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BEST PROXIMITY POINT RESULTS FOR Γ CLASS OF MAPPINGS

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Abstract. In this manuscript, we define the class of Γ consist of p-cyclic mapping with certain assumptions.We investigate the existence and uniqueness of a best proximity point in the setting of a metric space with an additional condition p-completeness. Further, we examine the both existence and uniqueness a best proximity point for the mapping that lies in this class in the context of strictly convex normed linear space.

1. Introduction

In nonlinear functional analysis, best proximity point theory is a crucial tool with dealing with the approximate solutions. Indeed, fixed point theory and best proximity point theory are like two halves of a whole and the same time, they are complements of the each other. In case the equation $Fx = x$ has an exact solution, we deal with the fixed point theory. On the other hand, if there is no exact solution of the equation $Fx = x$, the best point of proximity theory enters the discussion. Roughly speaking, for a self-mapping F on a metric space (X, d), existence of a fixed point theory is the equivalent of the equation $d(x, Fx) = 0$. In the case $d(x, Fx) > 0$ we investigate the best approximation. This problem turns into more interesting if consider two subsets A, B of a metric space in a way that $F : A \to B$. In this case, the best approximation is equivalent to the solution of the following equation $d(x, Fx) = d(A, B)$ where $d(A, B) := \inf\{d(a, b) : \ a \in A \text{ and } b \in B\}$. The solution of the last equality is called best proximity point of non-selfmapping F. It should be noted that almost all real world problems can be expressed as $Fx = x$, as fixed point problem. As it is expected, not all equations have an exact solution and that express the necessity of the best proximity point theory [1, 9, 12, 13, 18].

In what follows, we shall recall the cyclic mapping and cyclic contractions. For non-empty subset A, B of a metric space (X, d), a selfmapping T on $A \cup B$ is called cyclic if $T(A) \subset B$ and $T(B) \subset A$. In 2003, Kirk-Srinivasan-Veeramani [26] proposed a new contraction, namely cyclic contraction, based on the cyclic mappings. After this initial paper, existence and uniqueness of a cyclic mapping has been investigate heavily by many authors, see e.g. [2–7, 11, 14, 15, 19–23] and the related reference therein.

2. Preliminaries

Definition 2.1. Let (X, d) be a metric space. Let $A_1, A_2, ..., A_p (p \geq 2)$ be non empty subsets of X.

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(1) A map $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is called a **p-cyclic map** if $T(A_i) \subseteq A_{i+1}$ where $A_{p+i} = A_i$ ([24]).
(2) A point $x \in A_i$ is said to be a **best proximity point** of T in A_i, if $d(x, Tx) = dist(A_i, A_{i+1})$ ([24]).
(3) Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic map. T is said to be a **p-cyclic non expansive map** if for $x \in A_i,\ y \in A_{i+1},\ d(Tx, Ty) \leq d(x, y)$ ([24]).
(4) A p-cyclic map T is said to be p**-cyclic strict contraction**([17]) if, for all $x \in A_i$, $y \in A_{i+1}$, $1 \leq i \leq p$:
   (i) $\rho(x, y) > dist(A_i, A_{i+1}) \Rightarrow \rho(Tx, Ty) < \rho(x, y)$; and
   (ii) $\rho(x, y) = dist(A_i, A_{i+1}) \Rightarrow \rho(Tx, Ty) = \rho(x, y)$.
(5) A sequence $\{x_n\}_{n=1}^{\infty} \subseteq \cup_{i=1}^{p} A_i$ is called a p**–cyclic sequence**([17]) if $x_{pn+i} \in A_i$, for all $n \in \mathbb{N}_0$ and $i = 1, 2, ..., p$.
(6) We say that $\{x_n\}_{n=1}^{\infty}$ is a p**–cyclic Cauchy sequence**([17]), if for given $\epsilon > 0$ there exists an $N_0 \in \mathbb{N}$ such that *for some* $i \in \{1, 2, \ldots, p\}$, we have

$$\rho(x_{pn+i}, x_{pm+i+1}) < dist(A_i, A_{i+1}) + \epsilon,\ \forall\, m, n\ \geq\ N_0. \tag{2.1}$$

