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Ideal convergence in partial metric spaces

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Abstract
The aim of this paper was to develop the summability literature by introducing the concept of $\mathcal{I}_p$-convergence in the partial metric space (X, p). First, we give some properties of $\mathcal{I}_p$-convergence. Also, we introduce the concept of $\mathcal{I}_p^*$-convergence in the partial metric space (X, p) and examine relations between newly defined concepts. Then, we present the concepts of $\mathcal{I}_p$-Cauchy and $\mathcal{I}_p^*$-Cauchy sequence in the partial metric space (X, p) and investigate relations between these Cauchy sequences.

Keywords Ideal convergence · Statistical convergence · Partial metric space

1 Introduction

In mathematical analysis, especially in summability theory, the concept of convergence provides many applications and extensions that shed light on researchers. Therefore, over the years, many mathematicians have studied the concept of convergence and developed it in different areas such as statistical convergence, ideal convergence, fuzzy convergence and power-series summability (Fast 1951; Kostyrko et al. 2000; Baxhaku et al. 2022; Shukla et al. 2022).

Fast (1951) and Steinhaus (1951), independently, presented the idea of statistical convergence which is primarily based on the concept of the natural density. This concept has attracted the attention of many researchers, and many studies have been done on this concept (see, Schoenberg 1959; Šalát 1980; Fridy 1985). Then, the definition of $\mathcal{I}$-convergence, generalizing the concept of statistical convergence, was presented by Kostyrko et al. (2000). Also, this concept was developed and explained in detail with examples by Kostyrko et al. (2005). Besides, Nabiev et al. (2007) introduced the concepts of $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences. The concept of $\mathcal{I}$-convergence was studied from different perspectives such as for sequences of fuzzy numbers by Kumar and Kumar (2008) and in 2-normed spaces by Arslan and Dündar (2018).

The concept of partial metric space was introduced by Matthews (1994) as a generalization of the usual concept of metric space. In this study, Matthews presented the concept of partial metric space and gave some basic properties of this concept. Also, the concepts of convergence and Cauchy sequence were defined in this study. Recently, this concept has become the focus of researchers, and many studies have been done (see, Bukatin et al. 2009; Samet et al. 2013). In one of these studies, Aldemir et al. (2020) investigated the relationships of partial metric space and fuzzy metric space with partial fuzzy metric spaces. Also, Nuray (2022) presented the concept of statistical convergence in the partial metric space.

In this paper, we discuss the concept of $\mathcal{I}$-convergence in partial metric spaces.

2 Preliminaries

In this part, we remind some basic definitions, notations and properties which form the background for this paper (see, Matthews 1994; Kostyrko et al. 2000; Nabiev et al. 2007; Nuray 2022).

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3 DEFERRED STRONGLY CESARO SUMMABLE AND STATISTICALLY CONVERGENT FUNCTIONS

Nuray, F; Dündar, E and Ulusu, U

Dec 2022 | HONAM MATHEMATICAL JOURNAL 44 (4) , pp.560-571

20 References

In this paper, firstly we introduce the concepts of deferred Cesaro summable and deferred statistically convergent function, and sec- ondly we introduce the concepts of deferred almost summable and

Let S be a set of positive integers. The natural density of S is defined by

$$\delta(S) = \lim_{n\to\infty} \frac{1}{n}\left|\{k \in S : k \le n\}\right|.$$

A sequence (x_n) is said to be statistically convergent to L if for every $\varepsilon > 0$, the set $S(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \ge \varepsilon\}$ has natural density zero.

If (x_n) is a sequence, which satisfies a property P for all n except a set of natural density zero, then we say that (x_n) satisfies P for "almost all n", and we abbreviate this by "a.a. n".

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

I_1. $\emptyset \in \mathcal{I}$,
I_2. $E_1 \cup E_2 \in \mathcal{I}$ for each $E_1, E_2 \in \mathcal{I}$,
I_3. for each $E_1 \in \mathcal{I}$ and each $E_2 \subseteq E_1$, we have $E_2 \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and a non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

All ideals in this paper are assumed to be admissible.

