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SCIENTIFIC RESEARCH PAPER

Lacunary Statistical Equivalence of Order η for Double Sequences of Sets

Uğur Ulusu[1]

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Abstract In this study, by introducing the concepts of asymptotical lacunary statistical and asymptotical strong p-lacunary equivalence of order η $(0 < \eta \leq 1)$ in the Wijsman sense for double set sequences, some properties of these concepts are examined and also the relationship between these concepts is mentioned. Moreover, the relationships between these concepts and the asymptotical equivalence concepts previously given for double set sequences are investigated.

Keywords Asymptotical equivalence · Statistical convergence · Double lacunary sequence · Order η · Convergence in the Wijsman sense · Double set sequences

Mathematics Subject Classification 41A25 · 40G15 · 40A05

1 Introduction

Long after the concept of convergence for double sequences was introduced by Pringsheim [1], using the concepts of statistical convergence, double lacunary sequence etc., this concept was extended to new convergence concepts for double sequences [2, 3]. Recently, for double sequences, on two new convergence concepts called double almost statistical and double almost lacunary statistical convergence of order α were studied by Savaş [4, 5]. Also, for double sequences, the concept of asymptotical equivalence was introduced by Patterson [6]. After then, this concept was, respectively, extended to the concept of asymptotical double statistical and asymptotical double lacunary statistical equivalence [7, 8].

Over the years, on the various convergence concepts for set sequences have been studied, one of them is the concept of convergence in the Wijsman sense [9–11]. Using the concepts of statistical convergence, double lacunary sequence etc., this concept was extended to new convergence concepts for double set sequences [12–14]. Recently, Ulusu and Gülle [15] studied on some convergence concepts of order α in the Wijsman sense for double set sequences. Furthermore, for double set sequences, the concepts of asymptotical equivalence in the Wijsman sense were introduced by Nuray et al. [16] and then these concepts were studied by many authors. Lately, Gülle [17] studied on the concepts of asymptotical statistical and asymptotical strong Cesàro equivalence of order α in the Wijsman sense for double set sequences.

More information on the concepts of convergence and asymptotical equivalence for real and set sequences can be found in [18–26].

Significant Statement In this paper, we present the concepts of rate of convergence and degree of approximation of any two real or complex sequences, the asymptotical statistical and asymptotical lacunary statistical equivalence in the Wijsman sense for double sequences of sets and tried to explain them with examples.

✉ Uğur Ulusu
ugurulusu@cumhuriyet.edu.tr

[1] Sivas Cumhuriyet University, 58140 Sivas, Turkey

2 Preliminaries and Background

First of all, let us recall the basic notions [1, 3, 6, 9, 12–14, 16, 17].

A double sequence $(a_{mn}) \in \mathbb{R}$ is called convergent in the Pringsheim sense to $L \in \mathbb{R}$ if every $\xi > 0$, there exists

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2 **Lacunary Statistical Equivalence of Order ? for Double Sequences of Sets**

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3 **DEFERRED STRONGLY CESARO SUMMABLE AND STATISTICALLY CONVERGENT FUNCTIONS**

Nuray, F; Dündar, E and Ulusu, U

Dec 2022 | HONAM MATHEMATICAL JOURNAL 44 (4) , pp.560-571

20 References

In this paper, firstly we introduce the concepts of deferred Cesaro summable and deferred statistically convergent function, and secondly we introduce the concepts of deferred almost summable and

$N_\xi \in \mathbb{N}$ such that $|a_{mn} - L| < \xi$ whenever $m, n > N_\xi$. It is denoted by $P - \lim_{m,n\to\infty} a_{mn} = L$.

Non negative double sequences $(a_{mn}), (b_{mn}) \in \mathbb{R}$ are called asymptotically equivalent if

$$P - \lim_{m,n\to\infty} \frac{a_{mn}}{b_{mn}} = 1$$

and denoted by $a_{mn} \sim b_{mn}$.

For a metric space (Y, d), $\rho(y, V)$ denote the distance from y to V where

$$\rho(y, V) := \rho_y(V) = \inf_{v\in V} d(y, v)$$

for any $y \in Y$ and any non empty $V \subseteq Y$.

Throughout this study, (Y, d) is considered as a metric space and U_{mn}, V_{mn}, V are considered as any non empty closed subsets of Y.

For a non empty set Y, let a function $g : \mathbb{N} \to P_Y$ (the power set of Y) is defined by $g(m) = V_m \in P_Y$ for each $m \in \mathbb{N}$. Then the sequence $\{V_m\} = \{V_1, V_2, \dots\}$, which is the range's elements of g, is called set sequences.

