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Research article

Some properties and Vajda theorems of split dual Fibonacci and split dual Lucas octonions

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Abstract: In this paper, we introduce split dual Fibonacci and split dual Lucas octonions over the algebra $\widetilde{\widetilde{O}}(a,b,c)$, where a,b and c are real numbers. We obtain Binet formulas for these octonions. Also, we give many identities and Vajda theorems for split dual Fibonacci and split dual Lucas octonions including Catalan's identity, Cassini's identity and d'Ocagne's identity.

Keywords: octonions; dual Fibonacci and dual Lucas numbers; split dual Fibonacci and split dual Lucas octonions

Mathematics Subject Classification: 11B39, 11R52

1. Introduction

Quaternions and octonions were described by Sir William R. Hamilton in 1843 and Cayley-Dickson in 1845, respectively. In 1873, Clifford extended real numbers to dual numbers [16]. For the real numbers a and a^*, the form of the dual number d is

$$d = a + \varepsilon a^*$$

where ε is the dual unit and

$$\varepsilon^2 = 0, \varepsilon \neq 0.$$

The Binet formulas for the Fibonacci and Lucas numbers are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the characteristics equation

$$x^2 - x - 1 = 0.$$

The positive root α is known as golden ratio. The quaternions were introduced by Horadam [3]. Some authors studied Fibonacci or split Fibonacci quaternions [1, 2, 4–6, 8, 10–12, 17, 18].

Keçilioğlu and Akkuş gave the Fibonacci and Lucas octonions and their Binet formulas [7]. Halici also worked on dual Fibonacci octonions in [9] and obtained the Binet formula with its generator function. Ünal et al. investigated properties of dual Fibonacci and dual Lucas octonions [14]. In addition, Bilgici et al. obtained generalized Fibonacci and Lucas octonions, and gave a lot of identities to them [15].

In [13], Akkuş and Keiliolu took the form

$$a + lb$$

of a pair of quaternions, shown as (a, b), with a new imaginary unit l and defined the product rule as

$$(a + lb)(c + ld) = \left(ac + \lambda d\bar{b}\right) + l(\bar{a}d + cb)$$

where $\lambda^2 = 1$. When $\lambda = -1$ and $\lambda = 1$ above, octonions and split octonions are obtained, respectively. Split octonions form an eight-dimensional algebra on real numbers. The difference from standard octonions is that it contains elements that are different from zero. Accordingly, let us show the split dual octonion algebra on real numbers with $\widetilde{\widetilde{O}}$. In addition, Akkuş and Keiliolu have given the nth split Fibonacci and nth split Lucas octonions in the form of

$$Q_n = \sum_{i=1}^{7} F_{n+s} e_s$$

and

$$T_n = \sum_{i=1}^{7} L_{n+s} e_s$$

where F_n is nth Fibonacci number, L_n is nth Lucas number [13]. If we take $a = 1, b = 1$ and $c = -1$ in the generalized octonion multiplication table in [15], we get the split octonion table (see Table 1).

Table 1. Multiplication table of split octonion.

$\cdot$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	1	$-e_1$	$-e_2$	$-e_3$
e_5	e_5	e_4	$-e_7$	e_6	e_1	1	e_3	$-e_2$
e_6	e_6	e_7	e_4	$-e_5$	e_2	$-e_3$	1	e_1
e_7	e_7	$-e_6$	e_5	e_4	e_3	e_2	$-e_1$	1

We define split dual Fibonacci and split dual Lucas numbers by

$$\tilde{F}_n = F_n + \varepsilon F_{n+1} \tag{1.1}$$

and

$$\tilde{L}_n = L_n + \varepsilon L_{n+1} \tag{1.2}$$

respectively. In this paper, following Keçilioğlu and Akkuş, and Ünal et al., we define the split dual Fibonacci and split dual Lucas octonions over the split dual octonion algebra $\widetilde{\widetilde{O}}$. The nth split dual Fibonacci octonion $SDFO_n$ is

$$SDFO_n = \sum_{s=0}^{7} \tilde{F}_{n+s} e_s \tag{1.3}$$

and the nth split dual Lucas octonion $SDLO_n$ is

$$SDLO_n = \sum_{s=0}^{7} \tilde{L}_{n+s} e_s \tag{1.4}$$

where $\tilde{F}_n$ is nth split dual Fibonacci number and $\tilde{L}_n$ is nth split dual Lucas number. By using Eq (1.1), we obtain

