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Research Article

On a Classification of Almost $C(\alpha)$-Manifolds

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In this paper, pseudosymmetric and Ricci pseudosymmetric almost $C(\alpha)$-manifold are studied. For an almost $C(\alpha)$-manifold, Riemann pseudosymmetric, Riemann Ricci pseudosymmetric, Ricci pseudosymmetric, projective pseudosymmetric, projective Ricci pseudosymmetric, concircular pseudosymmetric, and concircular Ricci pseudosymmetric cases are considered and new results are obtained.

1. Introduction

In the period from the known past to the present, the place of geometry science in science and technology has always been preserved. Geometry was divided into various branches according to the needs of mankind over time and the studies were carried out more succinctly. One of these areas is differential geometry, where differential computation is applied to geometry. Differential geometry is one of the most popular fields of study in modern mathematics, as it finds applications in many disciplines. The beginning of differential geometry is based on Gauss's work on the curvature of surfaces. These works of Gauss pioneered the concept of Riemannian manifold. Differential geometry is very closely concerned with the properties of Riemannian manifolds where the derivative is defined.

In a Riemannian manifold, the Riemannian curvature tensor is R and for each U_1, $U_2 \in \chi(M)$, if $R(U_1, U_2).R = 0$, then the manifold is said to be semisymmetric. Similarly, if $R(U_1, U_2).S = 0$, the manifold is called Ricci semisymmetric, if $R(U_1, U_2).P = 0$; the manifold is called projective semisymmetric, if $R(U_1, U_2).\widetilde{Z} = 0$; and the manifold is called concircular semisymmetric, where S is the Ricci curvature tensor, P is the projective curvature tensor, and $\widetilde{Z}$ is concircular curvature tensor. Studies on the symmetric Riemannian manifolds started with Cartan [1]. In the following periods, many authors have studied the symmetry cases of various manifolds ([2–12]).

Again, interesting and important studies by many geometers continued to contribute to this field over time. In 2011, Dileo presented an important study on the geometry of the Kenmotsu type manifolds [13], while in 2019, a very important study was made for $\alpha-$ cosymplectic manifolds by Kupeli [14]. Many studies have also been made about almost $C(\alpha)-$ manifolds in [15–18] and important characterizations of this type of manifolds have been obtained ([19, 20]).

Let $M(\varphi, \xi, \eta, g)$ be the $(2n+1)-$ dimensional almost contact metric manifold given by the (φ, ξ, η, g) contact structure, and let the Riemann curvature tensor of the manifold be R. If manifold M satisfies the following condition:

$$\begin{aligned} R(U_1, U_2, U_3, U_4) = R(U_1, U_2, \varphi U_3, \varphi U_4) &+ \alpha\{-g(U_1, U_3)g(U_2, U_4) \\ &+ g(U_1, U_4)g(U_2, U_3) \\ &+ g(U_1, \varphi U_3)g(U_2, \varphi U_4) \\ &- g(U_1, \varphi U_4)g(U_2, \varphi U_3)\}. \end{aligned} \tag{1}$$

Then, manifold M is called almost $C(\alpha)-$ manifold, where every $U_1, U_2, U_3, U_4 \in \chi(M)$ and at least one $\alpha \in \mathbb{R}$ [21].

A $C(\alpha)-$ manifold is the general case of the co-Keahler, Sasakian, and Kenmotsu manifolds. That is, specially, if $\alpha = 0$, it is co-Keahler, if $\alpha = 1$, it is Sasakian, if $\alpha = -1$, it is the Kenmotsu manifold [21]. Although many studies have been conducted on these manifolds, studies on $C(\alpha)-$ manifold is quite limited.

In this article, the pseudosymmetry and Ricci pseudosymmetry properties of almost $C(\alpha)-$ manifold, which are a subclass of almost contact metric manifolds and the general case of the co-Keahler, Sasakian, and Kenmotsu manifolds, have been studied geometrically.

2. Preliminaries

Let M be a differentiable manifold with $(2n+1)$ dimensional. If the condition

$$\begin{gathered}\varphi^2 U_1 = -U_1 + \eta(U_1)\xi,\\ \eta(\xi) = 1,\end{gathered} \tag{2}$$

satisfies on M, where φ is tensor field with type $(1,1)$, ξ is a vector field and η is a $1-$ form, then we say that (φ, ξ, η) is an almost contact structure. Also, we say that (M, φ, ξ, η) is an almost contact manifold [22]. Let g be a metric with condition

$$\begin{gathered}g(\varphi U_1, \varphi U_2) = g(U_1, U_2) - \eta(U_1)\eta(U_2),\\ g(U_1, \xi) = \eta(U_1),\end{gathered} \tag{3}$$

for all $U_1, U_2 \in \chi(M)$ and $\xi \in \chi(M)$. In this case, we say that (φ, ξ, η, g) is an almost contact metric structure and $(M, \varphi, \xi, \eta, g)$ is almost contact metric manifold [22]. Moreover, we have the property

$$g(\varphi U_1, U_2) = -g(U_1, \varphi U_2), \tag{4}$$

for all $U_1, U_2 \in \chi(M)$ on M manifold with $(2n+1)$ dimensional. The fundamental 2-form of the (φ, ξ, η, g) almost contact metric structure is the Φ transformation such that

$$\Phi(U_1, U_2) = g(U_1, \varphi U_2), \tag{5}$$

for all $U_1, U_2 \in \chi(M)$, where

$$\eta \wedge \Phi^n \neq 0. \tag{6}$$

We can write the Riemann curvature tensor of an almost $C(\alpha)$-manifold which has a $c-$ constant sectional curvature by

$$\begin{aligned}R(U_1, U_2)U_3 = {}& \left(\frac{c+3\alpha}{4}\right)\{g(U_2, U_3)U_1 - g(U_1, U_3)U_2\}\\ &+\left(\frac{c-\alpha}{4}\right)\{g(U_1, \varphi U_3)\varphi U_2 - g(U_2, \varphi U_3)\varphi U_1\\ &+ 2g(U_1, \varphi U_2)\varphi U_3 + \eta(U_2)\eta(U_3)U_1\\ &+ g(U_1, U_3)\eta(U_2)\xi - g(U_2, U_3)\eta(U_1)\xi\}.\end{aligned} \tag{7}$$

If we take $U_1 = \xi$ in (4), then we obtain as follows:

$$R(\xi, U_2)U_3 = \alpha[g(U_2, U_3)\xi - \eta(U_3)U_2]. \tag{8}$$

If we take $U_3 = \xi$ in (4), then we have as follows:

$$R(U_1, U_2)\xi = \alpha[\eta(U_2)U_1 - \eta(U_1)U_2]. \tag{9}$$

Moreover, if we take $U_2 = \xi$ in (9), then we obtain as follows:

$$R(U_1, \xi)\xi = \alpha[U_1 - \eta(U_1)\xi]. \tag{10}$$

Let us take an inner product of (7) by $\xi \in \chi(M)$. Then, we obtain as follows:

$$\eta(R(U_1, U_2)U_3) = \alpha[g(U_2, U_3)\eta(U_1) - g(U_1, U_3)\eta(U_2)]. \tag{11}$$