An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to satisfy the property (AP) if for every countable family of mutually disjoint sets $\{E_1, E_2, ...\}$ belonging to $\mathcal{I}$, there exists a countable family of sets $\{F_1, F_2, ...\}$ such that the symmetric difference $E_j \Delta F_j$ is a finite set for $j \in \mathbb{N}$ and $F = \left(\bigcup_{j=1}^{\infty} F_j\right) \in \mathcal{I}$.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is called a filter if and only if

F_1. $\emptyset \notin \mathcal{F}$,
F_2. $E_1 \cap E_2 \in \mathcal{F}$ for each $E_1, E_2 \in \mathcal{F}$,
F_3. for each $E_1 \in \mathcal{F}$ and each $E_2 \supseteq E_1$, we have $E_2 \in \mathcal{F}$.

For any ideal $\mathcal{I}$, there is a corresponding filter $\mathcal{F}(\mathcal{I})$ defined by

$$\mathcal{F}(\mathcal{I}) = \left\{M \subset \mathbb{N} : (\exists E \in \mathcal{I})(M = \mathbb{N}\backslash E)\right\}.$$

A sequence (x_n) is said to be $\mathcal{I}$-convergent to L if for every $\varepsilon > 0$, the set $A_\varepsilon = \left\{n \in \mathbb{N} : |x_n - L| \ge \varepsilon\right\}$ belongs to $\mathcal{I}$. It is denoted by $\mathcal{I} - \lim x_n = L$.

A sequence (x_n) is said to be $\mathcal{I}^*$-convergent to L if there exists a set $M = \{m_1 < m_2 < ...\} \in \mathcal{F}(\mathcal{I})$ such that $\lim_{k\to\infty} x_{m_k} = L$. It is denoted by $\mathcal{I}^* - \lim x_n = L$.

Now, we recall the concept of partial metric and its properties. Then, we give the concepts of convergence and Cauchy sequence in the partial metric space.

Let X be a non-empty set. A function $p : X \times X \to \mathbb{R}$ is said to be a partial metric provided that for each $x, y, z \in X$

PM_1. If $p(x, x) = p(x, y) = p(y, y)$ then $x = y$
PM_2. $0 \le p(x, x) \le p(x, y)$
PM_3. $p(x, y) = p(y, x)$
PM_4. $p(x, y) \le p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) where X is a non-empty set and p is a partial metric on X.

It can be easily seen that if $p(x, y) = 0$, then from the conditions PM_1 and PM_2, $x = y$. But if $x = y$, $p(x, y)$ may not be 0.

Every partial metric space forms a metric space: A function $p^* : X \times X \to \mathbb{R}$ is a metric on X such that $p^*(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, where p is a partial metric on same space. So, a pair (X, p^*) is a metric space.

For the partial metric $p : X \times X \to \mathbb{R}$, $\circ_p \subseteq X \times X$ is a binary relation provided that for each $x, y \in X$, $x \circ_p y \Leftrightarrow p(x, x) = p(x, y)$.

A sequence (x_n) is said to be convergent to x in the partial metric space (X, p) if $\lim_{n\to\infty} p(x_n, x) = p(x, x)$.

A sequence (x_n) is said to be Cauchy sequence in the partial metric space (X, p) if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists.

Finally, we give the concepts of statistical convergence and statistical Cauchy sequence in the partial metric space.

A sequence (x_n) is said to be statistically convergent to x in the partial metric space (X, p) if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}\left|\{k \le n : |p(x_k, x) - p(x, x)| \ge \varepsilon\}\right| = 0.$$

A sequence (x_n) is said to be statistically Cauchy sequence in the partial metric space (X, p) if for every $\varepsilon > 0$, there exists a non-negative integer N and $L \ge 0$ such that

$$\lim_{n\to\infty} \frac{1}{n}\left|\{k \le n : |p(x_k, x_N) - L| \ge \varepsilon\}\right| = 0.$$

A point $a \in X$ is said to be proper statistical limit of a sequence (x_n) if this sequence is statistically convergent to a in the metric space (X, p^*).