A double set sequence $\{V_{mn}\}$ is called convergent to V in the Wijsman sense if each $y \in Y$,

$$P - \lim_{m,n\to\infty} \rho_y(V_{mn}) = \rho_y(V).$$

A double set sequence $\{V_{mn}\}$ is called statistically convergent to V in the Wijsman sense if every $\xi > 0$ and each $y \in Y$,

$$P - \lim_{i,j\to\infty} \frac{1}{ij}\Big|\big\{(m,n) : \ m \le i, n \le j, \ |\rho_y(V_{mn}) - \rho_y(V)| \ge \xi\big\}\Big| = 0.$$

A double set sequence $\{V_{mn}\}$ is called strongly Cesàro summable to V in the Wijsman sense if each $y \in Y$,

$$P - \lim_{i,j\to\infty} \frac{1}{ij} \sum_{m,n=1,1}^{i,j} |\rho_y(V_{mn}) - \rho_y(V)| = 0.$$

A double sequence $\theta_2 = \{(j_s, k_t)\}$ is called a double lacunary sequence if there exist increasing sequences (j_s) and (k_t) of the integers such that

$$j_0 = 0, \ h_s = j_s - j_{s-1} \to \infty \quad \text{and} \quad k_0 = 0,$$
$$\bar{h}_t = k_t - k_{t-1} \to \infty \ \text{ as } \ s, t \to \infty.$$

In general, the following notations are used for any double lacunary sequence:

$$\ell_{st} = j_s k_t, \ \ h_{st} = h_s \bar{h}_t,$$
$$I_{st} = \big\{(m,n) : j_{s-1} < m \le j_s \ \text{ and } \ k_{t-1} < n \le k_t\big\},$$

$$q_s = \frac{j_s}{j_{s-1}} \ \text{ and } \ \overline{q}_t = \frac{k_t}{k_{t-1}}.$$

Throughout this study, $\theta_2 = \{(j_s, k_t)\}$ is considered as a double lacunary sequence.

A double set sequence $\{V_{mn}\}$ is called lacunary statistically convergent to V in the Wijsman sense if every $\xi > 0$ and each $y \in Y$,

$$P - \lim_{s,t\to\infty} \frac{1}{h_{st}}\Big|\big\{(m,n) \in I_{st} \ : \ |\rho_y(V_{mn}) - \rho_y(V)| \ge \xi\big\}\Big| = 0$$

and denoted by $V_{mn} \xrightarrow{W_2(S_\theta)} V$.

A double set sequence $\{V_{mn}\}$ is called strongly lacunary summable to V in the Wijsman sense if each $y \in Y$,

$$P - \lim_{s,t\to\infty} \frac{1}{h_{st}} \sum_{(m,n)\in I_{st}} |\rho_y(V_{mn}) - \rho_y(V)| = 0$$

and denoted by $V_{mn} \xrightarrow{W_2[N_\theta]} V$.

The term $\rho_y\Big(\dfrac{U_{mn}}{V_{mn}}\Big)$ is defined as follows:

$$\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) = \begin{cases} \dfrac{\rho(y, U_{mn})}{\rho(y, V_{mn})} & , \ y \notin U_{mn} \cup V_{mn} \\ \\ \lambda & , \ y \in U_{mn} \cup V_{mn}. \end{cases}$$

Double set sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are called asymptotically equivalent in the Wijsman sense if each $y \in Y$,

$$P - \lim_{m,n\to\infty} \rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) = 1$$

and denoted by $U_{mn} \overset{W}{\sim} V_{mn}$.

Double set sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are called asymptotically statistically equivalent of order η to multiple λ in the Wijsman sense if every $\xi > 0$ and each $y \in Y$,

$$P - \lim_{i,j\to\infty} \frac{1}{(ij)^\eta}\bigg|\bigg\{(m,n) : \ m \le i, n \le j, \ \Big|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\Big| \ge \xi\bigg\}\bigg| = 0$$

where $0 < \eta \le 1$ and denoted by $U_{mn} \overset{W_2(S_\lambda^\eta)}{\sim} V_{mn}$.

Double set sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are called asymptotically strongly Cesàro equivalent of order η to multiple λ in the Wijsman sense if each $y \in Y$,

$$P - \lim_{i,j\to\infty} \frac{1}{(ij)^\eta} \sum_{m,n=1,1}^{i,j} \Big|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\Big| = 0$$

where $0 < \eta \le 1$ and denoted by $U_{mn} \overset{W_2[C_\lambda^\eta]}{\sim} V_{mn}$.