Let for $k \geq 1, \sigma$ be a $(0,k)$-type tensor field, a A be a $(0,2)$-type tensor field and M is a Riemannian manifold. We define $Q(A, \sigma)$-tensor field by

$$\begin{aligned}Q(A,\sigma)(X_1, X_2, \ldots, X_k; U_1, U_2) = {}& -\sigma((U_1 \Lambda_A U_2)X_1, X_2, \ldots, X_k)\ldots\\ &- \sigma(X_1, X_2, \ldots, X_{k-1}, (U_1 \Lambda_A U_2)X_k),\end{aligned} \tag{12}$$

for all $X_1, X_2, \ldots, X_k, U_1, U_2 \in \Gamma(TM)$, where

$$(U_1 \wedge_A U_2)U_3 = A(U_2, U_3)U_1 - A(U_1, U_3)U_2. \tag{13}$$

Lemma 1. *Let M be an almost $C(\alpha)-$ manifold with $(2n+1)-$ dimensional. Then, we have as follows:*

$$\begin{aligned}S(U_1, U_2) = {}& \left[\frac{\alpha(3n-1) + c(n+1)}{2}\right]g(U_1, U_2)\\ &+ \frac{(\alpha - c)(n+1)}{2}\eta(U_1)\eta(U_2),\end{aligned} \tag{14}$$

$$S(U_1, \xi) = 2n\alpha\eta(U_1), \tag{15}$$

$$QU_1 = \left[\frac{\alpha(3n-1) + c(n+1)}{2}\right]U_1 + \frac{(\alpha - c)(n+1)}{2}\eta(U_1)\xi, \tag{16}$$

$$Q\xi = 2n\alpha\xi, \tag{17}$$

for each $U_1, U_2, \in \chi(M)$, where Q is the Ricci operator and S is the Ricci tensor of manifold M.

If S and g are linearly dependent, that is, there is a constant λ provided

$$S(U_1, U_2) = \lambda g(U_1, U_2), \tag{18}$$

the almost $C(\alpha)-$ manifold is called the Einstein manifold [23]. In particular, if the S Ricci tensor satisfies the relation

$$S(U_1,U_2) = ag(U_1,U_2) + b\eta(U_1)\eta(U_2), \tag{19}$$

for each $U_1,U_2 \in \chi(M)$, M is called an $\eta-$ Einstein manifold [23].

Let $(M,\ g)$ be a Riemannian manifold. For two-dimensional subspace Π of the tangent, let space $T_p(M)$ be

$$g(U_1,U_1)g(U_2,U_2) - g(U_1,U_2)^2 \neq 0, \tag{20}$$

for each $U_1,U_2 \in \Pi$. In this case, $K(U_1\Lambda U_2)$ defined as follows

$$K(U_1 \Lambda U_2) = \frac{g(R(U_1,U_2)U_2,U_1)}{g(U_1,U_1)g(U_2,U_2) - g(U_1,U_2)^2}, \tag{21}$$

is called the sectional curvature of the plane Π. If $K(U_{1p}\Lambda U_{2p})$ is a constant for each $U_1,U_2 \in T_p(M)$, $p \in M$, M is called a $c-$ constant section curvature space or real space form [22]. In this case, if the M Riemann manifold is a real space form and $c-$ constant section curvature, the Riemann curvature tensor of M is as follows:

$$\begin{aligned} g(R(U_1,U_2)U_3,U_4) = c[&g(U_2,U_3)g(U_1,U_4) \\ &- g(U_1,U_3)g(U_2,U_4)], \end{aligned} \tag{22}$$

for each $U_1,U_2,U_3,U_4 \in \chi(M)$.

3. Pseudosymmetric and Ricci Pseudosymmetric Almost $C(\alpha)$-Manifold

In this section, the cases of pseudosymmetry and Ricci pseudosymmetry of an almost $C(\alpha)-$ manifold are investigated. According to Riemann, Ricci, projective, and concircular curvature tensors, the pseudosymmetrical and Ricci pseudosymmetrical cases of the almost $C(\alpha)-$ manifold can be given as follows.

Definition 1. Let M be an almost $C(\alpha)-$ manifold with $(2n+1)-$ dimensional, R be the Riemann curvature tensor of M, and S be the Ricci curvature tensor of M.

(i) If the pair $R\cdot R$ and $Q(g,R)$ are linearly dependent, that is, if a λ_1 function can be found on the set $M_1 = \{x \in M | g(x) \neq R(x)\}$ such that

$$R\cdot R = \lambda_1 Q(g,R), \tag{23}$$

the M manifold is called a Riemann pseudosymmetric manifold.

(ii) If the pair $R\cdot R$ and $Q(S,R)$ are linearly dependent, that is, if a λ_2 function can be found on the set $M_2 = \{x \in M | S(x) \neq R(x)\}$ such that

$$R\cdot R = \lambda_2 Q(S,R), \tag{24}$$

the M manifold is called a Riemann Ricci pseudosymmetric manifold.;

Particularly, if $\lambda_1 = 0$, then this manifold is said to be semisymmetric.

Let us now investigate the cases of Riemann pseudosymmetry and Riemann Ricci pseudosymmetry.

Theorem 1. *If a $(2n+1)-$ dimensional M almost $C(\alpha)-$ manifold is a Riemann pseudo-symmetric manifold, then $\lambda_1 = 0$.*

Proof. Let us assume that the manifold M is a Riemann pseudosymmetric manifold. Then, we can write

$$(R(U_1,U_2)\cdot R)(U_3,U_4,U_5) = \lambda_1 Q(g,R)(U_5,U_4,U_3;U_1,U_2), \tag{25}$$

for each $U_1,U_2,U_3,U_4,U_5 \in \chi(M)$. In this case, we obtain as follows:

$$\begin{aligned} &R(U_1,U_2)R(U_3,U_4)U_5 - R(R(U_1,U_2)U_3,U_4)U_5 \\ &- R(U_3,R(U_1,U_2)U_4)U_5 - R(U_3,U_4)R(U_1,U_2)U_5, \\ = &-\lambda_1\{R((U_1\Lambda_g U_2)U_5,U_4)U_3 + R(U_5,(U_1\Lambda_g U_2)U_4)U_3 \\ &+ R(U_5,U_4)(U_1\Lambda_g U_2)U_3\}. \end{aligned} \tag{26}$$