3 Main results

In this study, we introduce the concepts of $\mathcal{I}_p$-convergence and $\mathcal{I}_p^*$-convergence in a partial metric space (X, p). Also, we present the concepts of $\mathcal{I}_p$-Cauchy and $\mathcal{I}_p^*$-Cauchy sequence in the partial metric space (X, p). Moreover, we investigate relations between these concepts.

Definition 1 Let (X, p) be a partial metric space and (x_n) be a sequence in X.

i. The sequence (x_n) is said to be $\mathcal{I}_p$-convergent to $x \in X$ if for every $\varepsilon > 0$

$$A_p(\varepsilon) = \left\{n \in \mathbb{N} : |p(x_n, x) - p(x, x)| \ge \varepsilon\right\} \in \mathcal{I}.$$

The notation $\mathcal{I}_p - \lim p(x_n, x) = p(x, x)$ or $x_n \xrightarrow{\mathcal{I}_p} x$ is used.

ii. We say that $x \in X$ is a proper $\mathcal{I}_p$-limit of the sequence (x_n) if $\mathcal{I} - \lim x_n = x$ in (X, p^*). The notation $\mathcal{I}_p - \lim x_n = x\ (properly)$ is used. If a sequence has a proper $\mathcal{I}_p$-limit, then the sequence is said to be properly $\mathcal{I}_p$-convergent.

Theorem 1 *Let (X, p) be a partial metric space and (x_n) be a sequence in X. $\mathcal{I}_p - \lim x_n = x\ (properly)$ iff $\mathcal{I}_p - \lim p(x_n, x) = \mathcal{I}_p - \lim p(x_n, x_n) = p(x, x)$.*

Proof

$$
\begin{aligned}
&\mathcal{I}_p - \lim x_n = x\ (properly)\\
&\quad \Leftrightarrow p^*(x_n, x) < \varepsilon, \text{ for a.a. } n\\
&\quad \Leftrightarrow |2p(x_n, x) - p(x_n, x_n) - p(x, x)| < \varepsilon, \text{ for a.a. } n\\
&\quad \Leftrightarrow |p(x_n, x) - p(x, x)| < \frac{\varepsilon}{2}\\
&\quad\ \text{ and } |p(x_n, x_n) - p(x, x)| < \frac{\varepsilon}{2}, \text{ for a.a. } n\\
&\quad \Leftrightarrow \mathcal{I}_p - \lim p(x_n, x) = \mathcal{I}_p - \lim p(x_n, x_n) = p(x, x).
\end{aligned}
$$

□

Theorem 2 *Let (X, p) be a partial metric space and (x_n) be a sequence in X. If the sequence (x_n) is $\mathcal{I}_p$-convergent to x and there exists an element $x' \in X$ such that $x' \circ_p x$, then this sequence is $\mathcal{I}_p$-convergent to x'.*

Proof To prove this theorem, it is sufficient to show that

$$A'_p(\varepsilon) = \left\{n \in \mathbb{N} : |p\left(x_n, x'\right) - p\left(x', x'\right)| \geq \varepsilon\right\} \in \mathcal{I}.$$

Since the sequence (x_n) is $\mathcal{I}_p$-convergent to x, for every $\varepsilon > 0$

$$A_p(\varepsilon) = \left\{n \in \mathbb{N} : |p\left(x_n, x\right) - p\left(x, x\right)| \geq \varepsilon\right\} \in \mathcal{I}.$$