3 Main Results

In this section firstly, by introducing the concepts of asymptotical lacunary statistical and asymptotical strong p-lacunary equivalence of order η $(0 < \eta \le 1)$ in the Wijsman sense for double set sequences, some properties of these concepts are examined and also the relationship between these concepts is mentioned.

Definition 1 Double set sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are asymptotically lacunary statistically equivalent of order η to multiple λ in the Wijsman sense if every $\xi > 0$ and each $y \in Y$,

$$P - \lim_{s,t\to\infty} \frac{1}{h_{st}^{\eta}} \left| \left\{ (m,n) \in I_{st} : \left| \rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda \right| \geq \xi \right\} \right| = 0$$

where $0 < \eta \leq 1$. We denote this in $U_{mn} \overset{W_2^{\theta}(S_{\lambda}^{\eta})}{\sim} V_{mn}$ format and simply called asymptotically lacunary statistically equivalent of order η in the Wijsman sense if $\lambda = 1$.

The set of all asymptotically lacunary statistically equivalent double set sequences of order η to multiple λ in the Wijsman sense is denoted by $W_2^{\theta}(S_{\lambda}^{\eta})$.

Example 1 Let $Y = \mathbb{R}^2$ and double set sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ be defined as following:

$$U_{mn} := \begin{cases} \{(a,b) \in \mathbb{R}^2 : a^2 + b^2 - mna = 0\} & ; \text{ if } (m,n) \in I_{st} \text{ and } mn \text{ is square integer,} \\ \{(-1,1)\} & ; \text{ otherwise.} \end{cases}$$

and

$$V_{mn} := \begin{cases} \{(a,b) \in \mathbb{R}^2 : a^2 + b^2 + mna = 0\} & ; \text{ if } (m,n) \in I_{st} \text{ and } mn \text{ is square integer,} \\ \{(-1,1)\} & ; \text{ otherwise.} \end{cases}$$

In this case, the double set sequences are asymptotically lacunary statistically equivalent of order η $(0 < \eta \leq 1)$ in the Wijsman sense.

Remark 1 For $\eta = 1$, the concept of asymptotical lacunary statistical equivalence of order η in the Wijsman sense coincides with the concept of asymptotical lacunary statistical equivalence in the Wijsman sense for double set sequences in [16].

Theorem 1 *If $0 < \eta \leq \mu \leq 1$, then $W_2^{\theta}(S_{\lambda}^{\eta}) \subseteq W_2^{\theta}(S_{\lambda}^{\mu})$ for every double lacunary sequence $\theta_2 = \{(j_s, k_t)\}$.*

Proof Suppose that $0 < \eta \leq \mu \leq 1$ and $U_{mn} \overset{W_2^{\theta}(S_{\lambda}^{\eta})}{\sim} V_{mn}$. For every $\xi > 0$ and each $y \in Y$, we have

$$\frac{1}{h_{st}^{\mu}} \left| \left\{ (m,n) \in I_{st} : \left| \rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda \right| \geq \xi \right\} \right| \leq \frac{1}{h_{st}^{\eta}} \left| \left\{ (m,n) \in I_{st} : \left| \rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda \right| \geq \xi \right\} \right|.$$

Hence, by our assumption, we get $U_{mn} \overset{W_2^{\theta}(S_{\lambda}^{\mu})}{\sim} V_{mn}$. Consequently, $W_2^{\theta}(S_{\theta}^{\eta}) \subseteq W_2^{\theta}(S_{\theta}^{\mu})$. □

If $\mu = 1$ is taken in Theorem 1, then the following corollary obtained.

Corollary 1 *Let $\eta \in (0,1]$. If double set sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are asymptotically lacunary statistically equivalent of order η to multiple λ in the Wijsman sense, then the double set sequences are asymptotically lacunary statistically equivalent of multiple λ in the Wijsman sense.*

Definition 2 Double set sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are asymptotically strongly p-lacunary equivalent of order η to multiple λ in the Wijsman sense if each $y \in Y$,

$$P - \lim_{s,t\to\infty} \frac{1}{h_{st}^{\eta}} \sum_{(m,n)\in I_{st}} \left| \rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda \right|^p = 0$$

where $0 < \eta \leq 1$ and $0 < p < \infty$. We denote this in $U_{mn} \overset{W_2^{\theta}[N_{\lambda}^{\eta}]^p}{\sim} V_{mn}$ format and simply called asymptotically strongly p-lacunary equivalent of order η in the Wijsman sense if $\lambda = 1$.

