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Determination of a differential pencil from interior spectral data on a union of two closed intervals

İbrahim ADALAR*
Zara Veysel Dursun School of Applied Sciences, Sivas Cumhuriyet University, Sivas, Turkey

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Abstract: In this paper, we consider a quadratic pencil of Sturm–Liouville operator on closed sets. We study an interior-inverse problem for this kind operator and give a uniqueness theorem with an appropriate example.

Key words: Inverse problem, differential pencil, time scale

1. Introduction

The spectral problems appear in geophysics, mathematical physics, mechanics and other branches of natural sciences. Specifically, second order differential pencil, namely diffusion equation, originates from the problem of describing interactions between colliding particles in physics [17]. The reconstruction of an operator from its spectral characteristics is referred to as the inverse problem. For inverse problems of a differential pencil, such characteristics are one spectrum and normalizing constants, two given spectra, interior spectral data, nodal points, scattering data and the Weyl function.

The inverse problem for interior spectral data of the differential operator consists of reconstruction of this operator from the given eigenvalues and some information on eigenfunctions at an internal point. The interior-inverse Sturm–Liouville problem, which is one of the important subjects of inverse spectral theory, was first studied in by Mochizuki and Trooshin in [21]. They proved that the spectrum of the problem

$$\begin{gathered} -y'' + q(x)y = \lambda y,\ t \in (0,1) \\ y'(0) - hy(0) = y'(1) + Hy(1) = 0 \end{gathered}$$

and the logarithmic derivatives of the eigenfunctions at the point $1/2$ uniquely determine the potential $q(x)$ on the whole interval $[0,1]$ almost everywhere. This kind of problems for the differential operators on a continuous interval were studied in [8, 14],[18, 22, 23],[28, 34].

The time scale theory unifies continuous and discrete dynamic equations. This approach is used in many important applied problems, for example, heat transfer, dynamics of population, string theory. The theory of dynamic equations on closed sets was introduced in [16]. Nowadays, there are a few studies about the inverse problems of differential operators defined on time scales (see [1, 2, 7, 19, 20],[24, 27] and [35]).

If $\mathbb{T}$ is a closed subset of $\mathbb{R}$ it called a time scale. The forward and backward jump operators σ, $\rho : \mathbb{T} \to \mathbb{T}$

*Correspondence: iadalar@cumhuriyet.edu.tr
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are defined as follows:

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{ if } t \neq \sup \mathbb{T},$$
$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} \text{ if } t \neq \inf \mathbb{T}.$$

The delta derivative f^{Δ} and the nabla derivative f^{∇} for a function f defined on $\mathbb{T}$ was introduced in [4]. For basic concepts of the time scale theory and time scales notation, we refer to the textbooks [4, 5] and [6].

In this paper, we consider an interior-inverse problem for a differential pencil on a time scale and give a uniqueness theorem. The main result of this article is a generalization of the classical result for an interior-inverse problem on a continuous interval.

Throughout this paper, we assume that $\mathbb{T} = [0, a_1] \cup [a_2, a_3]$ is a bounded time scale for $0 < a_1 \leq a_2$ and $a_1 + a_2 = a_3$.

We consider following boundary value problem $L = L(q, p, h, H)$ on $\mathbb{T} = [0, a_1] \cup [a_2, a_3]$,

$$\ell y := -y^{\Delta\Delta}(t) + [q(t) + 2\lambda p(t)]y^{\sigma}(t) = \lambda^2 y^{\sigma}(t), \quad t \in \mathbb{T}, \tag{1.1}$$

$$U(y) := y^{\Delta}(0) - hy(0) = 0, \tag{1.2}$$

$$V(y) := y^{\Delta}(a_3) + Hy(a_3) = 0 \tag{1.3}$$

where $q(t)$ and $p(t)$ are real valued continuous functions on $\mathbb{T}$; $h,\ H \in \mathbb{R}$ and λ is a spectral parameter.