Also, using the fact that

$$x' \circ_p x \Rightarrow p\left(x', x'\right) = p\left(x', x\right),$$

we have

$$
\begin{aligned}
&|p\left(x_n, x'\right) - p\left(x', x'\right)|\\
&\quad \leq |p\left(x_n, x\right) + p\left(x, x'\right) - p(x, x) - p\left(x', x'\right)|\\
&\quad = |p\left(x_n, x\right) - p(x, x)|
\end{aligned}
$$

and so,

$$A'_p(\varepsilon) \subset A_p(\varepsilon).$$

Thus, from the condition I_3, we get that

$$A'_p(\varepsilon) \in \mathcal{I},$$

that is, $x_n \xrightarrow{\mathcal{I}_p} x'$. □

Theorem 3 *Let (X, p) be a partial metric space and (x_n) be a sequence in X. If the sequence (x_n) convergent to x, then this sequence is $\mathcal{I}_p$-convergent to same point.*

Proof Assume that the sequence (x_n) convergent to x in the partial metric space (X, p). Then, for every $\varepsilon > 0$, there exists a positive integer $n_0 = n_0(\varepsilon)$ such that for all $n > n_0$

$$|p(x_n, x) - p(x, x)| < \varepsilon.$$

Hence, we get

$$
\begin{aligned}
A_p(\varepsilon) &= \left\{n \in \mathbb{N} : |p(x_n, x) - p(x, x)| \geq \varepsilon\right\}\\
&\subset \{1, 2, \cdots, n_0\}.
\end{aligned}
$$

Since $\mathcal{I}$ is an admissible ideal and $\mathcal{I}_f = \{A \subset \mathbb{N} : A \text{ is finite}\} \subset \mathcal{I}$, $\{1, 2, \cdots, n_0\} \in \mathcal{I}$. So, $A_p(\varepsilon) \in \mathcal{I}$. Consequently, we get that $\mathcal{I}_p - \lim p(x_n, x) = p(x, x)$, that is, $x_n \xrightarrow{\mathcal{I}_p} x$. □

Remark 1 In the partial metric space (X, p), the sequence (x_n) is $\mathcal{I}_p$-convergent, but it not necessarily convergent. We can explain this with the following example:

Example 1 Let us take $\mathcal{I} = \mathcal{I}_\delta = \{A \subset \mathbb{N} : \delta(A) = 0\}$. The function $p : \mathbb{R}^- \times \mathbb{R}^- \to \mathbb{R}^+$, $p(x, y) = -\min\{x, y\}$ is the partial metric and the pair $(\mathbb{R}^-, p)$ is the partial metric space. In this partial metric space $(\mathbb{R}^-, p)$, let us take the sequence (x_n) as following:

$$x_n := \begin{cases} -n; & \text{if } n = k^2 \\ -1; & \text{if not.} \end{cases}$$

Since

$$
\begin{aligned}
\mathcal{I}_p - \lim p(x_n, -1) &= \mathcal{I}_p - \lim\left(-\min\{x_n, -1\}\right)\\
&= \mathcal{I}_p - \lim 1\\
&= 1 = p(-1, -1),
\end{aligned}
$$

this sequence $\mathcal{I}_p$-convergent to -1. But, it is not convergent.

Theorem 4 *Let (X, p) be a partial metric space and (x_n), (y_n) and (z_n) be three sequences in X. If the following conditions*

i. *$(x_n) \leq (y_n) \leq (z_n)$ for every $n \in H$ where $H \in \mathcal{F}(\mathcal{I})$ and*

ii. $x_n \xrightarrow{\mathcal{I}_p} x$ *and* $z_n \xrightarrow{\mathcal{I}_p} x$

are provided, then $y_n \xrightarrow{\mathcal{I}_p} x$.