If necessary arrangements are made in here, and we choose $U_1 = U_3 = \xi$ and using the expression (8) in (26), we obtain as follows:

$$\begin{aligned} &\alpha g(U_2,R(\xi,U_4)U_5)\xi - \alpha\eta(R(\xi,U_4)U_5)U_2 \\ &- \alpha^2\eta(U_2)g(U_4,U_5)\xi + \alpha^2\eta(U_5)\eta(U_2)U_4 \\ &+ \alpha R(U_2,U_4)U_5 + \alpha^2\eta(U_5)g(U_2,U_4)\xi \\ &- \alpha^2\eta(U_5)g(U_2,U_4)\xi + \alpha\eta(U_4)R(\xi,U_2)U_5 \\ &- \alpha^2\eta(U_4)g(U_2,U_5)\xi + \alpha^2 g(U_2,U_5)U_4 \\ &+ \alpha\eta(U_5)R(\xi,U_4)U_2 = -\lambda_1\{\alpha g(U_2,U_5)\eta(U_4)\xi \\ &- \alpha g(U_2,U_5)U_4 - \eta(U_5)R(U_2,U_4)\xi \\ &- \alpha g(U_2,U_4)\eta(U_5)\xi + \alpha g(U_2,U_4)U_5 \\ &- \eta(U_4)R(U_5,U_2)\xi + \alpha\eta(U_4)\eta(U_2)U_5 \\ &-\alpha\eta(U_5)\eta(U_2)U_4 - R(U_5,U_4)U_2\}. \end{aligned} \tag{27}$$

If we use the expression (8) again in (27) and make the necessary adjustments, then we obtain as follows:

$$\begin{aligned} &\alpha R(U_2,U_4)U_5 + \alpha^2 g(U_2,U_5)U_4 - \alpha^2 g(U_4,U_5)U_2 \\ &= -\lambda_1\{\alpha g(U_2,U_5)\eta(U_4)\xi \\ &\quad - \alpha g(U_2,U_4)\eta(U_5)\xi - \alpha g(U_2,U_5)U_4 \\ &\quad + \alpha g(U_2,U_4)U_5 - R(U_5,U_4)U_2\}. \end{aligned} \tag{28}$$

If we apply $\xi \in \chi(M)$ to both sides of (28) and make use of (11), then we obtain as follows:

$$\lambda_1\eta(R(U_5,U_4)U_2) = 0. \tag{29}$$

Since $\eta(R(U_5,U_4)U_2) \neq 0$, it is clear that

$$\lambda_1 = 0. \tag{30}$$

This completes our proof. □

Corollary 1. *If a $(2n+1)-$ dimensional M almost $C(\alpha)-$ manifold is a Riemann pseudosymmetric manifold, then M is a Riemann semisymmetric manifold.*

Corollary 2. *If a $(2n+1)-$ dimensional M almost $C(\alpha)-$ manifold is a Riemann pseudosymmetric manifold, then M is a real space form with a constant section curvature.*

Theorem 2. *If a $(2n+1)-$ dimensional M almost $C(\alpha)-$ manifold is a Riemann Ricci pseudosymmetric manifold, then M is either co-Keahler manifold or $\lambda_2 = 0$.*

Proof. Let us assume that the manifold M is a Riemann Ricci pseudosymmetric manifold. Then, we can write as follows:

$$(R(U_1,U_2)\cdot R)(U_3,U_4,U_5) = \lambda_2 Q(S,R)(U_5,U_4,U_3;U_1,U_2), \tag{31}$$

for each $U_1,U_2,U_3,U_4,U_5 \in \chi(M)$. In this case, we obtain as follows:

$$\begin{aligned} &R(U_1,U_2)R(U_3,U_4)U_5 - R(R(U_1,U_2)U_3,U_4)U_5 \\ &\quad - R(U_3,R(U_1,U_2)U_4)U_5 - R(U_3,U_4)R(U_1,U_2)U_5 \\ &= -\lambda_2\{R((U_1\Lambda_S U_2)U_5,U_4)U_3 + R(U_5,(U_1\Lambda_S U_2)U_4)U_3 \\ &\quad + R(U_5,U_4)(U_1\Lambda_S U_2)U_3\}. \end{aligned} \tag{32}$$

If necessary arrangements are made in here, and we choose $U_1 = U_3 = \xi$ and using (8), (15) in (32), we obtain as follows:

$$\begin{aligned} &\alpha g(U_2,R(\xi,U_4)U_5)\xi - \alpha\eta(R(\xi,U_4)U_5)U_2 \\ &- \alpha^2\eta(U_2)g(U_4,U_5)\xi + \alpha^2\eta(U_5)\eta(U_2)U_4 \\ &+ \alpha R(U_2,U_4)U_5 + \alpha^2\eta(U_5)g(U_2,U_4)\xi \\ &- \alpha^2\eta(U_5)g(U_2,U_4)\xi + \alpha\eta(U_4)R(\xi,U_2)U_5 \\ &- \alpha^2\eta(U_4)g(U_2,U_5)\xi + \alpha^2 g(U_2,U_5)U_4 \\ &+ \alpha\eta(U_5)R(\xi,U_4)U_2 = -\lambda_2\{\alpha S(U_2,U_5)\eta(U_4)\xi \\ &- \alpha S(U_2,U_5)U_4 - 2n\alpha\eta(U_5)R(U_2,U_4)\xi \\ &- \alpha S(U_2,U_4)\eta(U_5)\xi + \alpha S(U_2,U_4)U_5 \\ &- 2n\alpha\eta(U_4)R(U_5,U_2)\xi + 2n\alpha^2\eta(U_4)\eta(U_2)U_5 \\ &-2n\alpha^2\eta(U_5)\eta(U_2)U_4 - 2n\alpha R(U_5,U_4)U_2\}. \end{aligned} \tag{33}$$

If we use the expression (5) again in (33) and make the necessary adjustments, then we obtain as follows:

$$\begin{aligned} &\alpha R(U_2,U_4)U_5 + \alpha^2 g(U_2,U_5)U_4 - \alpha^2 g(U_4,U_5)U_2 \\ &= -\lambda_2\{\alpha S(U_2,U_5)\eta(U_4)\xi - \alpha S(U_2,U_4)\eta(U_5)\xi - \alpha S(U_2,U_5)U_4 \\ &\quad + \alpha S(U_2,U_4)U_5 - 2n\alpha R(U_5,U_4)U_2\}. \end{aligned} \tag{34}$$

If we apply $\xi \in \chi(M)$ to both sides of (34) and make use of (11), then we obtain as follows:

$$2n\alpha\lambda_2\eta(R(U_5,U_4)U_2) = 0. \tag{35}$$

Since $\eta(R(U_5,U_4)U_2) \neq 0$, it is clear that

$$\alpha\lambda_2 = 0. \tag{36}$$

Thus, either $\alpha = 0$ or $\lambda_2 = 0$ is obtained. This completes our proof. □