If $p = 1$, then double set sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are called asymptotically strongly lacunary equivalent of order η to multiple λ in the Wijsman sense, and we denote this $U_{mn} \overset{W_2^{\theta}[N_{\lambda}^{\eta}]}{\sim} V_{mn}$ format.

The set of all asymptotically strongly p-lacunary equivalent double set sequences of order η to multiple λ in the Wijsman sense is denoted by $W_2^{\theta}[N_{\lambda}^{\eta}]^p$.

Example 2 Let $Y = \mathbb{R}^2$ and double set sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ be defined as following:

$$U_{mn} := \begin{cases} \{(a,b) \in \mathbb{R}^2 : (a-1)^2 + b^2 = \frac{1}{mn}\} & ; \text{ if } (m,n) \in I_{st} \text{ and } mn \text{ is square integer,} \\ \{(0,1)\} & ; \text{ otherwise.} \end{cases}$$

and

$$V_{mn} := \begin{cases} \{(a,b) \in \mathbb{R}^2 : (a+1)^2 + b^2 = \frac{1}{mn}\} & ; \text{ if } (m,n) \in I_{st} \text{ and } mn \text{ is square integer,} \\ \{(0,1)\} & ; \text{ otherwise.} \end{cases}$$

In this case, the double set sequences are asymptotically strongly lacunary equivalent of order η $(0 < \eta \leq 1)$ in the Wijsman sense.

Remark 2 For $\eta = 1$, respectively, the concepts of asymptotical strong p-lacunary and asymptotical strong lacunary equivalence of order η in the Wijsman sense coincide with the concepts of asymptotical strong p-lacunary and asymptotical strong lacunary equivalence in the Wijsman sense for double set sequences in [16].

Theorem 2 *If* $0 < \eta \le \mu \le 1$, *then* $W_2^\theta[N_\lambda^\eta]^p \subseteq W_2^\theta[N_\lambda^\mu]^p$ *for every double lacunary sequence* $\theta_2 = \{(j_s, k_t)\}$.

Proof Suppose that $0 < \eta \le \mu \le 1$ and $U_{mn} \overset{W_2^\theta[N_\lambda^\eta]^p}{\sim} V_{mn}$. For each $y \in Y$, we have

$$\frac{1}{h_{st}^\mu} \sum_{(m,n)\in I_{st}} \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda\right|^p \le \frac{1}{h_{st}^\eta} \sum_{(m,n)\in I_{st}} \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda\right|^p.$$

Hence, by our assumption, we get $U_{mn} \overset{W_2^\theta[N_\lambda^\mu]^p}{\sim} V_{mn}$. Consequently, $W_2^\theta[N_\lambda^\eta]^p \subseteq W_2^\theta[N_\lambda^\mu]^p$. □

If $\mu = 1$ is taken in Theorem 2, then the following corollary is obtained.

Corollary 2 *Let* $\eta \in (0, 1]$. *If double set sequences* $\{U_{mn}\}$ *and* $\{V_{mn}\}$ *are asymptotically strongly p-lacunary equivalent of order* η *to multiple* λ *in the Wijsman sense, then the double set sequences are asymptotically strongly p-lacunary equivalent of multiple* λ *in the Wijsman sense.*

Now, we can state a theorem giving the relationship between $W_2^\theta[N_\lambda^\eta]^p$ and $W_2^\theta[N_\lambda^\eta]^q$, where $0 < \eta \le 1$ and $0 < p < q < \infty$.

Theorem 3 *Let* $0 < \eta \le 1$. *If* $0 < p < q < \infty$, *then* $W_2^\theta[N_\lambda^\eta]^q \subset W_2^\theta[N_\lambda^\eta]^p$ *for every double lacunary sequence* $\theta_2 = \{(j_s, k_t)\}$.

Proof Let $0 < \eta \le 1$ and $0 < p < q < \infty$. Also, we suppose that $U_{mn} \overset{W_2^\theta[N_\lambda^\eta]^q}{\sim} V_{mn}$. For each $y \in Y$, by Hölder inequality, we have

$$\frac{1}{h_{st}^\eta} \sum_{(m,n)\in I_{st}} \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda\right|^p < \frac{1}{h_{st}^\eta} \sum_{(m,n)\in I_{st}} \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda\right|^q.$$

Hence, by our assumption, we get $U_{mn} \overset{W_2^\theta[N_\lambda^\eta]^p}{\sim} V_{mn}$. Consequently, $W_2^\theta[N_\lambda^\eta]^q \subset W_2^\theta[N_\lambda^\eta]^p$. □

Theorem 4 *If double set sequences* $\{U_{mn}\}$ *and* $\{V_{mn}\}$ *are asymptotically strongly p-lacunary equivalent of order* η *to multiple* λ *in the Wijsman sense, then the double set sequences are asymptotically lacunary statistically equivalent of order* μ *to multiple* λ *in the Wijsman sense, where* $0 < \eta \le \mu \le 1$ *and* $0 < p < \infty$.