Proof Assume that $(x_n) \leq (y_n) \leq (z_n)$ for every $n \in H$ where $\mathbb{N} \supset H \in \mathcal{F}(\mathcal{I})$ and $x_n \xrightarrow{\mathcal{I}_p} x$ and $z_n \xrightarrow{\mathcal{I}_p} x$. Since $x_n \xrightarrow{\mathcal{I}_p} x$ and $z_n \xrightarrow{\mathcal{I}_p} x$, for every $\varepsilon > 0$, we have

$$\left\{n \in \mathbb{N} : |p(x_n, x) - p(x, x)| \geq \varepsilon\right\} \in \mathcal{I}$$
$$\text{and } \left\{n \in \mathbb{N} : |p(z_n, x) - p(x, x)| \geq \varepsilon\right\} \in \mathcal{I}.$$

This implies that the sets

$$G = \left\{n \in \mathbb{N} : |p(x_n, x) - p(x, x)| < \varepsilon\right\}$$
$$\text{and } K = \left\{n \in \mathbb{N} : |p(z_n, x) - p(x, x)| < \varepsilon\right\}$$

belong to $\mathcal{F}(\mathcal{I})$. Let

$$L = \left\{n \in \mathbb{N} : |p(y_n, x) - p(x, x)| < \varepsilon\right\}.$$

It is clear that $G \cap K \cap H \subset L$. Since $G \cap K \cap H \in \mathcal{F}(\mathcal{I})$ and $G \cap K \cap H \subset L$, from the condition F_3, we have $L \in \mathcal{F}(\mathcal{I})$ and so

$$\left\{n \in \mathbb{N} : |p(y_n, x) - p(x, x)| \geq \varepsilon\right\} \in \mathcal{I}.$$

Therefore, we get that $y_n \xrightarrow{\mathcal{I}_p} x$. □

Definition 2 Let (X, p) be a partial metric space and (x_n) be a sequence in X. The sequence (x_n) is said to be $\mathcal{I}_p^*$-convergent to $x \in X$ if there exist a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ and $M \in \mathcal{F}(\mathcal{I})$ such that

$$\lim_{k \to \infty} p(x_{m_k}, x) = p(x, x).$$

The notation $\mathcal{I}_p^* - \lim p(x, x_n) = p(x, x)$ or $x_n \xrightarrow{\mathcal{I}_p^*} x$ is used.

Theorem 5 *Let* (X, p) *be a partial metric space and* (x_n) *be a sequence in* X. *Then,* $x_n \xrightarrow{\mathcal{I}_p^*} x$ *implies that* $x_n \xrightarrow{\mathcal{I}_p} x$.

Proof Assume that the sequence (x_n) is $\mathcal{I}_p^*$-convergent to x in the partial metric space (X, p). Then, there exists a set $K \in \mathcal{I}$ such that for $M = \mathbb{N} \backslash K = \{m_1 < m_2 < \cdots < m_k < \cdots\}$, we have

$$\lim_{k} p(x_{m_k}, x) = p(x, x).$$

Thus, for every $\varepsilon > 0$, there exists a positive integer $k_0 = k_0(\varepsilon)$ such that for all $k > k_0$

$$|p(x_{m_k}, x) - p(x, x)| < \varepsilon,$$

for all $k \in M$. So, we have

$$A_p(\varepsilon) = \left\{n \in \mathbb{N} : |p(x_n, x) - p(x, x)| \geq \varepsilon\right\}$$
$$\subset K \cup \{m_1 < m_2 < \cdots < m_{k_0}\}.$$

Since $\mathcal{I}$ is an admissible ideal, we have $K \cup \{m_1 < m_2 < \cdots < m_{k_0}\} \in \mathcal{I}$ and so, $A_p(\varepsilon) \in \mathcal{I}$. Consequently, we get that $x_n \xrightarrow{\mathcal{I}_p} x$. □

Theorem 6 *Let* (X, p) *be a partial metric space,* (x_n) *be a sequence in* X *and* $\mathcal{I}$ *be an admissible ideal having the property* (AP). *Then,* $x_n \xrightarrow{\mathcal{I}_p} x$ *implies that* $x_n \xrightarrow{\mathcal{I}_p^*} x$.