$$\begin{aligned} SDFO_n &= \sum_{s=0}^{7} F_{n+s} e_s + \varepsilon \sum_{s=0}^{7} F_{n+s+1} e_s \\ &= \sum_{s=0}^{7} \left(F_{n+s} + \varepsilon F_{n+s+1}\right) e_s \\ &= SFO_n + \varepsilon SFO_{n+1} \end{aligned}$$

where $SFO_n = \sum_{s=0}^{7} F_{n+s} e_s$ is split Fibonacci octonion. Similarly, by using Eq (1.2), we have

$$SDLO_n = SLO_n + \varepsilon SLO_{n+1}$$

where $SLO_n = \sum_{s=0}^{7} L_{n+s} e_s$ is split Lucas octonion.

2. Binet formulas for split dual Fibonacci and Lucas octonions

In this section, we obtain Binet formulas and Vajda theorems for $SDFO_n$ and $SDLO_n$. There are three well-known identities for split dual Fibonacci and Lucas numbers, namely, Catalan's, Cassini's and d'Ocagne's identities. These types of identities for split dual Fibonacci and Lucas octonions can be obtained by using the Vajda theorems. The following theorem gives the Binet formulas for these octonions.

Theorem 1. *For $n \geq 0$, the nth split dual Fibonacci octonion is*

$$SDFO_n = \frac{\alpha' \alpha^n - \beta' \beta^n}{\alpha - \beta}$$

and nth split dual Lucas octonion is

$$SDLO_n = \alpha' \alpha^n + \beta' \beta^n$$

where $\alpha' = (1 + \varepsilon\alpha) \sum_{s=0}^{7} \alpha^s e_s$ *and* $\beta' = (1 + \varepsilon\beta) \sum_{s=0}^{7} \beta^s e_s$.

Proof. In [13], the Binet formulas for the split Fibonacci and split Lucas octonions are as follows:

$$Q_n = \frac{\alpha^* \alpha^n - \beta^* \beta^n}{\alpha - \beta} \tag{2.1}$$

and

$$T_n = \alpha^* \alpha^n + \beta^* \beta^n \tag{2.2}$$

where $\alpha^* = \sum_{s=0}^{7} \alpha^s e_s$ and $\beta^* = \sum_{s=0}^{7} \beta^s e_s$.

By using from the Eq (2.1), we get the Eq (1.3). We know that

$$SDFO_n = \sum_{s=0}^{7} \tilde{F}_{n+s} e_s = \sum_{s=0}^{7} (F_n + \varepsilon F_{n+1})\, e_s = SFO_n + \varepsilon SFO_{n+1}.$$

When the Binet formula of split Fibonacci octonion is used in this last equation, we get

$$\begin{aligned} SDFO_n &= \frac{\alpha^* \alpha^n - \beta^* \beta^n}{\alpha - \beta} + \varepsilon \frac{\alpha^* \alpha^{n+1} - \beta^* \beta^{n+1}}{\alpha - \beta} \\ &= \frac{\alpha^* \alpha^n}{\alpha - \beta} (1 + \varepsilon\alpha) - \frac{\beta^* \beta^n}{\alpha - \beta} (1 + \varepsilon\beta). \end{aligned}$$

If the expressions α^* and β^* given in [13] are used,

$$SDFO_n = \frac{\alpha^n}{\alpha - \beta} (1 + \varepsilon\alpha) \sum_{s=0}^{7} \alpha^s e_s - \frac{\beta^n}{\alpha - \beta} (1 + \varepsilon\beta) \sum_{s=0}^{7} \beta^s e_s = \frac{\alpha' \alpha^n - \beta' \beta^n}{\alpha - \beta}$$

is obtained. By using similar method, we get Binet formula of split dual Lucas octonion. □

Using Binet's formulas, we can easily derive the identities between split dual Fibonacci and split dual Lucas octonions.

Now, we give some useful identities that play very important roles throughout the paper for calculations.

