Miskolc Mathematical Notes
Vol. 24 (2023), No. 1, pp. 209–226
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.3644

DYNAMICAL COMPLEXITIES IN A DISCRETE-TIME PREDATOR-PREY SYSTEM AS CONSEQUENCES OF THE WEAK ALLEE EFFECT ON PREY

FİGEN KANGALGİL AND SEVAL IŞIK

Received 08 January, 2021

Abstract. In this paper, a two dimensional discrete-time predator-prey system with weak Allee effect, affecting the prey population, is considered. The existence of the positive fixed points of the system and topological classification of coexistence positive fixed point are examined. By using the bifurcation theory, it is shown that the discrete-time predator-prey system with Allee effect undergoes flip and Neimark-Sacker bifurcations depending on the parameter a. The parametric conditions for existence and direction of bifurcations are investigated. Numerical simulations including bifurcation diagrams, phase portraits and maximum Lyapunov exponents of the system are performed to validate analytical results. The computation of the maximum Lyapunov exponents confirm the presence of chaotic behaviour in the considered system. Finally,the OGY feedback control method is implemented to stabilize chaos existing in the system.

2010 *Mathematics Subject Classification:* 39A33; 37G35; 39A30

Keywords: predator-prey system, fixed point, stability, flip bifurcation, Neimark-Sacker bifurcation, chaotic behavior, OGY feedback control method

1. INTRODUCTION

The predator-prey systems which show interactions between two species are very dominant phenomenon in bio-mathematical literature. The first and the simplest of predator prey system is formulated by Lotka-Volterra [2, 22]. Since this system has neglected many real situations, significant changes in system by adding ecological factors such as functional responses, emigration, immigration, time delays, diffusion, Allee effect, etc. have been made by many researchers. In recent times, there have been increasing interests on study of the complex dynamical behaviors of predator prey systems [1, 10–19, 24].

The Allee effect can appear thanks to widespread range of biological phenomena, such as reduced anti-predator vigilance, genetic trends, mating difficulty and feeding deficiency at low population densities. The Allee effect introduced by Warder Clyde Allee can be classified into two types as a weak Allee effect and strong Allee effect to the basis of per capita growth rate at low density. The weak Allee effect represents that the unit growth rate is smaller when the population is small; but it will not be negative [23]. The strong Allee effect is called that the sign of the per capita growth rate in the limit of low density is negative. The strong Allee effect which is known as the Allee effect have been investigated intensively, but the researches on the weak Allee effect are relatively rare. Nowadays, it is well known that the Allee effect plays an important and fundamental role to understand the biological systems. So, many scholars have payed attention on Allee effect since it exhibits a rich dynamics [5, 6, 8, 9, 12, 20, 21, 23].

In [3], the author has investigated the following discrete-time predator-prey model

$$X_{t+1} = aX_t(1-X_t) - X_tY_t \qquad Y_{t+1} = \frac{1}{\beta}X_tY_t \tag{1.1}$$

where X_t and Y_t represent prey and predator population respectively with t th generation. The parameters a and β are positive real numbers. The authors have analyzed stability and Neimark-Sacker bifurcation of the discrete-time predator-prey system. In this paper, our aim is to discuss the dynamics of the following modified discrete-time predator-prey system (1.1) with adding weak Allee effect on prey population:

$$X_{t+1} = aX_t(1-X_t)(X_t-\theta) - X_tY_t \qquad Y_{t+1} = \frac{1}{\beta}X_tY_t \tag{1.2}$$

In the system (1.2), the term $aX_t(1-X_t)(X_t-\theta)$ represents the growth rate of the prey population with $-1 < \theta < 1$. In the above the system (1.2) if $0 < \theta < 1$ or $-1 < \theta < 0$, the Allee effect is considered to be strong or weak, respectively. In this study, we have investigated only type of weak Allee of discrete-time predator prey system on prey population [8, 12, 20]. We studied type of strong Allee effect of the presented model on prey population in our another paper.