Proof Assume that the sequence (x_n) is $\mathcal{I}_p$-convergent to x in the partial metric space (X, p). Then, for every $\varepsilon > 0$,

$$A_p(\varepsilon) = \left\{n \in \mathbb{N} : |p(x_n, x) - p(x, x)| \geq \varepsilon\right\} \in \mathcal{I}.$$

Now, let us take

$$A_p^1 = \left\{n \in \mathbb{N} : |p(x_n, x) - p(x, x)| \geq 1\right\}$$

and

$$A_p^n = \left\{n \in \mathbb{N} : \frac{1}{n} \leq |p(x_n, x) - p(x, x)| < \frac{1}{n-1}\right\}$$

for $n \geq 2$. It is clear that $A_p^s \cap A_p^t \neq \emptyset$ for $s \neq t$ and $A_p^s \in \mathcal{I}$ for each $s \in \mathbb{N}$. By the property (AP), there exists a sequence of sets $\{B_n\}_{n \in \mathbb{N}}$ such that $A_p^s \triangle B_s$ are finite sets and $B = \cup_{s=1}^{\infty} B_s \in \mathcal{I}$. It is sufficient to prove that

$$\lim_{\substack{n \to \infty \\ n \in M}} p(x_n, x) = p(x, x)$$

for $M \in \mathbb{N} \setminus B$. Let $\delta > 0$. Choose $m \in \mathbb{N}$ such that $\frac{1}{m} < \delta$. Then, we have

$$\left\{n \in \mathbb{N} : |p(x_n, x) - p(x, x)| \geq \delta\right\} \subset \bigcup_{s=1}^{m} A_p^s.$$

Since $A_p^s \triangle B_s$ are finite sets for $s = 1, 2, \cdots, m$, there exists an n_0 such that

$$\left(\bigcup_{s=1}^{m} B_s\right) \cup \{n \in \mathbb{N} : n \geq n_0\} = \left(\bigcup_{s=1}^{m} A_p^s\right) \cup \{n \in \mathbb{N} : n \geq n_0\}.$$

If $n > n_0$ and $n \notin B$, then

$$n \notin \bigcup_{s=1}^{m} B_s \;\Rightarrow\; n \notin \bigcup_{s=1}^{m} A_p^s.$$

Hence, we have

$$|p(x_n, x) - p(x, x)| < \frac{1}{n} < \delta$$

that is,

$$\lim_{\substack{n\to\infty \\ n\in M}} p(x_n, x) = p(x, x).$$

Therefore, we get that $x_n \xrightarrow{\mathcal{I}_p^*} x$. □

Theorem 7 *Let (X, p) be a partial metric space, (x_n) be a sequence in X and $\mathcal{I}$ be an admissible ideal having the property (AP). Then, the following conditions are equivalent:*

i. $x_n \xrightarrow{\mathcal{I}_p} x$
ii. *There exist two sequences (y_n) and (z_n) in the partial metric space (X, p) such that $x_n = y_n + z_n$, $\lim_{n\to\infty} p(y_n, x) = p(x, x)$ and $suppz \in \mathcal{I}$, where $suppz = \{n \in \mathbb{N} : z_n \neq 0\}$.*

Proof $(i) \Rightarrow (ii)$

Let $x_n \xrightarrow{\mathcal{I}_p} x$. Then, by Theorem (6), there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(\mathcal{I})$ such that

$$\lim_k p(x_{m_k}, x) = p(x, x).$$

Let us define a sequence (y_n) as following:

$$y_n := \begin{cases} x_n; & \text{if } n \in M \\ x; & \text{if } n \in \mathbb{N} \setminus M. \end{cases} \tag{1}$$

It is clear that $\lim_{n\to\infty} p(y_n, x) = p(x, x)$. Also, let

$$z_n = x_n - y_n, \quad n \in \mathbb{N}. \tag{2}$$

Since

$$\{n \in \mathbb{N} : x_n \neq y_n\} \subset \mathbb{N} \setminus M \in \mathcal{I},$$

we have

$$\{n \in \mathbb{N} : z_n \neq 0\} \in \mathcal{I}.$$

So, $suppz \in \mathcal{I}$ and by (1) and (2), we get that $x_n = y_n + z_n$.