Proof Let $0 < \eta \le \mu \le 1$ and $0 < p < \infty$. Also, we suppose that double sequences $\{U_{mn}\}$ and $\{V_{mn}\}$ are asymptotically strongly p-lacunary equivalent of order η to multiple λ in the Wijsman sense. For every $\xi > 0$ and each $y \in Y$, we have

$$\begin{aligned}\sum_{(m,n)\in I_{st}} \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda\right|^p &= \sum_{\substack{(m,n)\in I_{st}\\ \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right)-\lambda\right|\ge\xi}} \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda\right|^p \\ &\quad + \sum_{\substack{(m,n)\in I_{st}\\ \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right)-\lambda\right|<\xi}} \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda\right|^p \\ &\ge \sum_{\substack{(m,n)\in I_{st}\\ \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right)-\lambda\right|\ge\xi}} \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda\right|^p \\ &\ge \xi^p \left|\left\{(m,n) \in I_{st} : \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda\right| \ge \xi\right\}\right|\end{aligned}$$

and so

$$\begin{aligned}&\frac{1}{h_{st}^\eta} \sum_{(m,n)\in I_{st}} \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda\right|^p \\ &\ge \frac{\xi^p}{h_{st}^\eta} \left|\left\{(m,n) \in I_{st} : \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda\right| \ge \xi\right\}\right| \\ &\ge \frac{\xi^p}{h_{st}^\mu} \left|\left\{(m,n) \in I_{st} : \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda\right| \ge \xi\right\}\right|.\end{aligned}$$

Hence, by our assumption, we get that the double sequences are asymptotically lacunary statistically equivalent of order μ to multiple λ in the Wijsman sense. □

If $\mu = \eta$ is taken in Theorem 4, then the following corollary is obtained.

Corollary 3 *Let* $\eta \in (0, 1]$ *and* $0 < p < \infty$. *If double set sequences* $\{U_{mn}\}$ *and* $\{V_{mn}\}$ *are asymptotically strongly p-lacunary equivalent of order* η *to multiple* λ *in the Wijsman sense, then the double set sequences are asymptotically lacunary statistically equivalent of order* η *to multiple* λ *in the Wijsman sense.*

Now, secondly, the relationships between the new concepts that introduced above and the asymptotical equivalence concepts previously given for double set sequences are investigated.

Theorem 5 *If* $\liminf_s q_s^\eta > 1$ *and* $\liminf_t \overline{q}_t^\eta > 1$ *where* $0 < \eta \le 1$, *then*

$$U_{mn} \overset{W_2(S_\lambda^\eta)}{\sim} V_{mn} \Rightarrow U_{mn} \overset{W_2^\theta(S_\lambda^\eta)}{\sim} V_{mn}.$$

Proof Let $0 < \eta \le 1$. Also, we suppose that $\liminf_s q_s^\eta > 1$ and $\liminf_t \overline{q}_t^\eta > 1$. Then, there exist $\alpha, \beta > 0$ such that $q_s^\eta \ge 1 + \alpha$ and $\overline{q}_t^\eta \ge 1 + \beta$ for all s, t, which implies that

$$\frac{h_{st}^\eta}{\ell_{st}^\eta} \ge \frac{\alpha\beta}{(1+\alpha)(1+\beta)}.$$

For every $\xi > 0$ and each $y \in Y$, we have

$$\begin{aligned}
&\frac{1}{\ell_{st}^\eta}\left|\left\{(m,n) : m \le j_s, n \le k_t, \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda\right| \ge \xi\right\}\right| \\
&\quad \ge \frac{1}{\ell_{st}^\eta}\left|\left\{(m,n) \in I_{st} : \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda\right| \ge \xi\right\}\right| \\
&\quad = \frac{h_{st}^\eta}{\ell_{st}^\eta}\frac{1}{h_{st}^\eta}\left|\left\{(m,n) \in I_{st} : \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda\right| \ge \xi\right\}\right| \\
&\quad \ge \frac{\alpha\beta}{(1+\alpha)(1+\beta)}\frac{1}{h_{st}^\eta}\left|\left\{(m,n) \in I_{st} : \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda\right| \ge \xi\right\}\right|.
\end{aligned}$$