The aim of the present paper is to compare dynamical behaviors of the discrete-time predator- prey system with and without Allee effect and discuss richer and complex behaviors of the system (1.2) with weak Allee effect on prey population. Thereby we will investigate stability, flip and Neimark-Sacker bifurcations, and chaos control analyses of the system (1.2) at the coexistence fixed point. In Section 2, the conditions of the existence and stability of the fixed points are discussed. In Section 3, flip and Neimark-Sacker bifurcation analysis are studied by choosing a parameter as a bifurcation parameter. Furthermore, directions of both flip bifurcation and Neimark-Sacker bifurcation are obtained by using normal form theory [7]. In Section 4, OGY feedback control method is implemented for chaos control due to emergence of Neimark-Sacker bifurcation. Finally, some numerical simulations are carried out both to illustrate the analytic finding and to display new complex dynamical behaviors in Section 5.

2. THE FIXED POINTS, THEIR EXISTENCE AND STABILITY ANALYSIS

In this section, we present the existence and stability conditions of the fixed points of the system (1.2). So, we can obtain the following Lemma 1 about the existence of the fixed points of the considered system:

Lemma 1. *For the system* (1.2)*, the following statements hold true:*

(1) The trivial fixed point $E_0 = (0,0)$ *is always feasible.*

(2) The axial fixed points $E_1 = \left(\frac{a(\theta+1)\pm\sqrt{a^2(\theta-1)^2-4a}}{2a}, 0\right)$ *are feasible if* $-\frac{1}{a} < \theta < 0$.

(3) The unique positive fixed point $E_2 = (\beta, a(1-\beta)(\beta-\theta)-1)$ *is feasible if* $a > \frac{1}{(1-\beta)(\beta-\theta)}$ *and* $0 < \beta < 1$.

Proof. To find the fixed points of the system (1.2), we can solve Equation (2.1)

$$X^* = aX^*(1-X^*)(X^*-\theta) - X^*Y^*, \qquad Y^* = \frac{1}{\beta}X^*Y^*. \tag{2.1}$$

We first observe that for $X^* = 0$, we have the extinction fixed point $(0,0)$ for any values of parameters For $X^* \neq 0$ and $Y^* = 0$, we have the solution

$$E_1 = \left(\frac{a(\theta+1)\pm\sqrt{a^2(\theta-1)^2-4a}}{2a}, 0\right).$$

The fixed point E_1 has only biological meaning if $-\frac{1}{a} < \theta < 0$. For $X^* \neq 0$ and $Y^* \neq 0$, from the second equation of the system (1.2), we obtain $X^* = \beta$. Substituting $X^* = \beta$ into the function (2.1); it is obtained $Y^* = a(1-\beta)(\beta-\theta) - 1 > 0$ if $a > \frac{1}{(1-\beta)(\beta-\theta)}$ and $0 < \beta < 1$.

□

Now, we analyze the stability of the coexistence positive fixed point E_2 of the system (1.2) only. The Jacobian matrix of the system (1.2) evaluated at the unique positive fixed point E_2 is

$$J(E_2) = \begin{pmatrix} a\beta - 2a\beta^2 + a\beta\theta + 1 & -\beta \\ \frac{a\beta - a\beta^2 - a\theta + a\beta\theta - 1}{\beta} & 1 \end{pmatrix}.$$

Then characteristic polynomial of $J(E_2)$ is

$$F(\lambda) = \lambda^2 + \left(-2 - a\beta + 2a\beta^2 - a\beta\theta\right)\lambda + 2a\beta - 3a\beta^2 + 2a\beta\theta - a\theta. \tag{2.2}$$

In order to investigate the dynamics of a unique positive fixed point E_2 of the system (1.2), we give the following Definition 1 and Lemma 2 [3, 10, 24]:

Definition 1. Assume that λ_1 and λ_2 be roots of the characteristic polynomial at the positive fixed point (x,y). An fixed point (x,y) is called

(1) sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, and it is locally asymptotically stable,

(2) source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, and it is locally unstable,
(3) saddle if $|\lambda_1| > 1$ and $|\lambda_2| < 1$ or $|\lambda_1| < 1$ and $|\lambda_2| > 1$,
(4) non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

Lemma 2. *Let $F(\lambda) = \lambda^2 + B\lambda + C$, where B and C are two real constants and let $F(1) > 0$. Suppose λ_1 and λ_2 are two roots of $F(\lambda) = 0$. Then the following statements hold true:*

(1) $|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $F(-1) > 0$ and $C < 1$,
(2) $|\lambda_1| > 1$ and $|\lambda_2| > 1$ if and only if $F(-1) > 0$ and $C > 1$,
(3) $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$) if and only if $F(-1) < 0$,
(4) λ_1 and λ_2 are a pair of conjugate complex roots and $|\lambda_1| = |\lambda_2| = 1$ if and only if $B^2 - 4C < 0$ and $C = 1$,
(5) $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if $F(-1) = 0$ and $C \neq \pm 1$.