Corollary 3. *If a $(2n+1)-$ dimensional M almost $C(\alpha)-$ manifold is a Riemann Ricci pseudosymmetric manifold, then M is either co-Keahler manifold or a Riemann semisymmetric manifold.*

Corollary 4. *If a $(2n+1)-$ dimensional M almost $C(\alpha)-$ manifold is a Riemann Ricci pseudosymmetric manifold, then M is a real space form with a constant section curvature.*

Definition 2. Let M be an almost $C(\alpha)-$ manifold with $(2n+1)-$ dimensional, R be the Riemann curvature tensor of M and S be the Ricci curvature tensor of M. If the pair $R\cdot S$ and $Q(g,S)$ are linearly dependent, that is, if a λ_3 function can be found on the set $M_3 = \{x \in M | g(x) \neq S(x)\}$ such that

$$R\cdot S = \lambda_3 Q(g,S), \tag{37}$$

the M manifold is called a Ricci pseudosymmetric manifold.

Let us now investigate the case of Ricci pseudosymmetry.

Theorem 3. *If a $(2n+1)-$ dimensional M almost $C(\alpha)-$ manifold is a Ricci pseudosymmetric manifold, then M is either an Einstein manifold or $\lambda_3 = -\alpha$.*

Proof. Let us assume that the manifold M is a Ricci pseudosymmetric manifold. Then, we can write as follows:

$$(R(U_1,U_2)\cdot S)(U_5,U_4) = \lambda_3 Q(g,S)(U_5,U_4;U_1,U_2), \tag{38}$$

for each $U_1,U_2,U_4,U_5 \in \chi(M)$. In this case, we can write as follows:

$$\begin{aligned} &S(R(U_1,U_2)U_5,U_4) + S(U_5,R(U_1,U_2)U_4) \\ &= -\lambda_3\{S((U_1\Lambda_g U_2)U_5,U_4) + S(U_5,(U_1\Lambda_g U_2)U_4)\}. \end{aligned} \tag{39}$$

In equation (39), first $U_1 = \xi$ is chosen and then (8) and (15) are used, we obtain as follows:

$$\begin{aligned} &c2n\alpha^2 g(U_2,U_5)\eta(U_4) - \alpha\eta(U_5)S(U_2,U_4) \\ &+ 2n\alpha^2 g(U_2,U_4)\eta(U_5) - \alpha\eta(U_4)S(U_5,U_2) \\ &= -\lambda_3\{2n\alpha\eta(U_4)g(U_2,U_5) - \eta(U_5)S(U_2,U_4) \\ &\quad + 2n\alpha\eta(U_5)g(U_2,U_4) - \eta(U_4)S(U_5,U_2)\}. \end{aligned} \tag{40}$$

If we choose $U_5 = \xi$ in (40) and make the necessary adjustments, we obtain as follows:

$$-\alpha S(U_2,U_4) + 2n\alpha^2 g(U_2,U_4) = -\lambda_3\{-S(U_2,U_4) + 2n\alpha g(U_2,U_4)\}, \tag{41}$$

and from here, we can write that

$$(\alpha+\lambda_3)[-S(U_2,U_4) + 2n\alpha g(U_2,U_4)] = 0. \tag{42}$$

It is clear from equation (42) that either

$$\alpha + \lambda_3 = 0, \tag{43}$$

$$\text{or } S(U_2, U_4) = 2n\alpha g(U_2, U_4). \tag{44}$$

This completes the proof of the theorem.

The projective curvature tensor P is defined as follows:

$$P(U_1, U_2)U_3 = R(U_1, U_2)U_3 - \frac{1}{2n}[S(U_2, U_3)U_1 - S(U_1, U_3)U_2], \tag{45}$$

for all each $U_1, U_2, U_3 \in \chi(M)$, by Yano and Sawaki [24]. If $U_1 = \xi$, $U_2 = \xi$, and $U_3 = \xi$ are selected, respectively, in (45), then we obtain as follows:

$$P(\xi, U_2)U_3 = \alpha g(U_2, U_3)\xi - \frac{1}{2n}S(U_2, U_3)\xi, \tag{46}$$

$$P(U_1, \xi)U_3 = -\alpha g(U_1, U_3)\xi + \frac{1}{2n}S(U_1, U_3)\xi, \tag{47}$$

$$P(U_1, U_2)\xi = 0. \tag{48}$$

Let us take inner product of (45) by $\xi \in \chi(M)$, we get

$$\begin{aligned} cc\eta(P(U_1, U_2)U_3) &= \eta(U_1)\left[\alpha g(U_2, U_3) - \frac{1}{2n}S(U_2, U_3)\right] \\ &\quad - \eta(U_2)\left[\alpha g(U_1, U_3) - \frac{1}{2n}S(U_1, U_3)\right]. \end{aligned} \tag{49}$$

□

Definition 3. Let M be a $(2n+1)-$ dimensional almost $C(\alpha)-$ manifold, R be the Riemann curvature tensor of M, S be the Ricci curvature tensor of M, and P be the projective curvature tensor of M.

(i) If the pair $R \cdot P$ and $Q(g, P)$ are linearly dependent, that is, if a λ_4 function can be found on the set $M_4 = \{x \in M | g(x) \neq P(x)\}$ such that

$$R \cdot P = \lambda_4 Q(g, P), \tag{50}$$

the M manifold is called a projective pseudosymmetric manifold.

(ii) If the pair $R \cdot P$ and $Q(S, P)$ are linearly dependent, that is, if a λ_5 function can be found on the set $M_5 = \{x \in M | S(x) \neq P(x)\}$ such that

$$R \cdot P = \lambda_5 Q(S, P). \tag{51}$$

The M manifold is called a projective Ricci pseudosymmetric manifold.

Let us now investigate the case of projective pseudosymmetry and projective Ricci pseudosymmetry.