$(ii) \Rightarrow (i)$ Assume that there exist the sequences (y_n) and (z_n) in the partial metric space (X, p) such that

$$x_n = y_n + z_n, \quad \lim_{n\to\infty} p(y_n, x) = p(x, x) \text{ and } suppz \in \mathcal{I}.$$

Define a set $M = \{m_k\} \subset \mathbb{N}$ such that

$$M = \{m \in \mathbb{N} : z_m = 0\} = \mathbb{N} \setminus suppz.$$

Since $suppz \in \mathcal{I}$, we have $M \in \mathcal{F}(\mathcal{I})$, hence $x_n = y_n$ if $n \in M$. Thus, we conclude that there exists the set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(\mathcal{I})$ such that

$$\lim_k p(x_{m_k}, x) = p(x, x).$$

So, we get that $x_n \xrightarrow{\mathcal{I}_p^*} x$, and by Theorem (6), $x_n \xrightarrow{\mathcal{I}_p} x$. □

Definition 3 Let (X, p) be a partial metric space and (x_n) be a sequence in X.

i. The sequence (x_n) is said to be $\mathcal{I}_p$-Cauchy sequence if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ and $L \geq 0$ such that

$$\left\{n \in \mathbb{N} : |p(x_n, x_N) - L| \geq \varepsilon\right\} \in \mathcal{I}.$$

ii. The sequence (x_n) is said to be $\mathcal{I}_p^*$-Cauchy sequence if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ and $M \in \mathcal{F}(\mathcal{I})$ such that the subsequence $(x_M) = (x_{m_k})$ is Cauchy sequence, that is,

$$\lim p(x_{m_k}, x_{m_j})$$

exists.

Remark 2 Unlike the metric space, in the partial metric space (X, p), the sequence (x_n), which is $\mathcal{I}_p$-convergent, need not be $\mathcal{I}_p$-Cauchy sequence. We can explain this with the following example.

Example 2 Let take $\mathcal{I} = \mathcal{I}_\delta = \{A \subset \mathbb{N} : \delta(A) = 0\}$. The function $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $p(x, y) = \max\{x, y\}$ is the partial metric and the pair $(\mathbb{R}^+, p)$ is the partial metric space. In this partial metric space $(\mathbb{R}^+, p)$, let us take the sequence (x_n) as following:

$$x_n := \begin{cases} n; & \text{if } n = k^2 \\ 0; & \text{if } n \neq k^2 \text{ and } n \text{ is an even integer} \\ 1; & \text{if } n \neq k^2 \text{ and } n \text{ is an odd integer.} \end{cases}$$

Since

$$\begin{aligned} \mathcal{I}_p - \lim p(x_n, 1) &= \mathcal{I}_p - \lim\, (\max\{x_n, 1\}) \\ &= \mathcal{I}_p - \lim 1 \\ &= 1 = p(1, 1), \end{aligned}$$

this sequence is $\mathcal{I}_p$-convergent to 1. But, since $x_n = 0$ or $x_n = 1$ for $n \neq k^2$, $\mathcal{I} - \lim p(x_n, x_m)$ does not exist. Consequently, the sequence (x_n) is $\mathcal{I}_p$-convergent, but it is not $\mathcal{I}_p$-Cauchy sequence.

Theorem 8 *Let (X, p) be a partial metric space and (x_n) be a sequence in X. If the sequence (x_n) is $\mathcal{I}_p^*$-Cauchy sequence, then it is $\mathcal{I}_p$-Cauchy sequence.*

Proof Assume that the sequence (x_n) is an $\mathcal{I}_p^*$-Cauchy sequence in the partial metric space (X, p). Then, from the definition, there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ and $M \in \mathcal{F}(\mathcal{I})$ such that for every $\varepsilon > 0$ and $k, j > k_0$

$$|p(x_{m_k}, x_{m_j}) - L| < \varepsilon.$$

Let $N = N(\varepsilon) = m_{k_0} + 1$. Then, for every $\varepsilon > 0$, we have

$$|p(x_{m_k}, x_N) - L| < \varepsilon$$

for $k > k_0$. Now, take the set K as $K = \mathbb{N} \setminus M$. It is clear that $K \in \mathcal{I}$ and

$$A_p(\varepsilon) = \Big\{n \in \mathbb{N} : |p(x_n, x_N) - L| \geq \varepsilon\Big\} \subset K \cup \{m_1 < m_2 < \cdots < m_{k_0}\}.$$