Together with L, we consider a boundary value problem $\widetilde{L} = L(\widetilde{q}, \widetilde{p}, h, H)$ of the same form but with different coefficients $\widetilde{q}(t)$ and $\widetilde{p}(t)$. We assume that if a certain symbol x denotes an object related to L , then $\widetilde{x}$ will denote an analogous object related to $\widetilde{L}$. Also, we assume

$$h\left|y(0)\right|^2 + H\left|y(a_3)\right|^2 + \int_0^{a_3} \left(\left|y^{\Delta}(t)\right|^2 + q(t)\left|y^{\sigma}(t)\right|^2\right)\Delta t > 0.$$

A function y, defined on $\mathbb{T}$, is called a solution of equation (1) if $y \in C^2_{rd}(\mathbb{T})$ and y satisfies (1) for all $t \in \mathbb{T}$. The values of the λ parameter for which (1)–(3) has nonzero solutions are called eigenvalues, and the corresponding nontrivial solutions are called eigenfunctions.

We state the main result of this paper.

Let $\Lambda := \{\lambda_n,\ n \in \mathbb{Z}\}$ and $\widetilde{\Lambda} := \left\{\widetilde{\lambda}_n,\ n \in \mathbb{Z}\right\}$ be the eigenvalues sets of L and $\widetilde{L}$, $y_n(t)$ and $\widetilde{y_n}(t)$ are eigenfunctions related to this eigenvalues, respectively.

Theorem 1.1 *If* $\Lambda = \widetilde{\Lambda}$, *and for any* $n \in \mathbb{N}$,

$$\frac{y_n^{\nabla}(a_2)}{y_n(a_2)} = \frac{\widetilde{y}_n^{\nabla}(a_2)}{\widetilde{y}_n(a_2)} \tag{1.4}$$

then $q(t) = \widetilde{q}(t)$ *and* $p(t) = \widetilde{p}(t)$ *on* $\mathbb{T}$.

In case $a_1 = a_2$ and $p(t) = 0$, Theorem 1 coincides with the theorem in [21]. We note that, if it is taken $a_1 = a_2$ and $\mathbb{T} = [0, a_3]$ in L, the classical diffusion operator is obtained. In addition, since $y_n^{\Delta}(a_1) = -y_n^{\nabla}(a_2)$, the condition (1.4) can be replaced by

$$\frac{y_n^{\Delta}(a_1)}{y_n(a_1)} = \frac{\widetilde{y}_n^{\Delta}(a_1)}{\widetilde{y}_n(a_1)}.$$

2. Preliminaries

Let $\varphi(t,\lambda)$ and $\psi(t,\lambda)$ be the solution of (1.1) under the initial conditions

$$\begin{aligned}\varphi(0,\lambda) &= 1,\ \varphi^{\Delta}(0,\lambda) = h\\ \psi(a_3,\lambda) &= 1,\ \psi^{\Delta}(a_3,\lambda) = -H\end{aligned}$$

respectively. The following lemmas can be given by using a similar method to that in [24].

Lemma 2.1 *$\varphi(t,\lambda)$, $\psi(t,\lambda)$ and their Δ-derivatives are entire functions on λ for each fixed t.*

Lemma 2.2 *The eigenvalues of the problem (1.1)-(1.3) are real and algebraicly simple.*

It is clear that the characteristic function of L can be given as follows

$$\Delta(\lambda) = W[\psi,\varphi] = \varphi^{\Delta}(a_3,\lambda) + H\varphi(a_3,\lambda).$$

It follows from Lemma 1 that $\Delta(\lambda)$ is also an entire function and so L has a discrete spectrum.

