWORDS

inverse spectral problems, spectrum, Sturm–Liouville operator, uniqueness

1 | INTRODUCTION

We consider the quadratic pencils of Sturm–Liouville operator $L_\lambda(p, q, \alpha)$ of the form

$$L_\lambda y := -y'' + [2\lambda p(x) + q(x)]y = \lambda^2 y, \quad x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right], \tag{1.1}$$

with the boundary conditions

$$U(y) := y'(0) = 0, \quad V(y) := y'(\pi) = 0 \tag{1.2}$$

and with the discontinuous conditions

$$\begin{aligned} y\left(\frac{\pi}{2} + 0\right) &= \alpha y\left(\frac{\pi}{2} - 0\right), \\ y'\left(\frac{\pi}{2} + 0\right) &= \alpha^{-1} y'\left(\frac{\pi}{2} - 0\right), \end{aligned} \tag{1.3}$$

where λ is the spectral parameter, $p(x) \in W_2^1[0, \pi]$, $q(x) \in L_2[0, \pi]$ are real-valued functions, α is *a* real number, and $\alpha > 0$, $\alpha \neq 1$. Here we denote by $W_2^m[0, \pi]$ the space of functions $f(x)$, $x \in [0, \pi]$, such that the derivatives $f^{(m)}(x)$ $(m=\overline{0,n-1})$ are absolutely continuous and $f^{(n)}(x)\in L_2[0,\pi]$. Differential equations with potentials depending nonlinearly on the spectral parameter appear frequently in various models of quantum and classical mechanics (see Refs. [1–8] and references there in). For instance, the evolution equations that are used to model interactions between colliding relativistic spineless particles can be reduced to the form (1.1), here the parameter λ^2 can be regarded as the energy of this system [2, 5, 6]. Boundary value problems with discontinuous inside the interval often appear in mathematics, physics, geophysics, mechanics and other branches of natural properties (see Refs. [9–17] and references there in). The well-known works [9, 11] are the first result about discontinuous inverse spectral problems for the impulsive Sturm–Liouville operators. Direct and inverse problems for the pencil L_λ with impulse on a finite interval have been investigated in Ref. [1]. The theory of inverse problems for differential operators occupies an important position in the current developments of the spectral theory of linear operators. Inverse problems of spectral analysis consist in the recovery of operators from their spectral data. The interior spectral data used for reconstructing the differential operators contain the known eigenvalues and some information on eigenfunctions at some interior point in the defined interval. The similar problems for the Sturm–Liouville operators with impulse and differential pencils L_λ were considered [18–20]. When we prepared this work for publication, the work of the authors of Zhang et al. [21] was published. In this study, uniqueness theorems related to the half-inverse problem for the Sturm–Liouville equation with discontinuous coefficient have been proved. One of the important aspects of the study is that it did not use the Volterra integral equation. Unlike other publications, the subject examined in our study is not made by reducing the proof of the uniqueness theorem to the solution of Volterra type integral equation, but with the help of the uniqueness theorem by the Weyl function.

2 | PRELIMINARIES

Let the function $\varphi(x,\lambda)$ be the solution of Equation (1.1) with the initial conditions

$$\varphi(0,\lambda)=1,\varphi\prime(0,\lambda)=0 \tag{2.1}$$

and the impulse condition (1.3). It is shown in Ref. [1] if $q(x)\in L_2[0,\pi]$ and $p(x)\in W_2^1[0,\pi]$ that there exist functions $A(x,t)$ and $B(x,t)$ whose first-order partial derivatives are summable on $[0,\pi]$ for each $x\in[0,\pi]$ such that

$$\varphi(x,\lambda)=\varphi_0(x,\lambda)+\int_0^x A(x,t)\cos\lambda t\,dt+\int_0^x B(x,t)\sin\lambda t\,dt, \tag{2.2}$$

where

$$\varphi_0(x,\lambda)=l^+(x)\cos[\lambda x-\beta^+(x)]+l^-(x)\cos[\lambda(\pi-x)+\beta^-(x)] \tag{2.3}$$

and

$$l^\pm(x)=\frac{1}{2}\left(l(x)\pm\frac{1}{l(x)}\right),\quad l(x)=\begin{cases}1, & 0\le x\le\frac{\pi}{2}\\ \alpha, & \frac{\pi}{2}<x\le\pi\end{cases},\beta^\pm(x)=\int_{\frac{\frac{\pi}{2}\pm\frac{\pi}{2}}{2}}^x p(t)\,dt.$$