Theorem 4. *If a $(2n+1)-$ dimensional M almost $C(\alpha)-$ manifold is a projective pseudosymmetric manifold, then M is either an Einstein manifold or $\lambda_4 = \alpha$.*

Proof. Let us assume that the manifold M is a projective pseudosymmetric manifold. Then, we can write as follows:

$$\begin{aligned} &(R(U_1, U_2) \cdot P)(U_5, U_4, U_3) \\ &\quad = \lambda_4 Q(g, P)(U_5, U_4, U_3; U_1, U_2), \end{aligned} \tag{52}$$

for each $U_1, U_2, U_3, U_4, U_5 \in \chi(M)$. In this case, we obtain as follows:

$$\begin{aligned} &R(U_1, U_2)P(U_5, U_4)U_3 - P(R(U_1, U_2)U_5, U_4)U_3 \\ &\quad - P(U_5, R(U_1, U_2)U_4)U_3 - P(U_5, U_4)R(U_1, U_2)U_3 \\ &= -\lambda_4\{P((U_1\Lambda_g U_2)U_5, U_4)U_3 + P(U_5, (U_1\Lambda_g U_2)U_4)U_3 \\ &\quad + P(U_5, U_4)(U_1\Lambda_g U_2)U_3\}. \end{aligned} \tag{53}$$

In the last equation, if $U_1 = \xi$ is chosen and necessary corrections are made, we can write

$$\begin{aligned} &R(\xi, U_2)P(U_5, U_4)U_3 - P(R(\xi, U_2)U_5, U_4)U_3 \\ &\quad - P(U_5, R(\xi, U_2)U_4)U_3 - P(U_5, U_4)R(\xi, U_2)U_3 \\ &= -\lambda_4\{g(U_2, U_5)P(\xi, U_4)U_3 - g(\xi, U_5)P(U_2, U_4)U_3 \\ &\quad + g(U_2, U_4)P(U_5, \xi)U_3 - g(\xi, U_4)P(U_5, U_2)U_3 \\ &\quad + g(U_2, U_3)P(U_5, U_4)\xi - g(\xi, U_3)P(U_5, U_4)U_2\}. \end{aligned} \tag{54}$$

If (8), (46), and (47) are used in (54), we obtain follows:

$$\begin{aligned} &\alpha g(U_2, P(U_5, U_4)U_3)\xi - \alpha\eta(P(U_5, U_4)U_3)U_2 \\ &\quad - \alpha g(U_2, U_5)P(\xi, U_4)U_3 + \alpha\eta(U_5)P(U_2, U_4)U_3 \\ &\quad - \alpha g(U_2, U_4)P(U_5, \xi)U_3 + \alpha\eta(U_4)P(U, U_2)U_3 \\ &\quad - \alpha g(U_2, U_3)P(U_5, U_4)\xi + \alpha\eta(U_3)P(U_5, U_4)U_2 \\ &= -\lambda_4\left\{\alpha g(U_2, U_5)g(U_4, U_3)\xi - \frac{1}{2n}g(U_2, U_5)S(U_4, U_3)\xi\right. \\ &\quad - \eta(U_5)P(U_2, U_4)U_3 - \alpha g(U_2, U_4)g(U_5, U_3)\xi \\ &\quad + \frac{1}{2n}g(U_2, U_4)S(U_5, U_3)\xi - \eta(U_4)P(U_5, U_2)U_3 \\ &\quad \left. - \eta(U_3)P(U_5, U_4)U_2\right\}. \end{aligned} \tag{55}$$

If $U_5 = \xi$ is chosen in (55), equation (8) is used and necessary adjustments are made, we obtain as follows:

$$
\begin{aligned}
&l-\alpha^2 g(U_4,U_3)U_2+\alpha R(U_2,U_4)U_3\\
&+\frac{\alpha}{2n}S(U_2,U_3)U_4+\alpha^2 g(U_2,U_3)\eta(U_4)\xi\\
&-\frac{\alpha}{2n}\eta(U_4)S(U_2,U_3)\xi+\alpha^2\eta(U_3)g(U_4,U_2)\xi\\
&-\frac{1}{2n}\eta(U_2)S(U_4,U_3)\xi-R(U_2,U_4)U_3\\
&-\frac{\alpha}{2n}\eta(U_3)S(U_4,U_2)\xi=-\lambda_4\Big\{\alpha g(U_4,U_3)\eta(U_2)\xi\\
&+\frac{1}{2n}S(U_4,U_3)U_2-\frac{1}{2n}S(U_2,U_3)U_4\\
&-\alpha\eta(U_4)g(U_2,U_3)\xi+\frac{1}{2n}\eta(U_4)S(U_2,U_3)\xi\\
&-\alpha g(U_4,U_2)\eta(U_3)\xi+\frac{1}{2n}\eta(U_3)S(U_4,U_2)\xi\Big\}.
\end{aligned}
\tag{56}
$$

If we apply $\xi\in\chi(M)$ to both sides of equation (54) and make use of equation (11), we obtain as follows:

$$
(\alpha-\lambda_4)\alpha\eta(U_3)g(U_4,U_2)-\frac{1}{2n}(\alpha-\lambda_4)\eta(U_3)S(U_4,U_2)=0.
\tag{57}
$$

If $U_3=\xi$ is chosen in the last equation, we obtain as follows:

$$
(\alpha-\lambda_4)[-S(U_4,U_2)+2n\alpha g(U_4,U_2)]=0,
\tag{58}
$$

that is,

$$
\begin{aligned}
\alpha&=\lambda_4,\\
S(U_4,U_2)&=2n\alpha g(U_4,U_2).
\end{aligned}
\tag{59}
$$

Thus, the proof of the theorem is complete. □

Theorem 5. *If a $(2n+1)-$ dimensional M almost $C(\alpha)-$ manifold is a projective Ricci pseudosymmetric manifold, then M is either an Einstein manifold, co-Keahler or $\lambda_5=(1/2n)$.*

Proof. Let us assume that the manifold M is a projective Ricci pseudosymmetric manifold. Then, we can write

$$
(R(U_1,U_2)\cdot P)(U_5,U_4,U_3)=\lambda_5 Q(S,P)(U_5,U_4,U_3;U_1,U_2),
\tag{60}
$$

for each $U_1,U_2,U_3,U_4,U_5\in\chi(M)$. In this case, we obtain as follows:

$$
\begin{aligned}
&R(U_1,U_2)P(U_5,U_4)U_3-P(R(U_1,U_2)U_5,U_4)U_3\\
&-P(U_5,R(U_1,U_2)U_4)U_3-P(U_5,U_4)R(U_1,U_2)U_3\\
&=-\lambda_5\{P((U_1\Lambda_S U_2)U_5,U_4)U_3+P(U_5,(U_1\Lambda_S U_2)U_4)U_3\\
&+P(U_5,U_4)(U_1\Lambda_S U_2)U_3\}.
\end{aligned}
\tag{61}
$$

In the last equation, if $U_1=\xi$ is chosen and necessary corrections are made, we can write

$$
\begin{aligned}
&R(\xi,U_2)P(U_5,U_4)U_3-P(R(\xi,U_2)U_5,U_4)U_3\\
&-P(U_5,R(\xi,U_2)U_4)U_3-P(U_5,U_4)R(\xi,U_2)U_3\\
&=-\lambda_5\{S(U_2,U_5)P(\xi,U_4)U_3-S(\xi,U_5)P(U_2,U_4)U_3\\
&+S(U_2,U_4)P(U_5,\xi)U_3-S(\xi,U_4)P(U_5,U_2)U_3\\
&+S(U_2,U_3)P(U_5,U_4)\xi-S(\xi,U_3)P(U_5,U_4)U_2\}.
\end{aligned}
\tag{62}
$$