It is known from [15] that $\varphi(t,\lambda)$ satisfies the following representation on $[0,a_1]$

$$\varphi(t,\lambda) = \cos\left(\lambda t - \alpha(t)\right) + \int\limits_0^t A(t,x)\cos\lambda x dx + \int\limits_0^t B(t,x)\sin\lambda x dx$$

where the kernels $A(t,x)$ and $B(t,x)$ are the solution of the problem

$$\begin{aligned}&\frac{\partial^2 A(t,x)}{\partial t^2} - 2p(t)\frac{\partial B(t,x)}{\partial x} - q(t)A(t,x) = \frac{\partial^2 A(t,x)}{\partial x^2},\\ &\frac{\partial^2 B(t,x)}{\partial t^2} + 2p(t)\frac{\partial A(t,x)}{\partial x} - q(t)B(t,x) = \frac{\partial^2 B(t,x)}{\partial x^2},\\ &A(0,0) = h,\ B(t,0) = 0,\ \left.\frac{\partial A(t,x)}{\partial x}\right|_{x=0} = 0\end{aligned}$$

where

$$q(t) + p^2(t) = 2\frac{d}{dt}\left[A(t,t)\cos\alpha(t) + B(t,t)\sin\alpha(t)\right],\ \alpha(t) = \int\limits_0^t p(x)dx.$$

On the other hand, $\varphi^{\Delta}(t,\lambda)$ is continuous at a_1, and so the relation

$$a\varphi^{\Delta}(a_1-0,\lambda) = \varphi(a_2,\lambda) - \varphi(a_1,\lambda) \tag{2.1}$$

holds, where $a := a_2 - a_1$. Therefore, we obtain that if $a > 0,$, then the following asymptotic formula holds for $|\lambda| \to \infty$;

$$\varphi(a_2, \lambda) = -a\lambda \sin(\lambda a_1 - \alpha(a_1)) + O(\exp|\tau| a_1), \tag{2.2}$$

where $\tau = Im\lambda$.

Since $\varphi(t, \lambda)$ satisfies the equation (1.1) for $t = a_1$, it follows that

$$(a^2\lambda^2 + b\lambda + c)\varphi(a_2, \lambda) + a\varphi^{\Delta}(a_2 + 0, \lambda) + \varphi(a_1, \lambda) = 0, \tag{2.3}$$

where $a = a_2 - a_1,\ b = -2a^2 p(a_1),\ c = -a^2 q(a_1) - 1.$

Lemma 2.3 *The following asymptotic formulas hold for* $|\lambda| \to \infty$:

$$\varphi(t, \lambda) = \begin{cases} \cos\lambda t + \dfrac{h}{\lambda}\sin\lambda t + O\left(\dfrac{1}{\lambda}\exp|\tau| t\right), & t \in [0, a_1], \\ a^2\lambda^2 \sin\lambda a_1 \sin\lambda(t - a_2) + O(\lambda \exp|\tau|(t - a_2 + a_1)), & t \in [a_2, a_3], \end{cases} \tag{2.4}$$

$$\varphi^{\Delta}(t, \lambda) = \begin{cases} -\lambda\sin\lambda t + h\cos\lambda t + O(\exp|\tau| t), & t \in [0, a_1), \\ a^2\lambda^3 \sin\lambda a_1 \cos\lambda(t - a_2) + O\left(\lambda^2 \exp|\tau|(t - a_2 + a_1)\right), & t \in [a_2, a_3]. \end{cases} \tag{2.5}$$

Proof It is clear that

$$\varphi(t, \lambda) = \cos\lambda t + \frac{h}{\lambda}\sin\lambda t + \tag{2.6}$$

$$+\frac{1}{\lambda}\int_0^t \sin\lambda(t - \xi)\left[q(\xi) + 2\lambda p(\xi)\right]\varphi(\xi, \lambda)\Delta\xi,\ t \in [0, a_1]$$

On the other hand, the general solution of (1.1) on $[a_2, a_3]$ is given as follow.

$$\varphi(t, \lambda) = c_1(\lambda)\cos\lambda t + c_2(\lambda)\sin\lambda t + \tag{2.7}$$

$$+\frac{1}{\lambda}\int_{a_2}^t \sin\lambda(t - \xi)\left[q(\xi) + 2\lambda p(\xi)\right]\varphi(\xi, \lambda)\Delta\xi.$$