It follows from (2.2) and (2.3) that the characteristic function of the pencil $L(p,q,\alpha)$ can be reduced $\Delta(\lambda)$, where

$$\Delta(\lambda)=\varphi'(\pi,\lambda) \tag{2.5}$$

and

$$\Delta_0(\lambda) = \varphi_0'(\pi, \lambda) = -\alpha^+(\lambda - p(\pi))\sin(\lambda\pi - \beta^+(\pi)) + \alpha^-(\lambda + p(\pi))\sin(\beta^-(\pi))$$

or $\Delta(\lambda) = <\psi(x, \lambda), \varphi(x, \lambda)>$, where $<y, z> := y'z - yz'$ and $\psi(x, \lambda)$ be a solution of (1.1) with the initial conditions $\psi(\pi, \lambda) = 0$, $\psi'(\pi, \lambda) = -1$ and the impulse conditions (1.3).

One here supposes that the function $q(x)$ satisfies the additional condition

$$\int_0^\pi \{|y'(x)|^2 + q(x)|y(x)|^2\}dx > 0 \tag{2.6}$$

for all $y(x) \in W_2^2\left(\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]\right)$ such that $y(x) \neq 0$ and

$$y'(0)\overline{y(0)} - y'(\pi)\overline{y(\pi)} = 0. \tag{2.7}$$

Then it is shown in Ref. [1] that the eigenvalues of the boundary value problem $L(p, q, \alpha)$ are real nonzero, simple, and does not have associated functions. Additionally eigenfunctions corresponding to different eigenvalues of the problem $L(p, q, \alpha)$ are orthogonal in the sense of the equality

$$(\lambda_1 + \lambda_2)(y_1, y_2) - 2(py_1, y_2) = 0, \tag{2.8}$$

where $(.,.)$ denotes the inner product in $L_2[0, \pi]$. Denote by $G_\delta := \{\lambda : |\lambda - \lambda_n^0| \geq \delta > 0, n \in \mathbb{Z}\}$, where δ is sufficiently small positive number $\left(\delta \ll \frac{\beta}{2}\right)$. Then, [22] there exists a constant $C_\delta > 0$ such that

$$|\Delta(\lambda)| \geq (|\lambda|\alpha^+ - C)\exp(|\tau|\pi), \tau = \operatorname{Im}\lambda, \text{for all } \lambda \in \overline{G}_\delta. \tag{2.9}$$

Together with $L(p, q, \alpha)$, we consider the boundary value problem $\widetilde{L}(\widetilde{p}, \widetilde{q}, \widetilde{\alpha})$ of the same form but with different coefficients $(\widetilde{p}, \widetilde{q}, \widetilde{\alpha})$. It is assumed in what follows that is a certain symbol γ denotes an object related to the problem $L(p, q, \alpha)$, then $\widetilde{\gamma}$ will denote the corresponding object related to the problem $\widetilde{L}(\widetilde{p}, \widetilde{q}, \widetilde{\alpha})$. Moreover, for the solutions $\varphi(x, \lambda)$ and $\widetilde{\varphi}(x, \lambda)$ of the operators $L(p, q, \alpha)$ and $\widetilde{L}(\widetilde{p}, \widetilde{q}, \widetilde{\alpha})$, respectively, using (2.2)–(2.4) and by extending the range of $A(x, t)$, $\widetilde{A}(x, t)$ evenly with respect to the argument t and $B(x, t)$, $\widetilde{B}(x, t)$ oddly with respect to the argument t, and by some straight-forward calculations, we infer that there exist functions $R_1(x, t)$ and $R_2(x, t)$ whose first-order partial derivatives are summable on $[0, \pi]$ for each $x \in [0, \pi]$ such that for $0 \leq x < \frac{\pi}{2}$

$$\begin{aligned}\varphi(x, \lambda)\widetilde{\varphi}(x, \lambda) = {} & \frac{1}{2}[\cos(2\lambda x - \gamma_1^+(x)) + \cos(\gamma_1^-(x))] \\ & + \int_0^x [R_1(x, t)e^{2i\lambda t} + R_2(x, t)e^{-2i\lambda t}]\,dt,\end{aligned} \tag{2.10}$$