If $U_{mn} \overset{W_2(S_\lambda^\eta)}{\sim} V_{mn}$, then for each $y \in Y$ the term on the left side of the above inequality convergent to 0 and this implies that

$$\frac{1}{h_{st}^\eta}\left|\left\{(m,n) \in I_{st} : \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda\right| \ge \xi\right\}\right| \to 0.$$

Thus, we get $U_{mn} \overset{W_2^\theta(S_\lambda^\eta)}{\sim} V_{mn}$. □

Theorem 6 *If* $\limsup_s q_s^\eta < \infty$ *and* $\limsup_t \overline{q}_t^\eta < \infty$, *then*

$$U_{mn} \overset{W_2^\theta(S_\lambda^\eta)}{\sim} V_{mn} \Rightarrow U_{mn} \overset{W_2(S_\lambda^\eta)}{\sim} V_{mn}$$

where $0 < \eta \le 1$.

Proof Let $0 < \eta \le 1$ be given and suppose that $\limsup_s q_s^\eta < \infty$ and $\limsup_t \overline{q}_t^\eta < \infty$. Then, there exist $M, N > 0$ such that $q_s^\eta < M$ and $\overline{q}_t^\eta < N$ for all s, t. Also, we suppose that $U_{mn} \overset{W_2^\theta(S_\lambda^\eta)}{\sim} V_{mn}$ and let $\xi > 0$ and

$$\kappa_{st} := \left|\left\{(m,n) \in I_{st} : \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda\right| \ge \xi\right\}\right|.$$

Then, there exist $s_0, t_0 \in \mathbb{N}$ such that for all $s \ge s_0, t \ge t_0$

$$\frac{\kappa_{st}}{h_{st}^\eta} < \xi.$$

Now let

$$\gamma := \max\{\kappa_{st} : 1 \le s \le s_0,\ 1 \le t \le t_0\}$$

and let i and j be integers satisfying $j_{s-1} < i \le j_s$ and $k_{t-1} < j \le k_t$. Then, we have

$$\begin{aligned}
&\frac{1}{(ij)^\eta}\left|\left\{(m,n) : m \le i, n \le j, \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda\right| \ge \xi\right\}\right| \\
&\le \frac{1}{\ell_{(s-1)(t-1)}^\eta}\left|\left\{(m,n) : m \le j_s, n \le k_t, \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda\right| \ge \xi\right\}\right| \\
&= \frac{1}{\ell_{(s-1)(t-1)}^\eta}\left\{\kappa_{11} + \kappa_{12} + \kappa_{21} + \kappa_{22} + \cdots + \kappa_{s_0t_0} + \cdots + \kappa_{st}\right\} \\
&\le \frac{s_0t_0}{\ell_{(s-1)(t-1)}^\eta}\left(\max_{\substack{1\le m\le s_0\\ 1\le n\le t_0}}\{\kappa_{mn}\}\right) + \frac{1}{\ell_{(s-1)(t-1)}^\eta}\left\{h_{s_0(t_0+1)}^\eta\frac{\kappa_{s_0(t_0+1)}}{h_{s_0(t_0+1)}^\eta}\right. \\
&\quad \left. + h_{(s_0+1)t_0}^\eta\frac{\kappa_{(s_0+1)t_0}}{h_{(s_0+1)t_0}^\eta} + h_{(s_0+1)(t_0+1)}^\eta\frac{\kappa_{(s_0+1)(t_0+1)}}{h_{(s_0+1)(t_0+1)}^\eta} + \cdots + h_{st}^\eta\frac{\kappa_{st}}{h_{st}^\eta}\right\} \\
&\le \frac{s_0t_0\gamma}{\ell_{(s-1)(t-1)}^\eta} + \frac{1}{\ell_{(s-1)(t-1)}^\eta}\left(\sup_{\substack{s\ge s_0\\ t\ge t_0}}\frac{\kappa_{st}}{h_{st}^\eta}\right)\left(\sum_{m,n\ge s_0,t_0}^{s,t} h_{mn}^\eta\right) \\
&\le \frac{s_0t_0\gamma}{\ell_{(s-1)(t-1)}^\eta} + \xi q_s^\eta \overline{q}_t^\eta \\
&\le \frac{s_0t_0\gamma}{\ell_{(s-1)(t-1)}^\eta} + \xi MN.
\end{aligned}$$

Since $j_{s-1}, k_{t-1} \to \infty$ as $i, j \to \infty$, it follows that for each $y \in Y$

$$\frac{1}{(ij)^\eta}\left|\left\{(m,n) : m \le i, n \le j, \left|\rho_y\left(\frac{U_{mn}}{V_{mn}}\right) - \lambda\right| \ge \xi\right\}\right| \to 0.$$