(7) Let $\{x_n\}_{n=1}^{\infty}$ be a p-cyclic sequence in $\cup_{i=1}^{p} B_i$. If for some $j \in \{1, 2, \ldots, p\}$ the subsequence $\{x_{pn+j}\}$ of $\{x_n\}_{n=1}^{\infty}$ converges in A_j, then we say that $\{x_n\}_{n=1}^{\infty}$ is p**–cyclic convergent** ([17]).
(8) We say that $\cup_{i=1}^{p} A_i$ is p**-cyclic complete** if every p-cyclic Cauchy sequence in $\cup_{i=1}^{p} B_i$ is p-cyclic convergent ([17]).

Proposition 2.2 ([17]). *Let $A_1, A_2, ..., A_p$, $(p \geq 2)$ be non-empty convex subsets of a strictly convex normed linear space X such that $dist(A_i, A_{i+1}) > 0,\ i \in \{1, 2, \ldots, p\}$. Let $T : \cup_{i=1}^{p} A_i \longrightarrow \cup_{i=1}^{p} A_i$ be a p-cyclic map. Then, T has at most one best proximity point in A_i, $1 \leq i \leq p$.*

Definition 2.3 ([17]). For a non-empty set M, suppose $\rho : M \times M \to [0, \infty)$ forms a metric and $B_1, B_2, ..., B_p$ $(p \geq 2)$ are non-empty subsets of M. A p-cyclic map $T : \cup_{i=1}^{p} B_i \longrightarrow \cup_{i=1}^{p} B_i$ is said to belong to the class Ω if

(1) T is p-cyclic strict contraction.

(2) If $x, y \in B_i$, then $\lim_{n\to\infty} \rho(T^{pn}x, T^{pn+1}y) = dist(B_i, B_{i+1}), 1 \leq i \leq p$.

Definition 2.4 ([25]). Let (X, d) be a metric space. Let $A_1, A_2, ..., A_p$, be non empty subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic map such that for some $x \in A_i$, $(1 \leq i \leq p)$,the following inequality holds for each $k = 0, 1, 2, ..., (p-1)$

$$d(T^{pn+k}x, T^{k+1}y) \leq d(T^{pn+k-1}x, T^k y),\ \ for\ all\ y \in A_i\ and\ for\ all\ n \in \mathbb{N}. \tag{2.2}$$

Then the map T is called **p-cyclic orbital non expansive map**.

Definition 2.5 ([29]). Let (X, d) be a metric space. Let $A_1, A_2, ..., A_p$ be non empty subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic map such that for some $x \in A_i$, $(1 \leq i \leq p)$, for all $y \in A_i$ and for all $n \in \mathbb{N}$, the following inequality holds:

$$\begin{aligned} d(T^{pn+k}x, T^{k+1}y) \ &\leq \alpha[d(T^{pn+k-1}x, T^k y)]d(T^{pn+k-1}x, T^k y) \\ &\quad +[1 - \alpha(d(T^{pn+k-1}x, T^k y))dist(A_{i+k-1}, A_{i+k})] \end{aligned} \tag{2.3}$$

where $\alpha \in \mathbb{S}$. Then T is called, **p-cyclic orbital Geraghty type map**.