and for $\frac{\pi}{2} < x \leq \pi$

$$\begin{aligned}\varphi(x, \lambda)\widetilde{\varphi}(x, \lambda) = {} & \frac{(\alpha^+)^2}{2}[\cos(2\lambda x - \gamma_1^+(x)) + \cos(\gamma_1^-(x))] \\ & + \frac{(\alpha^-)^2}{2}[\cos(2\lambda(\pi - x) + \gamma_2^+(x)) + \cos(\gamma_2^-(x))] \\ & + \frac{\alpha^+\alpha^-}{2}[\cos(\lambda(2x - \pi) - \beta^+(x) - \widetilde{\beta}^-(x)) + \cos(\lambda\pi - \beta^+(x) + \widetilde{\beta}^-(x))] \\ & + \frac{\alpha^+\alpha^-}{2}[\cos(\lambda(2x - \pi) - \beta^-(x) - \widetilde{\beta}^+(x)) + \cos(\lambda\pi + \beta^-(x) - \widetilde{\beta}^+(x))] \\ & + \int_0^x [R_1(x, t)e^{2i\lambda t} + R_2(x, t)e^{-2i\lambda t}]\,dt\end{aligned} \tag{2.11}$$

where

$$\gamma_1^{\pm}(x) = \beta^+(x) \pm \widetilde{\beta}^+(x), \gamma_2^{\pm}(x) = \beta^-(x) \pm \widetilde{\beta}^-(x). \tag{2.12}$$

3 | MAIN RESULTS

Now we state the main result of this work. It is assumed in what follows that if a certain symbol s denotes an object related to $L_\lambda(p, q, \alpha)$, then the corresponding symbol $\widetilde{s}$ with tilde denote the analogous object related to $\widetilde{L}(p, q, \alpha)$. Let us denoted by $\varphi(x, \lambda_n)$, the eigenfunction which corresponds to λ_n.

Lemma 1 *If $\lambda_n = \widetilde{\lambda}_n, n = 0, \pm 1, \pm 2, \ldots$ then $\beta^+(\pi) = \widetilde{\beta}^+(\pi)$ and $\beta^-(\pi) = \widetilde{\beta}^-(\pi)$, that is, the sequence $\{\lambda_n\}$ uniquely determines $\beta^{\pm}(\pi)$.*

Proof of the Lemma is easily obtained from the asymptotic expression of λ_n.

Lemma 2 *If $\lambda_n = \widetilde{\lambda}_n$, $n = 0, \pm 1, \pm 2, \ldots$ then $\alpha = \widetilde{\alpha}$, that is, the sequence $\{\lambda_n\}$ uniquely determines number α.*

Proof. Since, $\lambda_n = \widetilde{\lambda}_n$ and $\Delta(\lambda), \widetilde{\Delta}(\lambda)$ are entire functions in λ of order one by Hadamard factorization theorem, for $\lambda \in \mathbb{C}$ ∎

$$\Delta(\lambda) \equiv C\widetilde{\Delta}(\lambda). \tag{3.1}$$

On the other hand, (1.1) can be written as

$$\Delta_0(\lambda) - C\widetilde{\Delta}_0(\lambda) = C[\widetilde{\Delta}(\lambda) - \widetilde{\Delta}_0(\lambda)] - [\Delta(\lambda) - \Delta_0(\lambda)]. \tag{3.2}$$

Hence,

$$\begin{aligned} C[\widetilde{\Delta}(\lambda) - \widetilde{\Delta}_0(\lambda)] - [\Delta(\lambda) - \Delta_0(\lambda)] &= -\alpha^+(\lambda - p(\pi)) \sin(\lambda\pi - \beta^+(\pi)) \\ &+ \alpha^-(\lambda + p(\pi)) \sin \beta^-(\pi) \\ &- C[-\widetilde{\alpha}^+(\lambda - \widetilde{p}(\pi)) \sin(\lambda\pi - \widetilde{\beta}^+(\pi)) + \widetilde{\alpha}^-(\lambda + \widetilde{p}(\pi)) \sin \widetilde{\beta}^-(\pi)], \end{aligned} \tag{3.3}$$

if we multiply both sides of (3.3) with $\sin(\lambda\pi - \beta^+(\pi))$ and integrate with respect to λ in $(0, T)$ for any positive real number T, then we get

$$\begin{aligned} &\int_0^T (C[\widetilde{\Delta}(\lambda) - \widetilde{\Delta}_0(\lambda)] - [\Delta(\lambda) - \Delta_0(\lambda)]) \sin(\lambda\pi - \beta^+(\pi))\, d\lambda \\ &= \int_0^T \big(-\alpha^+(\lambda - p(\pi)) \sin(\lambda\pi - \beta^+(\pi)) + \alpha^-(\lambda + p(\pi)) \sin \beta^-(\pi) \\ &- C[-\widetilde{\alpha}^+(\lambda - \widetilde{p}(\pi)) \sin(\lambda\pi - \widetilde{\beta}^+(\pi)) + \widetilde{\alpha}^-(\lambda + \widetilde{p}(\pi)) \sin \widetilde{\beta}^-(\pi)] \sin(\lambda\pi - \beta^+(\pi))\, d\lambda. \end{aligned}$$