Thus, we get $U_{mn} \overset{W_2(S_\lambda^\eta)}{\sim} V_{mn}$. □

Theorem 7 *If* $1 < \liminf_s q_s^\eta \le \limsup_s q_s^\eta < \infty$ and $1 < \liminf_t \overline{q}_t^\eta \le \limsup_t \overline{q}_t^\eta < \infty$ *where* $0 < \eta \le 1$, *then*

$$U_{mn} \overset{W_2^\theta(S_\lambda^\eta)}{\sim} V_{mn} \Leftrightarrow U_{mn} \overset{W_2(S_\lambda^\eta)}{\sim} V_{mn}.$$

Proof This can be obtained from Theorem 5 and Theorem 6, immediately. □

Theorem 8 *If* $\liminf_s q_s^\eta > 1$ *and* $\liminf_t \overline{q}_t^\eta > 1$ *where* $0 < \eta \le 1$, *then*

$$U_{mn} \overset{W_2[C_\lambda^\eta]}{\sim} V_{mn} \Rightarrow U_{mn} \overset{W_2^\theta[N_\lambda^\eta]}{\sim} V_{mn}.$$

Proof Let $0 < \eta \le 1$. Also, we suppose that $\liminf_s q_s^\eta > 1$ and $\liminf_t \overline{q}_t^\eta > 1$. Then, there exist $\alpha, \beta > 0$ such that $q_s^\eta \ge 1 + \alpha$ and $\overline{q}_t^\eta \ge 1 + \beta$ for sufficiently large s, t, which implies that

$$\frac{\ell_{st}^\eta}{h_{st}^\eta} \le \frac{(1+\alpha)(1+\beta)}{\alpha\beta} \quad \text{and} \quad \frac{\ell_{(s-1)(t-1)}^\eta}{h_{st}^\eta} \le \frac{1}{\alpha\beta}.$$

For each $y \in Y$, we have

$$\begin{aligned}
\frac{1}{h^{\eta}_{st}} \sum_{(m,n)\in I_{st}} \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right|
&= \frac{1}{h^{\eta}_{st}} \sum_{m,n=1,1}^{j_s,k_t} \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| \\
&\quad - \frac{1}{h^{\eta}_{st}} \sum_{m,n=1,1}^{j_{s-1},k_{t-1}} \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| \\
&= \frac{\ell^{\eta}_{st}}{h^{\eta}_{st}} \left(\frac{1}{\ell^{\eta}_{st}} \sum_{m,n=1,1}^{j_s,k_t} \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right|\right) \\
&\quad - \frac{\ell^{\eta}_{(s-1)(t-1)}}{h^{\eta}_{st}} \left(\frac{1}{\ell^{\eta}_{(s-1)(t-1)}} \sum_{m,n=1,1}^{j_{s-1},k_{t-1}} \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right|\right).
\end{aligned}$$

If $U_{mn} \overset{W_2[C^{\eta}_{\lambda}]}{\sim} V_{mn}$, then for each $y \in Y$ the following limits are hold

$$\frac{1}{\ell^{\eta}_{st}} \sum_{m,n=1,1}^{j_s,k_t} \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| \to 0$$

and

$$\frac{1}{\ell^{\eta}_{(s-1)(t-1)}} \sum_{m,n=1,1}^{j_{s-1},k_{t-1}} \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| \to 0.$$

Thus, when the above equality is considered, for each $y \in Y$ we get

$$\frac{1}{h^{\eta}_{st}} \sum_{(m,n)\in I_{st}} \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| \to 0,$$

that is, $U_{mn} \overset{W_2^{\theta}[N^{\eta}_{\lambda}]}{\sim} V_{mn}$. □

Theorem 9 *If* $\limsup_s q^{\eta}_s < \infty$ *and* $\limsup_t \overline{q}^{\eta}_t < \infty$, *then*

$$U_{mn} \overset{W_2^{\theta}[N^{\eta}_{\lambda}]}{\sim} V_{mn} \Rightarrow U_{mn} \overset{W_2[C^{\eta}_{\lambda}]}{\sim} V_{mn}$$

where $0 < \eta \leq 1$.