Lemma 1. *We have obtained followings*

$$\begin{aligned}
(\alpha^*)^2 &= \frac{1305}{2} + SPO_0 + \sqrt{5}\left(\frac{545}{2} + SFO_0\right), \\
(\beta^*)^2 &= \frac{1305}{2} + SPO_0 - \sqrt{5}\left(\frac{545}{2} + SFO_0\right), \\
\alpha^*\beta^* &= SPO_0 + \sqrt{5}\lambda, \\
\beta^*\alpha^* &= SPO_0 - \sqrt{5}\lambda,
\end{aligned}$$

where

$$\lambda = -e_1 - e_2 + 4e_3 - 3e_4 + 9e_5 + 6e_6 - 6e_7.$$

Proof. The proof is completed when $a = b = 1$ and $c = -1$ in Lemma 2 in [15]. □

Now we give Vajda theorems for split dual Fibonacci and split dual Lucas octonions that could be reduced to some special cases.

Theorem 2. *For any integers n, r and s, we get*

$$SDFO_{n+r}SDFO_{n+s} - SDFO_nSDFO_{n+r+s} = (-1)^n F_r (SPO_0F_s - \lambda L_s)(1 + \varepsilon)$$

and

$$SDLO_{n+r}SDLO_{n+s} - SDLO_nSDLO_{n+r+s} = 5(-1)^{n+1} F_r (SPO_0F_s - \lambda L_s)(1 + \varepsilon).$$

Proof. We make proof for split dual Fibonacci octnions. We can write

$$\begin{aligned}
SDFO_{n+r}SDFO_{n+s} - SDFO_nSDFO_{n+r+s} &= \tilde{Q}_{n+r}\tilde{Q}_{n+s} - \tilde{Q}_n\tilde{Q}_{n+r+s} \\
&+\varepsilon\left(\tilde{Q}_{n+r}\tilde{Q}_{n+s+1} + \tilde{Q}_{n+r+1}\tilde{Q}_{n+s} - \tilde{Q}_n\tilde{Q}_{n+r+s+1} - \tilde{Q}_{n+1}\tilde{Q}_{n+r+s}\right).
\end{aligned}$$

We calculate the following expression

$$\tilde{Q}_{n+r}\tilde{Q}_{n+s} - \tilde{Q}_n\tilde{Q}_{n+r+s} \tag{2.3}$$

and we get

$$\begin{aligned}
&\tilde{Q}_{n+r}\tilde{Q}_{n+s} - \tilde{Q}_n\tilde{Q}_{n+r+s} \\
&= \frac{1}{(\alpha-\beta)^2}\left[\begin{array}{c} (\alpha^*\alpha^{n+r} - \beta^*\beta^{n+r})(\alpha^*\alpha^{n+s} - \beta^*\beta^{n+s}) \\ -(\alpha^*\alpha^n - \beta^*\beta^n)(\alpha^*\alpha^{n+r+s} - \beta^*\beta^{n+r+s}) \end{array}\right] \\
&= \frac{1}{(\alpha-\beta)^2}\left[\begin{array}{c} \alpha^*\beta^*(-\alpha^{n+r}\beta^{n+s} + \alpha^n\beta^{n+r+s}) \\ +\beta^*\alpha^*(-\alpha^{n+s}\beta^{n+r} + \alpha^{n+r+s}\beta^n) \end{array}\right] \\
&= \frac{(-1)^n}{\alpha-\beta}F_r\left[P_0(\alpha^s - \beta^s) - \sqrt{5}\lambda(\alpha^s + \beta^s)\right] \\
&= (-1)^n F_r\left[P_0\left(\frac{\alpha^s - \beta^s}{\alpha-\beta}\right) - \frac{\sqrt{5}}{\alpha-\beta}\lambda(\alpha^s + \beta^s)\right] \\
&= (-1)^n F_r(P_0F_s - \lambda L_s).
\end{aligned}$$

If $s+1$ is written instead of s in Eq (2.3), then

$$\tilde{Q}_{n+r}\tilde{Q}_{n+s+1}-\tilde{Q}_{n}\tilde{Q}_{n+r+s+1}=(-1)^{n}F_{r}\left(P_{0}F_{s+1}-\lambda L_{s+1}\right)$$

is obtained. If $n+1$ is written instead of n and $s-1$ is written instead of s in equation given above, then we get

$$\tilde{Q}_{n+r+1}\tilde{Q}_{n+s}-\tilde{Q}_{n+1}\tilde{Q}_{n+r+s}=(-1)^{n+1}F_{r}\left(P_{0}F_{s-1}-\lambda L_{s-1}\right).$$