Since

$$\Delta(\lambda) - \Delta_0(\lambda) = O(e^{|\mathrm{Im}\lambda|\pi}), \widetilde{\Delta}(\lambda) - \widetilde{\Delta}_0(\lambda) = O(e^{|\mathrm{Im}\lambda|\pi})$$

for all λ in $(0, T)$

$$\frac{C\widetilde{\alpha}^+}{4} - \frac{\alpha^+}{4} = O\left(\frac{1}{T}\right).$$

By letting T tend to infinity, we see that

$$C = \frac{\alpha^{+}}{\widetilde{\alpha}^{+}}. \tag{3.4}$$

Similarly, if we multiply both sides of (3.3) with $\sin\beta^{-}(\pi)$ and integrate again with respect to λ in $(0, T)$, and by letting T tend to infinity, then we get

$$C = \frac{\alpha^{-}}{\widetilde{\alpha}^{-}}. \tag{3.5}$$

But since α and $\widetilde{\alpha}$ are positive we conclude that $C = 1$. We have therefore proved that $\alpha = \widetilde{\alpha}$.

Lemma 3 *The zeros $\{\lambda_n\}$ of the characteristic function $\Delta(\lambda)$ coincide with the eigenvalues of the boundary value problem $L(\alpha, a)$. The functions $\varphi(x, \lambda_n)$ and $\psi(x, \lambda_n)$ are eigenfunctions corresponding to the eigenvalue λ_n and there exists a sequence $\{\beta_n\}$ such that $\psi(x, \lambda_n) = \beta_n \varphi(x, \lambda_n)$, $\beta_n \neq 0$..*

Lemma 4 *The equality $\dot{\Delta}(\lambda_n) = -2\lambda_n \alpha_n \beta_n$ holds. Here $\dot{\Delta}\left(\frac{d}{d\lambda}\right)\Delta(\lambda)$.*

Lemma 5 *The problem $L(\alpha, a)$ has countable set of eigenvalues. If one denotes by λ_1, λ_2, … the positive eigenvalues arranged in increasing order and by λ_{-1}, λ_{-2}, … the negative eigenvalues arranged in decreasing order, then eigenvalues of the problem $L(\alpha, a)$ have the asymptotic behavior*

$$\lambda_n = \lambda_n^o + \frac{d_n}{\lambda_n^o} + \frac{\delta_n}{\lambda_n^o}, \quad n \to \pm\infty,$$

where $\delta_n \in l_2$ and d_n is a bounded sequence, $\lambda_n^o = n + \left(\frac{1}{\pi}\right)\beta^{+}(\pi) + h_n$, $\sup_n |h_n| < \infty$.

Similar to the proofs of Ref. 1, so we omit the details. Let $\Phi(x, \lambda)$ be the solution of (1.1) under the conditions $U(\Phi) = 1$, $V(\Phi) = 0$, and under the impulse conditions (1.3). One sets $M(\lambda) := \Phi(0, \lambda)$. The functions $\Phi(x, \lambda)$ and $M(\lambda)$ are called the Weyl solution and Weyl function for the boundary value problem $L_\lambda(p, q, \alpha)$, respectively. Using the solution $\varphi(x, \lambda)$ defined in the previous sections one has

$$\begin{aligned} \Phi(x, \lambda) &= -\frac{\psi(x, \lambda)}{\Delta(\lambda)} = S(x, \lambda) + M(\lambda)\varphi(x, \lambda), \\ M(\lambda) &= -\frac{\psi(0, \lambda)}{\Delta(\lambda)}, \quad \left(-\frac{\varphi(\pi, \lambda)}{\varphi'(\pi, \lambda)} = -\frac{\varphi(\pi, \lambda)}{\Delta(\lambda)}\right), \end{aligned} \tag{3.6}$$

where $\psi(x, \lambda)$ is a solution of (1.1) satisfying the conditions $\psi(0, \lambda) = 0$, $\psi'(0, \lambda) = -1$, and the impulse conditions (1.3) and $S(x, \lambda)$ is defined from the equality

$$\psi(x, \lambda) = \psi(0, \lambda)\varphi(x, \lambda) - \Delta(\lambda)S(x, \lambda). \tag{3.7}$$

Note that, by virtue of equalities $\langle \varphi(x, \lambda), S(x, \lambda)\rangle \equiv 1$ and (3.6), on has

$$\begin{aligned} &\langle \Phi(x, \lambda), \varphi(x, \lambda)\rangle \equiv 1, \\ &\langle \varphi(x, \lambda), \psi(x, \lambda)\rangle \equiv -\Delta(\lambda) \ \text{ for } \ x \neq \frac{\pi}{2}. \end{aligned} \tag{3.8}$$

The following theorem shows that the Weyl function uniquely determines the potentials and the coefficients of the boundary value problem $L_\lambda(p, q, \alpha)$.