Proposition 2.6 ([29, Proposition 2.1]). *Let (X,d) be a metric space and $A_1, A_2, ..., A_p$ be non empty subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic orbital Geraghty type map with an $x \in A_i$, $1 \leq i \leq p$, satisfying (2.6). Then the following hold:*

a) *T is a p - cyclic orbital non expansive map.*
b) $\lim_{n\to\infty} d(T^{pn+k}x, T^{pn+k+1}y) = dist(A_{i+k}, A_{i+k+1})$, *$y \in A_i$, for each $k \in \{0, 1, 2, ..., p\}$*

Definition 2.7 ([25]). Let (X,d) be a metric space. Let A_i, i = 1,...,p be non empty subsets of X.Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic map. Then T is called a **p-cyclic orbital Meir–Keeler contraction map**, if for some $x \in A_i$ $(1 \leq i \leq p)$, and for each $k = 0, 1, 2, ..., (p-1)$, the following holds: For every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\begin{aligned} &\text{if there holds } d(T^{pn+k-1}x, T^k y) < \mathrm{dist}(A_{i+k-1}, A_{i+k}) + \epsilon + \delta \\ &\text{then there holds } d(T^{pn+k}x, T^{k+1}y) < \mathrm{dist}(A_{i+k}, A_{i+k+1}) + \epsilon, \end{aligned} \tag{2.4}$$

for all $n \in \mathbb{N}$, for all $y \in A_i$.

Lemma 2.8 ([25]). *Let (X,d) be a metric space. Let A_i, i = 1,...,p be non empty subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p–cyclic orbital Meir–Keeler contraction map. Then there exists an L-function ϕ such that for an $x \in A_i$ satisfying (2.4), the following hold:*

$$\begin{aligned} &\textit{if there holds } d(T^{pn+k-1}x, T^k y) > \mathrm{dist}(A_{i+k-1}, A_{i+k}) \\ &\textit{then there holds } \lambda_{p,n,k}(x,y) < \phi(\lambda_{p,n,k-1}(x,y)), \end{aligned} \tag{2.5}$$

where we use the notation $\lambda_{p,n,k}(x,y) = d(T^{pn+k}x, T^{k+1}y) - \mathrm{dist}(A_{i+k}, A_{i+k+1})$ and

$$\begin{aligned} &\textit{if there holds } d(T^{pn+k-1}x, T^k y) = \mathrm{dist}(A_{i+k-1}, A_{i+k}) \\ &\textit{then there holds } d(T^{pn+k}x, T^{k+1}y) = \mathrm{dist}(A_{i+k}, A_{i+k+1}), \end{aligned} \tag{2.6}$$

for each $k = 1, 2, ..., p$, for all $n \in \mathbb{N}$ and for all $y \in A_i$.

Remark 2.9 ([25]). From Lemma 5.1, it follows that a p-cyclic orbital Meir–Keeler contraction map is p–cyclic orbital non expansive map.

Lemma 2.10 ([25]). *Let (X,d) be a metric space. Let A_i, i = 1,...,p be non empty subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic orbital Meir–Keeler contraction map with an $x \in A_i$ satisfying (2.4). Then for all $y \in A_i$ and for each $k \in \{0, 1, 2, ..., (p-1)\}$, the sequence $\{d(T^{pn+k}x, T^{pn+k+1}y)\}_{n=1}^{\infty}$ converges to* $\mathrm{dist}(A_{i+k}, A_{i+k+1})$.

Definition 2.11 ([10]). Let (X,d) be a metric space. Let $A_1, A_2, ..., A_p$ be non empty subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic map such that for some $x \in A_i$, $(1 \leq i \leq p)$, for all $y \in A_i$ and for all $n \in \mathbb{N}$, the following inequality holds:

$$\begin{aligned} d(T^{pn+k}x, T^{k+1}y) \leq \psi[&d(T^{pn+k-1}x, T^k y) - dist(A_{i+k-1}, A_{i+k})] \\ &+ dist(A_{i+k-1}, A_{i+k}) \end{aligned} \tag{2.7}$$

where $\psi : [0,\infty) \to [0,\infty)$ is upper semi continuous from the right and satisfies $\psi(t) < t$ for $t > 0$ and $\psi(0) = 0$. Then T is called **p-cyclic orbital contraction of Boyd-Wong type**.