Substituting (2.6) and (2.7) in (2.1) and (2.3) we obtain a linear system in which $c_1(\lambda)$ and $c_2(\lambda)$ are unknowns. Using the solution to the system, we show that the following integral equation is valid on $[a_2, a_3]$:

$$\varphi(t, \lambda) = A(\lambda)\sin\lambda(t - a_2) + B(\lambda)\cos\lambda(t - a_2) + \tag{2.8}$$

$$+\frac{1}{\lambda}\int_{a_2}^t \sin\lambda(t - \xi)\left[q(\xi) + 2\lambda p(\xi)\right]\varphi(\xi, \lambda)\Delta\xi$$

where

$$\begin{aligned} A(\lambda) &= \left((a^2\lambda^2+b\lambda+c)-\frac{(a^2\lambda+b\lambda+c+1)h}{a\lambda^2}\right)\sin\lambda a_1+ \\ &-\left(\frac{(a^2\lambda+b\lambda+c)(ah+1)+1}{a\lambda}\right)\cos\lambda a_1+ \\ &-\left(\frac{a^2\lambda^2+b\lambda+c}{\lambda}\right)\int_0^{a_1}\cos\lambda(a_1-\xi)\left[q(\xi)+2\lambda p(\xi)\right]\varphi(\xi)\Delta\xi+ \\ &-\left(\frac{a^2\lambda^2+b\lambda+c+1}{a\lambda^2}\right)\int_0^{a_1}\sin\lambda(a_1-\xi)\left[q(\xi)+2\lambda p(\xi)\right]\varphi(\xi)\Delta\xi \end{aligned}$$

and

$$\begin{aligned} B(\lambda) &= \left(\frac{h}{\lambda}-a\lambda\right)\sin\lambda a_1+(ah+1)\cos\lambda a_1+ \\ &+a\int_0^{a_1}\cos\lambda(a_1-\xi)\left[q(\xi)+2\lambda p(\xi)\right]\varphi(\xi)\Delta\xi+ \\ &+\frac{1}{\lambda}\int_0^{a_1}\sin\lambda(a_1-\xi)\left[q(\xi)+2\lambda p(\xi)\right]\varphi(\xi)\Delta\xi. \end{aligned}$$

Thus, we calculate from (2.6) and (2.8) our desired relations. □

It follows from $\Delta(\lambda)=\varphi^{\Delta}(a_3,\lambda)+H\varphi(a_3,\lambda)$ that the asymptotic relation

$$\Delta(\lambda)=\frac{a^2}{2}\lambda^3\sin 2\lambda a_1+O\left(\lambda^2\exp|\tau|\,2a_1\right) \tag{2.9}$$

is valid for $|\lambda|\to\infty$.

Now we are ready to prove our main result. We give proof in the case $a>0$. The other case is easier and similar.

3. Proof of the main result

Proof [Proof of the Theorem 1.] We give the proof by four steps.

Step 1: Let us write the equation (1.1) for φ and $\widetilde{\varphi}$

$$-\varphi^{\Delta\Delta}(t,\lambda)+[q(t)+2\lambda p(t)]\varphi^{\sigma}(t,\lambda)=\lambda^2\varphi^{\sigma}(t,\lambda) \tag{3.1}$$

$$-\widetilde{\varphi}^{\Delta\Delta}(t,\lambda)+[\widetilde{q}(t)+2\lambda\widetilde{p}(t)]\widetilde{\varphi}^{\sigma}(t,\lambda)=\lambda^2\widetilde{\varphi}^{\sigma}(t,\lambda). \tag{3.2}$$

It can be obtained from (3.1) and (3.2) that

$$\left[\varphi^{\Delta}(t,\lambda)\widetilde{\varphi}(t,\lambda)-\varphi(t,\lambda)\widetilde{\varphi}^{\Delta}(t,\lambda)\right]^{\Delta}=\left[q(t)-\widetilde{q}(t)+2\lambda(p(t)-\widetilde{p}(t))\right]\varphi^{\sigma}(t,\lambda)\widetilde{\varphi}^{\sigma}(t,\lambda). \tag{3.3}$$