Theorem 1 *If $M(\lambda) = \widetilde{M}(\lambda)$, then $L_\lambda(p, q, \alpha) = \widetilde{L}_\lambda(\widetilde{p}, \widetilde{q}, \widetilde{\alpha})$. Thus, the boundary value problem $L_\lambda(p, q, \alpha)$ is uniquely defined by the Weyl function.*

Proof. Since ∎

$$\psi^{(\nu)}(x, \lambda) = O(|\lambda|^{\nu-1} \exp(|\mathrm{Im}\lambda|(\pi - x))), \quad \lambda \in \widetilde{G}_\delta, \tag{3.9}$$

$$|\Delta(\lambda)| \geq C_\delta \exp(|\mathrm{Im}\lambda|\pi), \quad \lambda \in \widetilde{G}_\delta, \quad C_\delta > 0, \quad \nu = 0, 1, \tag{3.10}$$

it is easy to observe that

$$|\Phi^{(\nu)}(x, \lambda)| \leq C_\delta |\lambda|^{\nu-1} \exp(-|\mathrm{Im}\lambda| x), \quad \lambda \in G_\delta. \tag{3.11}$$

Let us define the matrix $P(x, \lambda) = [P_{jk}(x, \lambda)]_{j,\, k\, =\, 1,2}$, where

$$P_{j1}(x, \lambda) = \varphi^{(j-1)}(x, \lambda)\widetilde{\Phi}'(x, \lambda) - \Phi^{(j-1)}(x, \lambda)\widetilde{\varphi}'(x, \lambda),$$

$$P_{j2}(x, \lambda) = \Phi^{(j-1)}(x, \lambda)\widetilde{\varphi}'(x, \lambda) - \varphi^{(j-1)}(x, \lambda)\widetilde{\Phi}'(x, \lambda). \tag{3.12}$$

Then we have

$$\varphi(x, \lambda) = P_{11}(x, \lambda)\widetilde{\varphi}(x, \lambda) + P_{12}(x, \lambda)\widetilde{\varphi}'(x, \lambda),$$

$$\Phi(x, \lambda) = P_{11}(x, \lambda)\widetilde{\Phi}(x, \lambda) + P_{12}(x, \lambda)\widetilde{\Phi}'(x, \lambda). \tag{3.13}$$

According to (3.6) and (3.11), for each fixed x, the functions $P_{jk}(x, \lambda)$ are meromorphic in λ with poles and points λ_n and $\widetilde{\lambda}_n$. Denote $G_\delta^0 = G_\delta \cap \widetilde{G}_\delta$. By virtue of (3.11), (3.13), and

$$\varphi^{(\nu)}(x, \lambda) = O(|\lambda|^{\nu} \exp(|\mathrm{Im}\lambda| x)), \quad \lambda \in G_\delta^0, \quad \nu = 0, 1, \tag{3.14}$$

we get

$$|P_{12}(x, \lambda)| \leq C_\delta |\lambda|^{-1}, \quad |P_{12}(x, \lambda)| \leq C_\delta, \lambda \in G_\delta^0. \tag{3.15}$$

It follows from (3.6) and (3.12) that if $M(\lambda) \equiv \widetilde{M}(\lambda)$, then for each fixed x the functions P_{1k} are entire in x. Together with (3.15) this yields $P_{12}(x, \lambda) \equiv 0$, $P_{11}(x, \lambda) \equiv A(x)$. Now using (3.13), we obtain

$$\varphi(x, \lambda) \equiv A(x)\widetilde{\varphi}(x, \lambda), \quad \Phi(x, \lambda) \equiv A(x)\widetilde{\Phi}(x, \lambda). \tag{3.16}$$

Therefore, for $|\lambda| \to \infty$, $\arg\lambda \in [\varepsilon, \pi - \varepsilon]$ $(\varepsilon > 0)$, we have

$$\varphi(x, \lambda) = \frac{B}{2} \exp(-i(\lambda x - \beta_1(x))) \left(1 + O\left(\frac{1}{|\lambda|}\right)\right), \tag{3.17}$$

where $B = 1$, for $x < a$ and $B = \alpha^+$ for $x > a$. Similarly, one can calculate

$$\Phi(x, \lambda) = (i\lambda B)^{-1} \exp(i(\lambda x - \beta_1(x))) \left(1 + O\left(\frac{1}{|\lambda|}\right)\right), \quad |\lambda| \to \infty, \ \arg\lambda \in [\varepsilon, \pi - \varepsilon]. \tag{3.18}$$