Proposition 2.12 ([10, Proposition 11]). *Let (X,d) be a metric space and $A_1, A_2, ..., A_p$ be non empty subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic orbital contraction map of Boyd-Wong type, with an $x \in A_i$, $1 \leq i \leq p$, satisfying (2.6). Then the following hold:*

a) *T is a p - cyclic orbital non expansive map.*
b) $\lim_{n\to\infty} d(T^{pn+k}x, T^{pn+k+1}y) = dist(A_{i+k}, A_{i+k+1})$, *$y \in A_i$, for each $k \in \{0, 1, 2, ..., p\}$*

Definition 2.13 ([16]). Let $A_1, A_2, \ldots, A_p (p \in \mathbb{N}, p \geq 2)$ be non empty subsets of a metric space X and $\phi : [0,\infty) \longrightarrow [0,\infty)$ be a strictly increasing map. We say that a p-cyclic map $T : \bigcup_{i=1}^{p} A_i \longrightarrow \bigcup_{i=1}^{p} A_i$ is **p-cyclic orbital ϕ-contraction** if for each $k = 0, 1, 2, \ldots, (p-1)$ and for some $x \in A_i$ $(1 \leq i \leq p)$, the following inequality holds:

(2.8)

$$d(T^{pn+k}x, T^{k+1}y) \leq d(T^{pn+k-1}x, T^k y) - \phi(d(T^{pn+k-1}x, T^k y)) + \phi(d(A_{i+k-1}, A_{i+k}))$$

for all $y \in A_i$ and $n \in \mathbb{N}$.

Proposition 2.14 ([16, Proposition 2.2]). *Every p-cyclic orbital ϕ-contraction map is p-cyclic orbital non-expansive.*

Proposition 2.15 ([16, Proposition 2.3]). *Let $A_1, A_2, \ldots, A_p (p \in \mathbb{N}, p \geq 2)$ be non empty subsets of a metric space X. Let $\phi : [0,\infty) \longrightarrow [0,\infty)$ be a strictly increasing map. If $T : \bigcup_{i=1}^{p} A_i \longrightarrow \bigcup_{i=1}^{p} A_i$ is a p-cyclic orbital ϕ-contraction map such that equation(2.8) holds for some $x \in A_i, (1 \leq i \leq p)$, then*

$$\lim_{n\to\infty} d(T^{pn+k}x, T^{pn+k+1}y) = dist(A_{i+k}, A_{i+k+1})$$

for every $y \in A_i$ and $k \in \{0, 1, 2, \ldots, p\}$.

Lemma 2.16 ([8]). *Let $(X, \|.\|)$ be a uniformly convex Banach space. Let A and B be non empty, closed and convex subsets of X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in A and $\{z_n\}$ be a sequence in B such that the following holds:*

- $\lim_{n\to\infty} \|x_n - z_n\| = dist(A, B)$
- $\lim_{n\to\infty} \|y_n - z_n\| = dist(A, B)$

then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

3. Γ class of mappings

Definition 3.1. Let (X, d) be a metric space and $A_1, A_2, ..., A_p$ be non empty subsets of X. A p-cyclic map $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is said to belong to class Γ if T satisfies the following conditions:

(C1) T is p-cyclic orbital non expansive map.
(C2) For each $x \in A_i$ $(1 \leq i \leq p)$ satisfying (2.2),

$$lim_{n\to\infty} d(T^{pn+k}x, T^{pn+k+1}y) = dist(A_{i+k}, A_{i+k+1})$$

for all $n \in \mathbb{N}$, for all $y \in A_i$ and $k \in \{0, 1, 2, ..., p\}$

If $T \in \Gamma$ then T entails the distances between the adjacent sets to be equal. This is proved in the following proposition.