If (8), (46), and (47) are used in (62), we obtain as follows:

$$
\begin{aligned}
&\alpha g(U_2,P(U_5,U_4)U_3)\xi-\alpha\eta(P(U_5,U_4)U_3)U_2\\
&-\alpha g(U_2,U_5)P(\xi,U_4)U_3+\alpha\eta(U_5)P(U_2,U_4)U_3\\
&-\alpha g(U_2,U_4)P(U_5,\xi)U_3+\alpha\eta(U_4)P(U_5,U_2)U_3\\
&-\alpha g(U_2,U_3)P(U_5,U_4)\xi+\alpha\eta(U_3)P(U_5,U_4)U_2\\
&=-\lambda_5\Big\{\alpha S(U_2,U_5)g(U_4,U_3)\xi-\frac{1}{2n}S(U_2,U_5)S(U_4,U_3)\xi\\
&-2n\alpha\eta(U_5)P(U_2,U_4)U_3-\alpha S(U_2,U_4)g(U_5,U_3)\xi\\
&+\frac{1}{2n}S(U_2,U_4)S(U_5,U_3)\xi-2n\alpha\eta(U_4)P(U_5,U_2)U_3\\
&-2n\alpha\eta(U_3)P(U_5,U_4)U_2\Big\}.
\end{aligned}
\tag{63}
$$

If $U_5=\xi$ is chosen in (63), equation (8) is used and necessary adjustments are made, we obtain as follows:

$$
\begin{aligned}
&l-\alpha^2 g(U_4,U_3)U_2+\alpha R(U_2,U_4)U_3\\
&+\frac{\alpha}{2n}S(U_2,U_3)U_4+\alpha^2 g(U_2,U_3)\eta(U_4)\xi\\
&-\frac{\alpha}{2n}S(U_2,U_3)U_4+\alpha^2\eta(U_3)g(U_4,U_2)\xi\\
&-\frac{\alpha}{2n}\eta(U_3)S(U_4,U_2)\xi=-\lambda_5\{2n\alpha^2 g(U_4,U_3)\eta(U_2)\xi\\
&-\alpha\eta(U_2)S(U_4,U_3)\xi-2n\alpha R(U_2,U_4)U_3\\
&+\alpha S(U_4,U_3)U_2-\alpha S(U_2,U_3)U_4\\
&-2n\alpha^2\eta(U_4)g(U_2,U_3)\xi+\alpha\eta(U_4)S(U_2,U_3)\xi\\
&-2n\alpha^2 g(U_4,U_2)\eta(U_3)\xi+\alpha\eta(U_3)S(U_4,U_2)\xi\}.
\end{aligned}
\tag{64}
$$

If we apply $\xi\in\chi(M)$ to both sides of equation (65) and make use of equation (11), we obtain as follows:

$$2n\alpha^2(1-2n\lambda_5)\eta(U_3)g(U_4,U_2)-\alpha(1-2n\lambda_5)\eta(U_3)S(U_4,U_2)=0. \tag{65}$$

If $U_3=\xi$ is chosen in the last equation, we yield as follows:

$$\alpha(1-2n\lambda_5)[-S(U_4,U_2)+2n\alpha g(U_4,U_2)]=0, \tag{66}$$

that is,

$$\begin{aligned}\alpha=0, 1-2n\lambda_5=0,\\ \text{or } S(U_4,U_2)=2n\alpha g(U_4,U_2).\end{aligned} \tag{67}$$

Thus, the proof of the theorem is complete.

Let M be a Riemannian manifold with $(2n+1)-$ dimensional. Then, the concircular curvature tensor $\widetilde{Z}$ is defined as

$$\widetilde{Z}(U_1,U_2)U_3=R(U_1,U_2)U_3 - \frac{r}{2n(2n+1)}[g(U_2,U_3)U_1-g(U_1,U_3)U_2], \tag{68}$$

for each $U_1,U_2,U_3\in\chi(M)$ [25]. If we choose, respectively, $U_1=\xi$, $U_2=\xi$, and $U_3=\xi$ in (68), then we obtain as follows:

$$\widetilde{Z}(\xi,U_2)U_3=\left(\alpha-\frac{r}{2n(2n+1)}\right)[g(U_2,U_3)\xi-\eta(U_3)U_2], \tag{69}$$

$$\widetilde{Z}(U_1,\xi)U_3=\left(\alpha-\frac{r}{2n(2n+1)}\right)[-g(U_1,U_3)\xi+\eta(U_3)U_1], \tag{70}$$

$$\widetilde{Z}(U_1,U_2)\xi=\left(\alpha-\frac{r}{2n(2n+1)}\right)[\eta(U_2)U_1-\eta(U_1)U_2]. \tag{71}$$

In addition, we choose $U_3=\xi$ in (69), we obtain as follows:

$$\widetilde{Z}(\xi,U_2)\xi=\left(\alpha-\frac{r}{2n(2n+1)}\right)[\eta(U_2)\xi-U_2]. \tag{72}$$

and finally, we take inner product of (68) by $\xi\in\chi(M)$, we have

$$\eta\left(\widetilde{Z}(U_1,U_2)U_3\right)=\eta(R(U_1,U_2)U_3)-\frac{r}{2n(2n+1)} \cdot [g(U_2,U_3)\eta(U_1)-g(U_1,U_3)\eta(U_2)]. \tag{73}$$

□

Definition 4. Let M be a $(2n+1)-$ dimensional almost $C(\alpha)-$ manifold, R be the Riemann curvature tensor of M, S be the Ricci curvature tensor of M, and $\widetilde{Z}$ be the concircular curvature tensor of M.

(i) If the pair $R\cdot\widetilde{Z}$ and $Q(g,\widetilde{Z})$ are linearly dependent, that is, if a λ_6 function can be found on the set $M_6=\{x\in M|g(x)\neq\widetilde{Z}(x)\}$ such that

$$R\cdot\widetilde{Z}=\lambda_6 Q(g,\widetilde{Z}), \tag{74}$$

the M manifold is called a concircular pseudosymmetric manifold.

(ii) If the pair $R\cdot\widetilde{Z}$ and $Q(S,\widetilde{Z})$ are linearly dependent, that is, if a λ_7 function can be found on the set $M_7=\{x\in M|S(x)\neq\widetilde{Z}(x)\}$ such that

$$R\cdot\widetilde{Z}=\lambda_7 Q(S,\widetilde{Z}), \tag{75}$$

the M manifold is called a concircular Ricci pseudosymmetric manifold.