Finally, taking into account the relations $\langle \Phi(x, \lambda), \varphi(x, \lambda) \rangle \equiv 1$ and (3.11), we have $\alpha^+ = \widetilde{\alpha}^+$, $A(x) \equiv 1$, that is, $\varphi(x, \lambda) \equiv \widetilde{\varphi}(x, \lambda)$, $\Phi(x, \lambda) \equiv \widetilde{\Phi}(x, \lambda)$ for all x and λ. Consequently, $L_\lambda(p, q, \alpha) = \widetilde{L}_\lambda(p, p, \alpha)$. The theorem is proved.

Theorem 2 *If $\lambda_n = \widetilde{\lambda}_n$ for all $n \in \mathbb{Z}$ and $p(x) = \widetilde{p}(x)$ and $q(x) = \widetilde{q}(x)$ on $\left(\frac{\pi}{2}, \pi\right)$, then $p(x) = \widetilde{p}(x)$ and $q(x) = \widetilde{q}(x)$ almost everywhere on $(0, \pi)$.*

Proof. Let $\widetilde{\varphi}(x,\lambda)$ be the solution of the equation ∎

$$-\widetilde{\varphi}''(x,\lambda)+[2\lambda\widetilde{p}(x)+\widetilde{q}(x)]\widetilde{\varphi}(x,\lambda)=\lambda^2\widetilde{\varphi}(x,\lambda), \tag{3.19}$$

with the initial value conditions

$$\widetilde{\varphi}(0,\lambda)=1, \widetilde{\varphi}'(0,\lambda)=0 \tag{3.20}$$

and the impulse conditions (1.3). Multiplying (1.1) by $\widetilde{\varphi}(x,\lambda)$ and (3.19) by $\varphi(x,\lambda)$, respectively, and subtracting, we get

$$\widetilde{\varphi}''(x,\lambda)\varphi(x,\lambda)-\varphi''(x,\lambda)\widetilde{\varphi}(x,\lambda)=[2\lambda(p(x)-\widetilde{p}(x))+(q(x)-\widetilde{q}(x))]\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda). \tag{3.21}$$

Integrating the above equality from 0 to π with respect to x, using the initial conditions at $x=0$ and impulse conditions (1.3), we have

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}[2\lambda(\widetilde{p}(x)-p(x))+(\widetilde{q}(x)-q(x))]\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda)\,dx\\&=(\widetilde{\varphi}'(x,\lambda)\varphi(x,\lambda)-\varphi'(x,\lambda)\widetilde{\varphi}(x,\lambda))\left(\Big|_0^{\frac{\pi}{2}-0}+\Big|_{\frac{\pi}{2}+0}^{\pi}\right),\end{aligned}$$

from the hypothesis $\widetilde{p}(x)=p(x), \widetilde{q}(x)=q(x)$ on $\left(\frac{\pi}{2},\pi\right)$. Then we obtain

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}[2\lambda(\widetilde{p}(x)-p(x))+(\widetilde{q}(x)-q(x))]\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda)\,dx\\&=\widetilde{\varphi}'(\pi,\lambda)\varphi(\pi,\lambda)-\varphi'(\pi,\lambda)\widetilde{\varphi}(\pi,\lambda).\end{aligned} \tag{3.22}$$

Denote

$$P(x)=\widetilde{p}(x)-p(x), Q(x)=\widetilde{q}(x)-q(x)$$

and

$$F_0(\lambda)=2\lambda\int_0^{\frac{\pi}{2}}P(x)\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda)\,dx+\int_0^{\frac{\pi}{2}}Q(x)\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda)\,dx. \tag{3.23}$$

It follows from (2.2) to (2.3) and (2.10) that $F_0(\lambda)$ is an entire function of exponential type, and there are some positive constants c_1 and c_2 such that

$$|F_0(\lambda)|\le(c_1+c_2|\lambda|)\exp(|\mathrm{Im}\lambda|\pi)\ \text{ for all }\ \lambda\in\mathbb{C}. \tag{3.24}$$

It is clear from the properties of $\varphi(x,\lambda)$, $\varphi'(x,\lambda)$ and the boundary conditions (1.2)

$$F_0(\lambda_n)=0,\quad n\in\mathbb{Z}, \tag{3.25}$$

for each eigenvalue λ_n.