Proof Let $0 < \eta \leq 1$ be given and suppose that $\limsup_s q^{\eta}_s < \infty$ and $\limsup_t \overline{q}^{\eta}_t < \infty$. Then, there exist $M, N > 0$ such that $q^{\eta}_s < M$ and $\overline{q}^{\eta}_t < N$ for all s, t. Also, we suppose that $U_{mn} \overset{W_2^{\theta}[N^{\eta}_{\lambda}]}{\sim} V_{mn}$. Then, for given $\xi > 0$ and each $y \in Y$, we can find $s_0, t_0 > 0$ and $\vartheta > 0$ such that

$$\sup_{m\geq s_0, n\geq t_0} \tau_{mn} < \xi \quad \text{and} \quad \tau_{mn} < \vartheta \quad \text{for all} \quad m,n = 1,2,\ldots$$

where

$$\tau_{st} := \frac{1}{h^{\eta}_{st}} \sum_{(m,n)\in I_{st}} \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right|.$$

If i and j are any integers satisfying $j_{s-1} < i \leq j_s$ and $k_{t-1} < j \leq k_t$ where $s > s_0$ and $t > t_0$, then for each $y \in Y$ we have

$$\begin{aligned}
\frac{1}{(ij)^{\eta}} \sum_{m,n=1,1}^{i,j} \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right|
&\leq \frac{1}{\ell^{\eta}_{(s-1)(t-1)}} \sum_{m,n=1,1}^{j_s,k_t} \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| \\
&= \frac{1}{\ell^{\eta}_{(s-1)(t-1)}} \Bigg(\sum_{I_{11}} \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| + \sum_{I_{12}} \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| \\
&\quad + \sum_{I_{21}} \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| + \sum_{I_{22}} \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| \\
&\quad + \cdots + \sum_{I_{st}} \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| \Bigg) \\
&= \frac{h^{\eta}_{11}}{\ell^{\eta}_{(s-1)(t-1)}} \tau_{11} + \frac{h^{\eta}_{12}}{\ell^{\eta}_{(s-1)(t-1)}} \tau_{12} + \frac{h^{\eta}_{21}}{\ell^{\eta}_{(s-1)(t-1)}} \tau_{21} \\
&\quad + \frac{h^{\eta}_{22}}{\ell^{\eta}_{(s-1)(t-1)}} \tau_{22} + \cdots + \frac{h^{\eta}_{st}}{\ell^{\eta}_{(s-1)(t-1)}} \tau_{st}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{m,n=1,1}^{s_0,t_0} \frac{h^{\eta}_{mn}}{\ell^{\eta}_{(s-1)(t-1)}} \tau_{mn} + \sum_{m,n\geq s_0,t_0}^{s,t} \frac{h^{\eta}_{mn}}{\ell^{\eta}_{(s-1)(t-1)}} \tau_{mn} \\
&\leq \Big(\sup_{\substack{1\leq m\leq s_0\\ 1\leq n\leq t_0}} \tau_{mn}\Big) \sum_{m,n=1,1}^{s_0,t_0} \frac{h^{\eta}_{mn}}{\ell^{\eta}_{(s-1)(t-1)}} + \Big(\sup_{\substack{m\geq s_0\\ n\geq t_0}} \tau_{mn}\Big) \sum_{m,n\geq s_0,t_0}^{s,t} \frac{h^{\eta}_{mn}}{\ell^{\eta}_{(s-1)(t-1)}} \\
&\leq \frac{\vartheta\, \ell^{\eta}_{s_0 t_0}}{\ell^{\eta}_{(s-1)(t-1)}} + \xi\, q^{\eta}_s\, \overline{q}^{\eta}_t \\
&\leq \frac{\vartheta\, \ell^{\eta}_{s_0 t_0}}{\ell^{\eta}_{(s-1)(t-1)}} + \xi\, M N.
\end{aligned}$$

Since $j_{s-1}, k_{t-1} \to \infty$ as $i, j \to \infty$, it follows that for each $y \in Y$

$$\frac{1}{(ij)^{\eta}} \sum_{m,n=1,1}^{i,j} \left|\rho_y\Big(\frac{U_{mn}}{V_{mn}}\Big) - \lambda\right| \to 0,$$

thus, we get $U_{mn} \overset{W_2[C^{\eta}_{\lambda}]}{\sim} V_{mn}$. □

Theorem 10 *If* $1 < \liminf_s q^{\eta}_s \leq \limsup_s q^{\eta}_s < \infty$ and $1 < \liminf_t \overline{q}^{\eta}_t \leq \limsup_t \overline{q}^{\eta}_t < \infty$ *where* $0 < \eta \leq 1$, *then*

$$U_{mn} \overset{W_2^{\theta}[N^{\eta}_{\lambda}]}{\sim} V_{mn} \Leftrightarrow U_{mn} \overset{W_2[C^{\eta}_{\lambda}]}{\sim} V_{mn}.$$

Proof This can be obtained from Theorem 8 and Theorem 9, immediately. □

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