Let us now investigate the cases of concircular pseudosymmetry and concircular Ricci pseudosymmetry.

Theorem 6. *If a $(2n+1)-$ dimensional M almost $C(\alpha)-$ manifold is a concircular pseudosymmetric manifold, then we can see either $\lambda_6=0$ or $\alpha=(r/2n(2n+1))$.*

Proof. Let us assume that the manifold M is a concircular pseudosymmetric manifold. Then, we can write

$$\left(R(U_1,U_2)\cdot\widetilde{Z}\right)(U_5,U_4,U_3)=\lambda_6 Q(g,\widetilde{Z})(U_5,U_4,U_3;U_1,U_2), \tag{76}$$

for each $U_1,U_2,U_3,U_4,U_5\in\chi(M)$. In this case, we yield as follows:

$$\begin{aligned}&R(U_1,U_2)\widetilde{Z}(U_5,U_4)U_3\\ &-\widetilde{Z}(R(U_1,U_2)U_5,U_4)U_3\\ &-\widetilde{Z}(U_5,R(U_1,U_2)U_4)U_3-\widetilde{Z}(U_5,U_4)R(U_1,U_2)U_3\\ &=-\lambda_6\left\{\widetilde{Z}\left(\left(U_1\Lambda_g U_2\right)U_5,U_4\right)U_3+\widetilde{Z}\left(U_5,\left(U_1\Lambda_g U_2\right)U_4\right)U_3\right.\\ &\left.+\widetilde{Z}(U_5,U_4)\left(U_1\Lambda_g U_2\right)U_3\right\}.\end{aligned} \tag{77}$$

If necessary arrangements are made in here and we choose $U_1=U_5=\xi$ and using (8), (69) in (77), we obtain as follows:

$$\begin{aligned}
&\alpha g\left(U_2,\widetilde{Z}(\xi,U_4)U_3\right)\xi-\alpha\eta\left(\widetilde{Z}(\xi,U_4)U_3\right)U_2\\
&\quad-\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)\eta(U_2)g(U_4,U_3)\xi+\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)\eta(U_3)\eta(U_2)U_4\\
&\quad+\alpha\widetilde{Z}(U_2,U_4)U_3+\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)\eta(U_3)g(U_2,U_4)\xi\\
&\quad-\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)\eta(U_3)g(U_2,U_4)\xi+\alpha\eta(U_4)\widetilde{Z}(\xi,U_2)U_3\\
&\quad-\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)\eta(U_4)g(U_2,U_3)\xi+\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_2,U_3)U_4\\
&\quad+\alpha\eta(U_3)\widetilde{Z}(\xi,U_4)U_2=-\lambda_6\left\{\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_4,U_3)\eta(U_2)\xi\right.\\
&\quad-\left(\alpha-\frac{r}{2n(2n+1)}\right)\eta(U_2)\eta(U_3)U_4-\widetilde{Z}(U_2,U_4)U_3\\
&\quad-\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_2,U_4)\eta(U_3)\xi+\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_2,U_4)\eta(U_3)\xi\\
&\quad-\eta(U_4)\widetilde{Z}(\xi,U_2)U_3+\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_2,U_3)\eta(U_4)\xi\\
&\quad\left.-\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_2,U_3)U_4-\eta(U_3)\widetilde{Z}(\xi,U_4)U_2\right\}.
\end{aligned}\tag{78}$$

If we use the expression (69) again in (78) and make the necessary adjustments, then we obtain as follows:

$$\begin{aligned}
&-\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_4,U_3)U_2+\alpha\widetilde{Z}(U_2,U_4)U_3\\
&-\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_2,U_3)U_4=-\lambda_6\left\{\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_4,U_3)\eta(U_2)\xi\right.\\
&-\widetilde{Z}(U_2,U_4)U_3+\left(\alpha-\frac{r}{2n(2n+1)}\right)\eta(U_4)\eta(U_3)U_2\\
&\left.-\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_2,U_3)U_4-\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_4,U_2)\eta(U_3)\xi\right\}.
\end{aligned}\tag{79}$$

If we replace the expression (68) in (79) and make the necessary adjustments, we have the following:

$$
\begin{aligned}
&-\alpha^2 g(U_4,U_3)U_2+\alpha^2 g(U_2,U_3)U_4+\alpha R(U_2,U_4)U_3 \\
&=-\lambda_6\left\{\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_4,U_3)\eta(U_2)\xi-R(U_2,U_4)U_3\right. \\
&+\frac{\alpha r}{2n(2n+1)}g(U_4,U_3)U_2-\frac{\alpha r}{2n(2n+1)}g(U_2,U_3)U_4 \\
&+\left(\alpha-\frac{r}{2n(2n+1)}\right)\eta(U_4)\eta(U_3)U_2 \\
&-\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_2,U_3)U_4 \\
&\left.-\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_4,U_2)\eta(U_3)\xi\right\}.
\end{aligned} \tag{80}
$$

If we apply $\xi\in\chi(M)$ to both sides of (80) and make use of (11), then we obtain as follows:

$$
\lambda_6\left(\alpha-\frac{r}{2n(2n+1)}\right)\left[-g(U_4,U_2)+\eta(U_4)\eta(U_2)\right]=0. \tag{81}
$$

On the other hand, we know that there is a g metric such that

$$
g(\varphi U_4,\varphi U_2)=g(U_4,U_2)-\eta(U_4)\eta(U_2), \tag{82}
$$

on a $(2n+1)$ dimensional almost contact manifold [26]. Therefore, equation (81) can be written as follows:

$$
\lambda_6\left(\alpha-\frac{r}{2n(2n+1)}\right)g(\varphi U_4,\varphi U_2)=0. \tag{83}
$$

It is clear from the last equation that

$$
\begin{aligned}
&\lambda_6=0, \\
&\text{or } \alpha=\frac{r}{2n(2n+1)}.
\end{aligned} \tag{84}
$$

This completes our proof. □

Corollary 5. *If a $(2n+1)-$ dimensional M almost $C(\alpha)-$ manifold is a concircular pseudosymmetric manifold, then M is either $\alpha=(r/2n(2n+1))$ or a concircular semi-symmetric manifold.*

Corollary 6. *If a $(2n+1)-$ dimensional M almost $C(\alpha)-$ manifold is a concircular pseudosymmetric manifold, then M is a real space form with a constant section curvature.*

Theorem 7. *If a $(2n+1)-$ dimensional M almost $C(\alpha)-$ manifold is a concircular Ricci pseudosymmetric manifold, then M is either a co-Keahler, $\alpha=(r/2n(2n+1))$ or $\lambda_7=0$.*