Proposition 3.2. *Let $X, A_1, A_2, ..., A_p, T$ be as in definition* 3.1. *Let $T \in \Gamma$. Then*

$$dist(A_1, A_2) = dist(A_2, A_3) = ... = dist(A_p, A_1).$$

Proof. Without loss of generality, let $x \in A_1$ satisfy 2.2. Then

$$dist(A_k, A_{k+1}) \leq d(T^{pn+k}x, T^{pn+k+1}y) \leq d(T^{pn+k-1}x, T^{pn+k}y),$$

for all $n \in \mathbb{N}$, for all $y \in A_1$ and $k \in \{0, 1, 2, ..., p\}$. Taking limit as $n \to \infty$ and by condition (C2),

$$dist(A_k, A_{k+1}) \leq \lim_{n\to\infty} d(T^{pn+k-1}x, T^{pn+k}y) = dist(A_{k-1}, A_k).$$

Since this is true for each $k \in \{0, 1, 2, ..., p\}$ we have

$$dist(A_1, A_2) = dist(A_{p+1}, A_{p+2}) \leq dist(A_p, A_{p+1}) \leq ... \leq dist(A_2, A_1).$$

Hence the proof. □

Remark 3.3. If $T \in \Omega$ then T is p-cyclic non expansive and this condition alone is enough for the distances between the adjacent sets to be equal. But if $T \in \Gamma$ then both the conditions (C1) and (C2) are needed for the distances between the adjacent sets to be equal.

4. Some known maps which belong to class Γ

Some p-cyclic maps which satisfy various orbital contractive conditions fall under class Γ.

Example 4.1. Let (X, d) be a metric space and $A_1, A_2, ..., A_p$ be non empty subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic orbital Meir–Keeler contraction (definition 2.7). Then $T \in \Gamma$.

Proof. By remark □

Example 4.2. Let (X, d) be a metric space and $A_1, A_2, ..., A_p$ be non empty subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic orbital contraction of Boyd-Wong type (definition 2.11). By proposition 2.12 (a) and (b), conditions (C1) and (C2) are satisfied respectively. Therefore $T \in \Gamma$.

Example 4.3. Let (X, d) be a metric space and $A_1, A_2, ..., A_p$ be non empty subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic orbital Geraghty type map (definition 2.5). By proposition 2.6 (a) and (b), conditions (C1) and (C2) are satisfied respectively. Therefore $T \in \Gamma$.

Example 4.4. Let (X, d) be a metric space and $A_1, A_2, ..., A_p$ be non empty subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic orbital ϕ-contraction (definition 2.13). By propositions 2.14 and 2.15, conditions (C1) and (C2) are satisfied respectively. Therefore $T \in \Gamma$.

Remark 4.5. Let $T \in \Gamma$ and $x = x_0 \in A_i$ satisfy (C1) and (C2). Set $x_n = T^n x$. Then we see that the sequence of orbit of $x \in A_i$ is a p-cyclic sequence.

5. Best proximity point results for Γ class of mappings

Lemma 5.1. *Let (X,d) be a metric space and $A_1, A_2, ..., A_p$ be non empty and closed subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic map which belongs to Γ. Assume that for some $k \in \mathbb{N}$ and $x \in A_i$ $(1 \le i \le p)$, the sequence $\{T^{pn+k}x\}$ converges to $\xi \in A_{i+k}$ then ξ is a best proximity point of T in A_{i+k}.*

Proof. Let $x \in A_i$ satisfy conditions (C1) and (C2). By proposition 3.2,

$$\begin{aligned} dist(A_{i+k}, A_{i+k+1}) &= dist(A_{i+k-1}, A_{i+k}) \\ &\le d(T^{pn+k-1}x, \xi), \ for\ each\ n \in \mathbb{N} \\ &\le d(T^{pn+k-1}x, T^{pn+k}x) + d(T^{pn+k}x, \xi) \end{aligned}$$

Since $T \in \Gamma$,

$$\lim_{n\to\infty} \{d(T^{pn+k-1}x, T^{pn+k}x) + d(T^{pn+k}x, \xi)\} = dist(A_{i+k-1}, A_{i+k}).$$