Taking $P(t)=p(t)-\widetilde{p}(t),\ Q(t)=q(t)-\widetilde{q}(t)$ and $\Delta-$integrating both sides of (3.3) on $[0,a_1]$,

$$\begin{aligned}\left[\varphi^{\Delta}(t,\lambda)\widetilde{\varphi}(t,\lambda)-\varphi(t,\lambda)\widetilde{\varphi}^{\Delta}(t,\lambda)\right]_0^{a_1} &= \int_0^{a_1}\left[Q(t)+2\lambda P(t)\right]\varphi^{\sigma}(t,\lambda)\widetilde{\varphi}^{\sigma}(t,\lambda)\Delta t\\ &= \int_0^{a_1}\left[Q(t)+2\lambda P(t)\right]\varphi(t,\lambda)\widetilde{\varphi}(t,\lambda)\Delta t.\end{aligned}$$

It is obvious that

$$\varphi^{\Delta}(a_1,\lambda)\widetilde{\varphi}(a_1,\lambda)-\varphi(a_1,\lambda)\widetilde{\varphi}^{\Delta}(a_1,\lambda)=\int_0^{a_1}\left[Q(t)+2\lambda P(t)\right]\varphi(t,\lambda)\widetilde{\varphi}(t,\lambda)\Delta t.$$

Let

$$K(\lambda):=\int_0^{a_1}\left[Q(t)+2\lambda P(t)\right]\varphi(t,\lambda)\widetilde{\varphi}(t,\lambda)\Delta t.$$

Since

$$\frac{y_n^{\Delta}(a_1)}{y_n(a_1)}=\frac{\widetilde{y}_n^{\Delta}(a_1)}{\widetilde{y}_n(a_1)},$$

we get that $K(\lambda_n)=0$ for all $\lambda_n\in\Lambda$ and so $\chi(\lambda):=\dfrac{K(\lambda)}{\Delta(\lambda)}$ is an entire function on λ.

On the other hand, we obtain

$$K(\lambda)=O(\lambda\exp 2\left|\tau\right|a_1)$$

for all complex λ by using the asymptotics (2.4). From (2.9) it can be calculated that

$$\left|\chi(\lambda)\right|\leq C\left|\lambda\right|^{-2}.$$

for suffciently large $|\lambda|$. By the Liouville's theorem, $\chi(\lambda)=0$ for all λ. Hence, $K(\lambda)\equiv 0$.

Step 2: By integrating again both sides of the equality (3.3) on $(0,a_1)$, we get

$$\varphi^{\Delta}(a_1,\lambda)\widetilde{\varphi}(a_1,\lambda)-\varphi(a_1,\lambda)\widetilde{\varphi}^{\Delta}(a_1,\lambda)=K(\lambda)=0$$

and so

$$\varphi^{\Delta}(a_1,\lambda)\widetilde{\varphi}(a_1,\lambda)=\varphi(a_1,\lambda)\widetilde{\varphi}^{\Delta}(a_1,\lambda). \tag{3.4}$$

Put $\psi(t,\lambda):=\varphi(a_1-t,\lambda)$. It is clear that $\psi(t,\lambda)$ is the solution of the following initial value problem

$$\begin{aligned}&-y^{\Delta\Delta}+[q(a_1-t)+2\lambda p(a_1-t)]y^{\sigma}=\lambda y^{\sigma},\ t\in(0,a_1)\\ &y(a_1)=1,y^{\Delta}(a_1)=-h\end{aligned}$$

It follows from (3.4) that

$$\psi^{\Delta}(0,\lambda)\widetilde{\psi}(0,\lambda)=\psi(0,\lambda)\widetilde{\psi}^{\Delta}(0,\lambda).$$

Taking into account Theorem 4.1. in [32], it is concluded that $q(t) = \widetilde{q}(t)$ and $p(t) = \widetilde{p}(t)$ on $[0, a_1]$.