Proof. Let us assume that the manifold M is a concircular Ricci pseudosymmetric manifold. Then, we can write as follows:

$$
\left(R(U_1,U_2)\cdot\widetilde{Z}\right)(U_5,U_4,U_3)=\lambda_7 Q(S,\widetilde{Z})(U_5,U_4,U_3;U_1,U_2), \tag{85}
$$

for each $U_1,U_2,U_3,U_4,U_5\in\chi(M)$. In this case, we yield the following:

$$
\begin{aligned}
&R(U_1,U_2)\widetilde{Z}(U_5,U_4)U_3-\widetilde{Z}(R(U_1,U_2)U_5,U_4)U_3 \\
&\quad-\widetilde{Z}(U_5,R(U_1,U_2)U_4)U_3 \\
&\quad-\widetilde{Z}(U_5,U_4)R(U_1,U_2)U_3 \\
&\quad=-\lambda_7\left\{\widetilde{Z}((U_1\Lambda_S U_2)U_5,U_4)U_3\right. \\
&\quad+\widetilde{Z}(U_5,(U_1\Lambda_S U_2)U_4)U_3 \\
&\quad\left.+\widetilde{Z}(U_5,U_4)(U_1\Lambda_S U_2)U_3\right\}.
\end{aligned} \tag{86}
$$

If necessary arrangements are made in here and we choose $U_1=U_5=\xi$ and using (8), (69), and (70) in (86), then we obtain as follows:

$$
\begin{aligned}
&\alpha g\left(U_2,\widetilde{Z}(\xi,U_4)U_3\right)\xi-\alpha\eta\left(\widetilde{Z}(\xi,U_4)U_3\right)U_2 \\
&-\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)\eta(U_2)g(U_4,U_3)\xi \\
&+\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)\eta(U_3)\eta(U_2)U_4 \\
&+\alpha\widetilde{Z}(U_2,U_4)U_3+\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)\eta(U_3)g(U_2,U_4)\xi \\
&-\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)\eta(U_3)g(U_2,U_4)\xi+\alpha\eta(U_4)\widetilde{Z}(\xi,U_2)U_3 \\
&-\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)\eta(U_4)g(U_2,U_3)\xi \\
&+\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_2,U_3)U_4 \\
&+\alpha\eta(U_3)\widetilde{Z}(\xi,U_4)U_2= \\
&-\lambda_7\left\{\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_4,U_3)S(U_2,\xi)\xi\right. \\
&-\left(\alpha-\frac{r}{2n(2n+1)}\right)S(U_2,\xi)\eta(U_3)U_4-2n\alpha\widetilde{Z}(U_2,U_4)U_3 \\
&-\left(\alpha-\frac{r}{2n(2n+1)}\right)S(U_2,U_4)\eta(U_3)\xi \\
&+\left(\alpha-\frac{r}{2n(2n+1)}\right)S(U_2,U_4)\eta(U_3)\xi \\
&-2n\alpha\eta(U_4)\widetilde{Z}(\xi,U_2)U_3+\left(\alpha-\frac{r}{2n(2n+1)}\right)S(U_2,U_3)\eta(U_4)\xi \\
&\left.-\left(\alpha-\frac{r}{2n(2n+1)}\right)S(U_2,U_3)U_4-2n\alpha\eta(U_3)\widetilde{Z}(\xi,U_4)U_2\right\}.
\end{aligned} \tag{87}
$$

If we use the expression (69) again in the last equation and make the necessary adjustments, then we obtain as follows:

$$\begin{aligned}
&-\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_4,U_3)U_2+\alpha\widetilde{Z}(U_2,U_4)U_3\\
&+\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_2,U_3)U_4=-\lambda_7\left\{2n\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_4,U_3)\eta(U_2)\xi\right.\\
&-2n\alpha\widetilde{Z}(U_2,U_4)U_3-2n\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_2,U_3)\eta(U_4)\xi\\
&+2n\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)\eta(U_4)\eta(U_3)U_2+\left(\alpha-\frac{r}{2n(2n+1)}\right)S(U_2,U_3)\eta(U_4)\xi\\
&\left.-\left(\alpha-\frac{r}{2n(2n+1)}\right)S(U_2,U_3)U_4-2n\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_4,U_2)\eta(U_3)\xi\right\}.
\end{aligned}\tag{88}$$

If we replace the expression (68) in (88) and make the necessary adjustments, we have as follows:

$$\begin{aligned}
&-\alpha^2g(U_4,U_3)U_2+\alpha^2g(U_2,U_3)U_4\\
&+\alpha R(U_2,U_4)U_3=-\lambda_7\left\{2n\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_4,U_3)\eta(U_2)\xi\right.\\
&-2n\alpha R(U_2,U_4)U_3+\frac{2n\alpha r}{2n(2n+1)}g(U_4,U_3)U_2\\
&-\frac{2n\alpha r}{2n(2n+1)}g(U_2,U_3)U_4-2n\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_2,U_3)\eta(U_4)\xi\\
&+2n\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)\eta(U_4)\eta(U_3)U_2+\left(\alpha-\frac{r}{2n(2n+1)}\right)S(U_2,U_3)\eta(U_4)\xi\\
&\left.-\left(\alpha-\frac{r}{2n(2n+1)}\right)S(U_2,U_3)U_4-2n\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)g(U_4,U_2)\eta(U_3)\xi\right\}.
\end{aligned}\tag{89}$$

If we apply $\xi\in\chi(M)$ to both sides of (89) and make use of (11), then we get

$$2n\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)\lambda_7\left[-g(U_4,U_2)+\eta(U_4)\eta(U_2)\right]=0.\tag{90}$$

On the other hand, we know that there is a g metric such that

$$g(\varphi U_4,\varphi U_2)=g(U_4,U_2)-\eta(U_4)\eta(U_2),\tag{91}$$

on a $(2n+1)$ dimensional almost contact manifold [26]. Therefore, equation (90) can be written as follows:

$$2n\alpha\left(\alpha-\frac{r}{2n(2n+1)}\right)\lambda_7 g(\varphi U_4,\varphi U_2)=0.\tag{92}$$

It is clear from the last equation that

$$\begin{aligned}
&\alpha=0,\\
&\lambda_7=0,\\
&\text{or }\alpha=\frac{r}{2n(2n+1)}.
\end{aligned}\tag{93}$$

This completes our proof. □

Corollary 7. *If a* $(2n+1)-$ *dimensional* M *almost* $C(\alpha)-$ *manifold is a concircular Ricci pseudosymmetric manifold, then* M *is either a co-Keahler,* $\alpha = (r/2n(2n+1))$ *or a concircular semisymmetric manifold.*

Corollary 8. *If a* $(2n+1)-$ *dimensional* M *almost* $C(\alpha)-$ *manifold is a concircular Ricci pseudosymmetric manifold, then* M *is a real space form with a constant section curvature.*

Data Availability

The citation data used to support the findings of this study are included in the article and will be submitted by the author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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