Now

$$\begin{aligned} dist(A_{i+k}, A_{i+k+1}) \le d(\xi, T\xi) &= \lim_{n\to\infty} d(T^{pn+k}x, T\xi) \\ &\le \lim_{n\to\infty} d(T^{pn+k-1}x, \xi) \\ &= \lim_{n\to\infty} \{d(T^{pn+k-1}x, T^{pn+k}x)\} \\ &= dist(A_{i+k-1}, A_{i+k}) \\ &= dist(A_{i+k}, A_{i+k+1}) \end{aligned}$$

Therefore $d(\xi, T\xi) = dist(A_{i+k}, A_{i+k+1})$. Hence the proof. □

Lemma 5.2. *Let (X,d) be a metric space and $A_1, A_2, ..., A_p$ be non empty subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic map which belongs to Γ such that $x \in A_1$ satisfies conditions (C1) and (C2). If for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for some $j \in \{0, 1, 2, ..., p\}$,*

$$d(T^{pn+j}x, T^{pm+j+1}x) < dist(A_{j+1}, A_{j+2}) + \epsilon \ m, n \ge n_0$$

then for given epsilon > 0 there exists an $n_1 \in \mathbb{N}$ such that for every $k = 1, 2, ..., p$ there holds

$$d(T^{pn+j+k}x, T^{pm+j+k+1}x) < dist(A_{j+k+1}, A_{j+k+2}) + \epsilon \ m, n \ge n_1.$$

Proof. Let $\epsilon > 0$ be given. Applying p-cyclic orbital non expansive condition for each $k = 1, 2, ..., p$ we get

$$\begin{aligned} d(T^{pn+j+k}x, T^{pm+j+k+1}x) &\le d(T^{pn+j}x, T^{pm+j+1}x) \\ &< dist(A_{j+1}, A_{j+2}) + \epsilon, \ \ m, n \ge n_0. \\ &= dist(A_{j+k+1}, A_{j+k+2}) + \epsilon \ m, n \ge n_0. \end{aligned}$$

Hence the proof. □

Theorem 5.3. *Let (X,d) be a metric space and $A_1, A_2, ..., A_p$ be non empty and closed subsets of X such that $X = \cup_{i=1}^{p} A_i$. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic map which belongs to Γ such that $x \in A_i$ $(1 \le i \le p)$ satisfies conditions (C1) and*

(C2). Let X be p-cyclic complete. Then the sequence $\{T^{pn+k}x\}$ converges to ξ_{i+k} which is a best proximity point of T in A_{i+k}, for each $k = 0, 1, 2, ..., p-1$.

Proof. Let $x = x_0 \in A_i$ satisfy conditions (C1) and (C2). Define a sequence $\{x_n\}$ in X such that $x_n = T^n x, n \in \mathbb{N}$.
Claim: $\{T^n x\}$ is a p-cyclic Cauchy sequence.
Let $m, n \in \mathbb{N}$ such that $m > n$. Now

$$\begin{aligned} d(T^{pm}x, T^{pn+1}x) &= d(T^{p(n+r)}x, T^{pn+1}x) \text{ where } m = n + r,\ r \in \mathbb{N} \\ &= d(T^{pn}y, T^{pn+1}x),\ \text{where } y = T^{pr}x \in A_i \\ &\longrightarrow dist(A_i, A_{i+1}) \text{ as } n \to \infty \text{ by } (C2). \end{aligned}$$

This implies that for every $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that

$$d(T^{pm}x, T^{pn+1}x) < dist(A_i, A_{i+1}) + \epsilon,\ m, n \geq n_0.$$

By lemma 5.2, for a given $\epsilon > 0$ there exists an $n_1 \in \mathbb{N}$ such that

$$d(T^{pm+k}x, T^{pn+k+1}x) < dist(A_{i+k}, A_{i+k+1}) + \epsilon,\ m, n \geq n_1 \quad for\ each\ k = 0, 1, 2, ..., p-1.$$

Hence $\{T^{pn+k}x\}$ are p-cyclic Cauchy sequences for each $k = 0, 1, 2, ..., p-1$. Since X is p-cyclic complete, each sequence converges to ξ_{i+k} in A_{i+k}. By lemma 5.1 ξ_{i+k} is a best proximity point of T in A_{i+k} for each $k = 0, 1, 2, ..., p-1$. □

Even though there exists a best proximity point ξ_k in each set A_k, it is interesting to ask whether $\xi_{k+1} = T\xi_k$?