Define

$$F(\lambda):=\frac{F_0(\lambda)}{\Delta(\lambda)},$$

which is an entire function from the above arguments and it follows from (2.9) and (3.24) that

$$F(\lambda)=O(1),$$

for sufficiently large $|\lambda|$, $\lambda \in G_\delta$. Using Liouville's theorem, [23] we obtain for all λ that

$$F(\lambda) = C,$$

where C is a constant.

Let us show that the $C = 0$. Now, we can rewrite the equation $F_0(\lambda) = C\Delta(\lambda)$ as

$$2\lambda \int_0^{\frac{\pi}{2}} P(x)\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda)\,dx + \int_0^{\frac{\pi}{2}} Q(x)\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda)\,dx$$
$$= C\lambda[\alpha^- \sin(\beta^-(\pi)) - \alpha^+ \sin(\lambda\pi - \beta^+(x))] + O(e^{|\mathrm{Im}\lambda|\pi}).$$

By use of Riemann–Lebesgue Lemma, [23] we see that the limit of the left-hand side of the above equality exists as $\lambda \to \infty$, $\lambda \in \mathbb{R}$. Therefore, we get that $C = 0$. So, we have

$$F_0(\lambda) = 0, \text{ for all } \lambda \in \mathbb{C}.$$

Then, from the equality (3.23) we obtain

$$\widetilde{\varphi}'(\pi,\lambda)\varphi(\pi,\lambda) - \varphi'(\pi,\lambda)\widetilde{\varphi}(\pi,\lambda) = 0,$$

for all $\lambda \in \mathbb{C}$. Since

$$-\frac{\varphi(\pi,\lambda)}{\varphi'(\pi,\lambda)} = -\frac{\widetilde{\varphi}(\pi,\lambda)}{\widetilde{\varphi}'(\pi,\lambda)},$$

we obtain

$$M(\lambda) = \widetilde{M}(\lambda),$$

for all $\lambda \in \mathbb{C}$. The function $M(\lambda) = -\frac{\varphi(\pi,\lambda)}{\varphi'(\pi,\lambda)}$ is the Weyl function of the boundary value problem for Equation (1.1) on the $\left(\frac{\pi}{2}, \pi\right)$, with jump conditions (1.3) and boundary conditions $y(0) = 0$, $V(y) = 0$.

Theorem 3 *If* $\lambda_n = \widetilde{\lambda}_n$ *for all* $n \in \mathbb{Z}$ *and* $p(x) = \widetilde{p}(x)$, $q(x) = \widetilde{q}(x)$ *on* $\left(0, \frac{\pi}{2}\right)$, *then* $p(x) = \widetilde{p}(x)$ *and* $q(x) = \widetilde{q}(x)$ *almost everywhere on* $(0, \pi)$.

Proof. If the operations in the proof of Theorem 2 are also performed for the function $\psi(x,\lambda)$ which provides the initial conditions $\psi(\pi,\lambda) = 0$, $\psi'(\pi,\lambda) = -1$ and (1.3) discontinuity of Equation (1.1) ■

$$\int_{\frac{\pi}{2}}^{\pi} [2\lambda(\widetilde{p}(x) - p(x)) + (\widetilde{q}(x) - q(x))]\psi(x,\lambda)\widetilde{\psi}(x,\lambda)\,dx$$
$$= \widetilde{\psi}'(0,\lambda)\psi(0,\lambda) - \psi'(0,\lambda)\widetilde{\psi}(0,\lambda)$$

equality is obtained.

$$H(\lambda) = \int_{\frac{\pi}{2}}^{\pi} [2\lambda(\widetilde{p}(x) - p(x)) + (\widetilde{q}(x) - q(x))]\psi(x,\lambda)\widetilde{\psi}(x,\lambda)dx$$

function, $H(\lambda) \equiv 0$ condition for $\forall \lambda \in \mathbb{C}$

$$\widetilde{\psi'}(0,\lambda)\psi(0,\lambda) - \psi'(0,\lambda)\widetilde{\psi}(0,\lambda) = 0, \quad \forall \lambda \in \mathbb{C}$$

obtained. From here for each $\lambda \in \mathbb{C}$

$$\frac{\psi(0,\lambda)}{\psi'(0,\lambda)} = \frac{\widetilde{\psi}(0,\lambda)}{\widetilde{\psi}'(0,\lambda)},$$

equality is obtained. The function $M(\lambda) = -\frac{\psi(0,\lambda)}{\psi'(0,\lambda)}$ is the Weyl function of the boundary value problem for equation on $\left(0, \frac{\pi}{2}\right)$ with boundary conditions $U(y) = 0$, $V(y) = 0$ and without discontinuity. [1] The Weyl function uniquely determined $p(x)$ and $q(x)$ on $(0, \pi)$.