Since $\mathcal{I}$ is an admissible ideal, we get

$$K \cup \{m_1 < m_2 < \cdots < m_{k_0}\} \in \mathcal{I}$$

and so, $A_p(\varepsilon) \in \mathcal{I}$. Hence, the sequence (x_n) is $\mathcal{I}_p$-Cauchy sequence. □

Theorem 9 *Let (X, p) be a partial metric space, (x_n) be a sequence in X and $\mathcal{I}$ be an admissible ideal having the property (AP). If the sequence (x_n) is $\mathcal{I}_p$-Cauchy sequence, then it is $\mathcal{I}_p^*$-Cauchy sequence.*

Proof Assume that the sequence (x_n) is $\mathcal{I}_p$-Cauchy sequence in the partial metric space (X, p). Then, for every $\varepsilon > 0$, there is a positive number $N = N(\varepsilon)$ such that

$$\Big\{n \in \mathbb{N} : |p(x_n, x_N) - L| \geq \varepsilon\Big\} \in \mathcal{I}.$$

Let us take

$$P_i = \Big\{n \in \mathbb{N} : |p(x_n, x_{m_i}) - L| < \frac{1}{i}\Big\},$$

where $m_i = s\left(\frac{1}{i}\right)$ for $i = 1, 2, \ldots$. It is clear that $P_i \in \mathcal{F}(\mathcal{I})$. Since the admissible ideal $\mathcal{I}$ has the property (AP), there exists a set $P \subset \mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{I})$ and $P \setminus P_i$ are finite for all i. It is sufficient to prove that following limit

$$\lim_{\substack{m,n\to\infty \\ m,n\in P}} p(x_m, x_n)$$

exists. To prove this, let $\varepsilon > 0$ and $j \in \mathbb{N}$ be such that $j > \frac{3}{\varepsilon}$. If $m, n \in P$, since $P \setminus P_i$ are finite sets, there exists $k = k_j$ such that $m \in P_j$ and $n \in P_j$ for all $m, n > k_j$. Hence,

$$|p(x_m, x_{m_j}) - L| < \frac{1}{j} \text{ and } |p(x_n, x_{m_j}) - L| < \frac{1}{j}$$

for all $m, n > k_j$. Therefore, using the conditions of PM_1 and PM_4, we write

$$|p(x_m, x_n) - L| \leq |p(x_m, x_{m_j}) + p(x_n, x_{m_j}) - p(x_{m_j}, x_{m_j}) - L| < \frac{3}{j} < \varepsilon.$$

Thus, for every $\varepsilon > 0$, there exists $k = k_\varepsilon$ such that for $m, n > k_\varepsilon$ and $m, n \in P \in \mathcal{F}(\mathcal{I})$

$$|p(x_m, x_n) - L| < \varepsilon.$$

This shows that the sequence (x_n) is an $\mathcal{I}_p^*$-Cauchy sequence. □

4 Conclusion

The number of studies on the concepts of convergence and summability in the partial metric spaces, which has been studied in many fields since its definition, is very few. To fill this gap in the literature, in this study, we introduced and studied the concepts of $\mathcal{I}_p$ and $\mathcal{I}_p^*$-convergence, $\mathcal{I}_p$-Cauchy and $\mathcal{I}_p^*$-Cauchy sequence in the partial metric space (X, p). Also, we investigated some relations between these. This work can shed many future study on the concepts of convergence and summability in the partial metric spaces that will be extended and developed by using the concepts of lacunary sequence, invariant mean, deferred Cesàro mean and asymptotical equivalence.

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