Corollary 5.4. *Let (X, d) be a metric space and $A_1, A_2, ..., A_p$ be non empty and closed subsets of X such that $X = \cup_{i=1}^p A_i$. Let $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a p-cyclic map which belongs to Γ such that $x \in A_i$ $(1 \leq i \leq p)$ satisfies conditions (C1) and (C2). Let X be p-cyclic complete and $dist(A_i, A_{i+1}) = 0$ $(1 \leq i \leq p)$. Then there exists a unique fixed point ξ of T such that $\{T^{pn}x\}$ converges to ξ which belongs to $\cap_{i=1}^p A_i$.*

Proof. By proposition 3.2 $dist(A_i, A_{i+1}) = 0$ for all $i = 1, 2, ..., p$. From theorem 5.3, there exists a $\xi \in A_i$ such that $\{T^{pn}x\}$ converges to ξ and $d(\xi, T\xi) = 0$. That is $\xi = T\xi$. Since T is p-cyclic, $\xi \in \cap_{i=1}^p A_i$ and is a fixed point of T. To prove that xi is unique, let η be such that $\eta = T\eta$. Then

$$d(\xi, \eta) = \lim_{n\to\infty} d(T^{pn}x, T^{pn+1}\eta) = 0,$$

which implies $\eta = \xi$. □

Theorem 5.5. *Let X be a strictly convex normed linear space. Let $A_1, A_2, ..., A_p$ be non empty, closed and convex subsets such that $X = \cup_{i=1}^p A_i$. Let $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a p-cyclic map which belongs to Γ. If X is p-cyclic complete, then there exists a unique best proximity point of T in each subset.*

Proof. By theorem 5.3, there exists a best proximity point $T_j \in A_j$ for each $j = 1, 2, ..., p$. By proposition 2.2 the obtained best proximity point is unique in each set A_j. □

Theorem 5.6. *Let X be a uniformly convex Banach space. Let $A_1, A_2, ..., A_p$ be non empty, closed and convex subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic map which belongs to class Γ with an $x \in A_i$, $(1 \leq i \leq p)$ satisfying conditions (C1) and (C2). Then the sequence $\{T^{pn+k}x\}$ converges to ξ_k which is a unique best proximity point of T in each subset A_{i+k} for each $k = 0, 1, 2, ..., p-1$.*

Proof. Let $x \in A_i$, $(1 \leq i \leq p)$ satisfy condition(C1). Let $m > n$, $m, n \in \mathbb{N}$. By condition (C2) we have

$$\lim_{m\to\infty} \|T^{pm+k}x - T^{pm+k+1}x\| = dist(A_{i+k}, A_{i+k+1}).$$

$$\begin{aligned} \lim_{m,n\to\infty} \|T^{pn+k}x - T^{pm+k+1}x\| &= \lim_{n\to\infty} \|T^{pn+k}x - T^{pn+k+1}y\|, \ y = T^{pr}x \ r \in \mathbb{N} \\ &= dist(A_{i+k}, A_{i+k+1}). \end{aligned}$$

By Lemma 2.16, $\lim_{m,n\to\infty} \|T^{pm+k}x - T^{pn+k}x\| = 0$. This implies sequence $\{T^{pn+k}x\}$ is a Cauchy sequence in A_{i+k} for each $k = 0, 1, 2, ..., p-1$. Since X is complete, each sequence converge to a ξ_k in A_{i+k}. By Lemma 5.1, ξ_k is a best proximity point of T in each A_{i+k}. By Proposition 2.2, the obtained best proximity points are unique. □

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