ORCID

Abdullah Ergun https://orcid.org/0000-0002-2795-8097

REFERENCES

[1] R. K. Amirov and A. A. Nabiev, *Inverse problems for the quadratic pencil of the Sturm–Liouville equations with impulse*, Abstr. Appl. Anal. 2013 (2013, 361989), 10.
[2] P. Jonas, *On the spectral theory of operators associated with perturbed Klein–Gordon and wave type equations*, J. Operat. Theory 29 (1993), 207–224.
[3] M. V. Keldysh, *On the eigenvalues and eigenfunctions of some classes of nonselfadjoint equations*, Dokl. Akad. Nauk SSSR 77 (1951), 11–14.
[4] A. G. Kostyuchenko and A. A. Shkalikov, *Selfadjoint quadratic operator pencils and elliptic problems*, Funct. Anal. Appl. 17 (1983), 109–128.
[5] A. S. Markus, Introduction to the spectral theory of polynomial operator pencils, Shtinitsa, Kishinev, 1986; English transl., AMS, Providence, 1988.
[6] M. Yamamoto, *Inverse eigenvalue problem for a vibration of a string with viscous drag*, J. Math. Anal. Appl. 152 (1990), 20–34.
[7] V. A. Yurko, *An inverse problem for systmes of differential equations with nonlinear dependence on the spectral parameter, Diff. Uravneniya, 33 (1997), 390-395; English transl*, Diff. Eq. 33 (1997), 388–394.
[8] H. Koyunbakan and E. S. Panakhov, *Half inverse problem for diffusion operators on the finite interval*, JMAA 326 (2007), 1024–1030.
[9] R. K. Amirov, *On Sturm–Liouville operators with discontinuity conditions inside an interval*, J. Math. Anal. Appl. 317 (2006), 163–176.
[10] G. Freiling and V. A. Yurko, *Inverse spectral problems for singular non-selfdjoint differential operators with discontinuities in an interior point*, Inverse Probl. 18 (2002), 757–773.
[11] O. H. Hald, *Discontinuous inverse eigenvalue problems*, Commun. Pure Appl. Math. 37 (1984), 539–577.
[12] R. J. Krueger, *Inverse problems for nonabsorbing media with discontinuous material properties*, J. Math. Phys. 23 (1982), 396–404.
[13] A. S. Ozkan, B. Keskin, and Y. Cakmak, *Uniqueness of the solution of half inverse problem for the impulsive Sturm–Liouville operator*, Bull. Korean Math. Soc. 50 (2013), 499–506.
[14] C.-T. Shieh and V. A. Yurko, *Inverse nodal and inverse spectral problems for discontinuous boundary value problems*, J. Math. Anal. Appl. 347 (2008), 266–272.
[15] C. Willis, *Inverse Sturm–Liouville problems with two discontinuities*, Inverse Probl. 1 (1985), 263–289.
[16] X.-C. Xu and C.-F. Yang, *Inverse spectral problems for the Sturm–Liouville operator with discontinuity*, J. Differ. Eq. 262 (2017), 3093–3106.
[17] C.-F. Yang, *Inverse problems for the Sturm–Liouville operator with discontinuity*, Inverse Probl. Sci. Eng. 22 (2014), 232–244.
[18] K. Mochizuki and I. Trooshin, *Inverse problem for interior spectral data of Sturm–Liouville operator*, J. Inverse III-Posed Probl. 9 (2001), 425–433.
[19] C.-F. Yang, *An interior inverse problem for discontinuous boundary-value problems*, Integr. Eq. Oper. Theory 65 (2009), 593–604.
[20] C.-F. Yang and Y. Guo, *Determination of a differential pencil from interior spectral data*, J. Math. Anal. Appl. 375 (2011), 284–293.
[21] R. Zhang et al., *Determination of the impulsive Sturm–Liouville operator from a set of eigenvalues*, J. Inverse III-Posed Probl 28 (2019), no. 3, 341–348.
[22] R. Bellman and K. L. Cooke, *Differential-difference equations*, Academic Press, New York, 1963.

[23] B. Y. Levin, Levin, lectures on entire functions, in *Translations of mathematical monographs*, Vol 150, American Mathematical Society, Providence, RI, 1996.

How to cite this article: Amirov R, Ergun A, Durak S. Half-inverse problems for the quadratic pencil of the Sturm–Liouville equations with impulse. *Numer Methods Partial Differential Eq.* 2020;1–10. https://doi.org/10.1002/num.22559