Step 3: To prove that $q(t) = \widetilde{q}(t)$ and $p(t) = \widetilde{p}(t)$ on $[a_2, a_3]$, we will consider the supplementary problem L_1 :

$$\begin{gathered} -y^{\nabla\nabla} + [q_1(t) + 2\lambda p_1(t)]y^{\rho} = \lambda y^{\rho},\ t \in \mathbb{T}, \\ y^{\nabla}(0) - Hy(0) = y^{\nabla}(a_3) + hy(a_3) = 0, \end{gathered}$$

where $q_1(t) = q(a_3 - t)$ and $p_1(t) = p(a_3 - t)$.

By using chain rule for nabla deriative in [3], we have that $\varphi_1(t, \lambda) = \varphi(a_3 - t, \lambda)$ satisfies the equation

$$-\varphi_1^{\nabla\nabla} + [q_1(t) + 2\lambda p_1(t)]\varphi_1^{\rho} = \lambda\varphi_1^{\rho}$$

and the initial conditions

$$\varphi_1(a_3, \lambda) = 1, \varphi_1^{\nabla}(a_3, \lambda) = -h.$$

Furthermore, the assumption of the theorem

$$\frac{\varphi_1^{\nabla}(a_2, \lambda_n)}{\varphi_1(a_2, \lambda_n)} = \frac{\tilde{\varphi}_1^{\nabla}(a_2, \lambda_n)}{\tilde{\varphi}_1(a_2, \lambda_n)}$$

holds.

If we repeat the calculations in the Step 1, then we replace equation (3.3) by

$$\left[\varphi_1(t, \lambda)\widetilde{\varphi}_1^{\nabla}(t, \lambda) - \varphi_1^{\nabla}(t, \lambda)\widetilde{\varphi}_1(t, \lambda)\right]^{\nabla} = \left[q_1(t) - \widetilde{q}_1(t) + 2\lambda(p_1(t) - \widetilde{p}_1(t))\right]\varphi_1^{\rho}(t, \lambda)\widetilde{\varphi}_1^{\rho}(t, \lambda). \tag{3.5}$$

By integrating (in the sense of ∇-integral) both sides of this equality on $[0, a_2]$, we obtain

$$\begin{aligned} \left[\varphi_1(t, \lambda)\widetilde{\varphi}_1^{\nabla}(t, \lambda) - \varphi_1^{\nabla}(t, \lambda)\widetilde{\varphi}_1(t, \lambda)\right]_0^{a_2} &= \int_0^{a_2} \left[q_1(t) - \widetilde{q}_1(t) + 2\lambda(p_1(t) - \widetilde{p}_1(t))\right]\varphi_1^{\rho}(t, \lambda)\widetilde{\varphi}_1^{\rho}(t, \lambda)\nabla t \\ &= \int_0^{a_1} \left[q_1(t) - \widetilde{q}_1(t) + 2\lambda(p_1(t) - \widetilde{p}_1(t))\right]\varphi_1(t, \lambda)\widetilde{\varphi}_1(t, \lambda)\nabla t \\ &\quad + \int_{a_1}^{a_2} \left[q_1(t) - \widetilde{q}_1(t) + 2\lambda(p_1(t) - \widetilde{p}_1(t))\right]\varphi_1^{\rho}(t, \lambda)\widetilde{\varphi}_1^{\rho}(t, \lambda)\nabla t \end{aligned}$$

From Step 2, since $q(a_1) = \widetilde{q}(a_1)$ and $p(a_1) = \widetilde{p}(a_1)$,then $q_1(a_2) = \widetilde{q}_1(a_2)$ and $p_1(a_2) = \widetilde{p}_1(a_2)$. Thus we get

$$\int_{a_1}^{a_2} \left[q_1(t) - \widetilde{q}_1(t)\right]\varphi^{\rho}(t, \lambda)\widetilde{\varphi}^{\rho}(t, \lambda)\nabla t = \left[q_1(a_2) - \widetilde{q}_1(a_2)\right]\varphi_1^{\rho}(a_2, \lambda)\widetilde{\varphi}_1^{\rho}(a_2, \lambda)\left(a_2 - a_1\right) = 0.$$