In this case,

$$\tilde{Q}_{n+r}\tilde{Q}_{n+s+1}-\tilde{Q}_{n}\tilde{Q}_{n+r+s+1}+\tilde{Q}_{n+r+1}\tilde{Q}_{n+s}-\tilde{Q}_{n+1}\tilde{Q}_{n+r+s}=(-1)^{n}F_{r}\left(P_{0}F_{s}-\lambda L_{s}\right)$$

and so,

$$SDFO_{n+r}SDFO_{n+s}-SDFO_{n}SDFO_{n+r+s}=(-1)^{n}F_{r}\left(SPO_{0}F_{s}-\lambda L_{s}\right)(1+\varepsilon).$$

Similarly, we can prove the Vajda theorem for split dual Lucas octonions. □

From Vajda theorems, we also have the following special identities for these octonions:

Corollary 1. *If $-r$ is taken instead of s in the Vajda theorem, Catalan's identities are obtained for split dual Fibonacci and split dual Lucas octonions, respectively:*

$$SDFO_{n+r}SDFO_{n-r}-SDFO_{n}^{2}=(-1)^{n-r+1}\left(SPO_{0}F_{r}^{2}+\lambda F_{2r}\right)(1+\varepsilon)$$

and

$$SDLO_{n+r}SDLO_{n-r}-SDLO_{n}^{2}=5(-1)^{n-r}\left(SPO_{0}F_{r}^{2}+\lambda F_{2r}\right)(1+\varepsilon).$$

Corollary 2. *If $-r$ is taken instead of s and then 1 is taken instead of r in the Vajda theorem, Cassini's identities are obtained for split dual Fibonacci and split dual Lucas octonions, respectively :*

$$SDFO_{n+1}SDFO_{n-1}-SDFO_{n}^{2}=(-1)^{n}\left(SPO_{0}+\lambda\right)(1+\varepsilon)$$

and

$$SDLO_{n+1}SDLO_{n-1}-SDLO_{n}^{2}=5(-1)^{n-1}\left(SPO_{0}+\lambda\right)(1+\varepsilon)$$

where $F_0=0, F_1=1$ and $F_2=1$.

Corollary 3. *If m is taken instead of n, $n-m$ is taken instead of s and then 1 is taken instead of r in the Vajda theorem, d'Ocagne's identities are obtained for split dual Fibonacci and split dual Lucas octonions, respectively:*

$$SDFO_{m+1}SDFO_{n}-SDFO_{m}SDFO_{n+1}=(-1)^{m}\left(SPO_{0}F_{n-m}-\lambda L_{n-m}\right)(1+\varepsilon)$$

and

$$SDLO_{m+1}SDLO_{n}-SDLO_{m}SDLO_{n+1}=5(-1)^{m+1}\left(SPO_{0}F_{n-m}-\lambda L_{n-m}\right)(1+\varepsilon).$$

3. Some results for split dual Fibonacci and split dual Lucas octonions

In this section, after deriving famous three identities Catalan's, Cassini's and d'Ocagne's by using the Vajda theorems, we present some other identities for the split dual Fibonacci and split dual Lucas octonions.

Theorem 3. *Split dual Fibonacci octonions satisfy the following identities;*

i) $SDFO_{m+n} + (-1)^n SDFO_{m-n} = SDFO_m L_n$,

ii) $SDFO_{n+r}F_{n+r} - SDFO_{n-r}F_{n-r} = F_{2r}SDFO_{2n}$,

iii) $SDFO_n SDFO_m - SDFO_m SDFO_n = 2(-1)^{m+1}\lambda F_{n-m}(1+\varepsilon)$.

Proof. We prove only the first identity of this theorem. We need the Binet formulas for the split dual Fibonacci octonions.

i) $SDFO_{m+n} + (-1)^n SDFO_{m-n}$
$= \left(\tilde{Q}_{m+n} + \varepsilon\tilde{Q}_{m+n+1}\right) + (\alpha\beta)^n\left(\tilde{Q}_{m-n} + \varepsilon\tilde{Q}_{m-n+1}\right)$
$= (\alpha^n + \beta^n)\left(\tilde{Q}_m + \varepsilon\tilde{Q}_{m+1}\right) = (\alpha^n + \beta^n)SDFO_m = SDFO_m L_n$.