Therefore we have

$$\varphi_1^{\nabla}(a_2, \lambda)\widetilde{\varphi}_1(a_2, \lambda) - \varphi_1(a_2, \lambda)\widetilde{\varphi}_1^{\nabla}(a_2, \lambda) = \int_0^{a_1} \left[q_1(t) - \widetilde{q}_1(t) + 2\lambda(p_1(t) - \widetilde{p}_1(t))\right]\varphi_1(t, \lambda)\widetilde{\varphi}_1(t, \lambda)\nabla t.$$

Let

$$K_1(\lambda) := \int_0^{a_1} [q_1(t) - \widetilde{q}_1(t) + 2\lambda(p_1(t) - \widetilde{p}_1(t))]\, \varphi_1(t,\lambda)\widetilde{\varphi}_1(t,\lambda)\nabla t.$$

It is obvious that $K_1(\lambda_n) = 0$ for all $\lambda_n \in \Lambda$ and so $\omega(\lambda) := \dfrac{K_1(\lambda)}{\Delta(\lambda)}$ is entire on λ. Similar to the calculations in the last part of Step 1, we obtain

$$K_1(\lambda) \equiv 0.$$

Step 4: By integrating again both sides of the equality (3.5) on $(0, a_1)$, we get

$$\varphi_1^{\nabla}(a_1,\lambda)\widetilde{\varphi}_1(a_1,\lambda) - \varphi_1(a_1,\lambda)\widetilde{\varphi}_1^{\nabla}(a_1,\lambda) = K_1(\lambda) = 0.$$

Repeating the Step 2 for the supplementary problem L_1 and $\varphi_1(t,\lambda)$, it is concluded that $q_1(t) = \widetilde{q}_1(t)$ and $p_1(t) = \widetilde{p}_1(t)$ on $[0, a_1]$, that is $q(t) = \widetilde{q}(t)$ and $p(t) = \widetilde{p}(t)$ on $[a_2, a_3]$. This completes the proof. $\square$

Example 3.1 *Consider the following problems on* $\mathbb{T} = [0, 1/2] \cup [1, 3/2]$*,*

$$L_0 : \begin{cases} -y^{\Delta\Delta}(t) = \lambda y^{\sigma}(t), \quad t \in [0,1/2] \cup [1,3/2] \\ y^{\Delta}(0) = 0 \\ y^{\Delta}(3/2) = 0 \end{cases}$$

and

$$\widetilde{L}_0 : \begin{cases} -y^{\Delta\Delta}(t) + [q(t) + 2\lambda p(t)]y^{\sigma}(t) = \lambda y^{\sigma}(t), \quad t \in [0,1/2] \cup [1,3/2] \\ y^{\Delta}(0) = 0 \\ y^{\Delta}(3/2) = 0. \end{cases}$$

Let $\Lambda_0 := \{\lambda_n,\, n \in \mathbb{Z}\}$ *and* $\widetilde{\Lambda}_0 := \left\{\widetilde{\lambda}_n,\, n \in \mathbb{Z}\right\}$ *be the eigenvalues sets of* L *and* $\widetilde{L}_0$*,* $y_n(t)$ *and* $\widetilde{y}_n(t)$ *are eigenfunctions related to this eigenvalues, respectively. According to Theorem-1, if* $\Lambda_0 = \widetilde{\Lambda}_0$ *and for any* $n \in \mathbb{N}$*,*

$$\frac{y_n^{\nabla}(1)}{y_n(1)} = \frac{\widetilde{y}_n^{\nabla}(1)}{\widetilde{y}_n(1)}$$

then $q(t) \equiv p(t) = 0$ *on* $\mathbb{T}$*.*

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