The second and third identities can be proved similarly. □

Theorem 4. *Split dual Lucas octonions satisfy the following identities;*

i) $SDLO_{n+r}L_{n+r} - SDLO_{n-r}L_{n-r} = 5F_{2r}SDFO_{2n}$,

ii) $SDLO_{n+r}L_{n+r} + SDLO_{n-r}L_{n-r} = L_{2r}SDLO_{2n} + 2(-1)^{n+r}SDLO_0$,

iii) $SDLO_n SDLO_m - SDLO_m SDLO_n = 10(-1)^m\lambda F_{n-m}(1+\varepsilon)$.

Proof. We prove only the first identity of this theorem. We need the Binet formulas for the split dual Lucas octonions

i) $SDLO_{n+r}L_{n+r} - SDLO_{n-r}L_{n-r} = \left(\tilde{P}_{n+r} + \varepsilon\tilde{P}_{n+r+1}\right)L_{n+r} - \left(\tilde{P}_{n-r} + \varepsilon\tilde{P}_{n-r+1}\right)L_{n-r}$
$= \left(\alpha^{2r} - \beta^{2r}\right)\left(\alpha^*\alpha^{2n} - \beta^*\beta^{2n}\right) + \varepsilon\left(\alpha^{2r} - \beta^{2r}\right)\left(\alpha^*\alpha^{2n+1} - \beta^*\beta^{2n+1}\right)$
$= (\alpha-\beta)^2\left(\dfrac{\alpha^{2r} - \beta^{2r}}{\alpha-\beta}\right)\left[\left(\dfrac{\alpha^*\alpha^{2n} - \beta^*\beta^{2n}}{\alpha-\beta}\right) + \varepsilon\left(\dfrac{\alpha^*\alpha^{2n+1} - \beta^*\beta^{2n+1}}{\alpha-\beta}\right)\right]$
$= 5F_{2r}\left(\tilde{Q}_{2n} + \varepsilon\tilde{Q}_{2n+1}\right) = 5F_{2r}SDFO_{2n}$.

The other two identities can be proved similar to the theorem proved by using split dual Fibonacci octonion. □

Theorem 5. *Split dual Fibonacci and split dual Lucas octonions satisfy the following identities;*

i) $SDFO_m SDLO_n - SDFO_n SDLO_m = 2(-1)^{m+1}SPO_0 F_{n-m}(1+\varepsilon)$, *ii)* $SDFO_mSDLO_n - SDLO_mSDFO_n = 2(-1)^{m+1}(SPO_0F_{n-m} - \lambda L_{n-m})(1+\varepsilon)$,

iii) $SDLO_{n+r}SDFO_{n+s} - SDLO_{n+s}SDFO_{n+r} = 2(-1)^{n+r}SPO_0F_{s-r}(1+\varepsilon)$,

iv) $SDLO_{n+r}F_{n+t} + SDLO_{n+t}F_{n+r} = 2SDFO_{2n+r+t} - (-1)^{n+1}L_{r-t}SDFO_0$,

v) $SDFO_{n+r}L_{n+r} + SDFO_{n-r}L_{n-r} = L_{2r}SDFO_{2n} + 2(-1)^{n+r}SDFO_0$.

Proof. The proof of the theorem can be done similarly by the Binet formulas for the nth split dual Fibonacci and nth split dual Lucas octonions proved above. □

4. Conclusions

In the articles about Quaternions and Octonions in the literature, Catalan's, Cassini's and d'Ocagne's identities have been obtained by using their Binet formulas. In this paper, the Vajda theorems are obtained for split dual Fibonacci and split dual Lucas octonions. We aimed to introduce Vajda theorem, which is not included in the literature and will contribute to finding Catalan's, Cassini's and d'Ocagne's identities. In the Vajda theorem, when we write the criteria in Corollaries 1–3 in our study, Catalan's, Cassini's and d'Ocagne's identities are obtained, respectively. Thus, when we obtain the Vajda theorem by using the Binet formula, we get these three identities, which are well known in the literature, without the need for any other calculations.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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