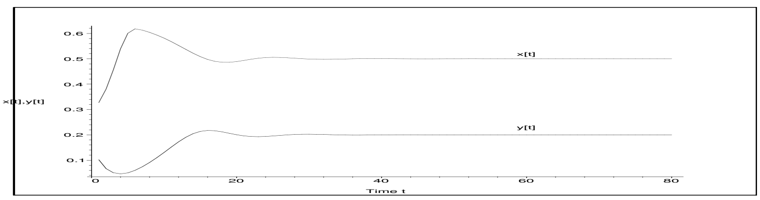
Now we will discuss the topological classification of the unique positive fixed point E_2 of the system (1.2) and we will apply Lemma 2 to prove the following Lemma 3. From Lemma 2, we have $F(1) = a\beta - a\theta - a\beta^2 + a\beta\theta - 1 > 0$. Since $a > \frac{1}{(1-\beta)(\beta-\theta)}$ and $0 < \beta < 1$, $F\ (1) > 0$. $F(-1) = Ka + 3$ and $F(0) = Sa$ where $K = 3\beta - 5\beta^2 + 3\beta\theta - \theta$, $S = -3\beta^2 + 2\beta\theta - \theta + 2\beta$.

Lemma 3. *Assume that $a > a_3$ and $0 < \beta < 1$ then for unique positive fixed point E_2 of the system (1.2) the following holds true.*

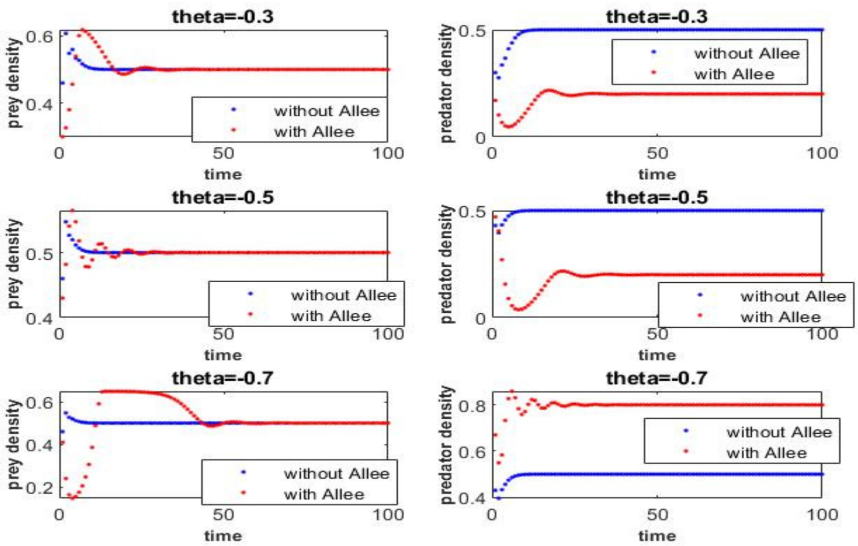
(1) E_2 is a sink fixed point if the following conditions hold:
  (i) $K > 0$, $S > 0$ and $\max\{a_1, a_3\} < a < a_2$,
  (ii) $K < 0$, $S < 0$ and $\max\{a_2, a_3\} < a < a_1$,
  (iii) $K < 0$, $S > 0$ and $a_3 < a < \min\{a_1, a_2\}$,
(2) E_2 is a source fixed point if the following conditions hold:
  (i) $K > 0$, $S > 0$ and $a > \max\{a_1, a_2, a_3\}$,
  (ii) $K < 0$, $S < 0$ and $a_3 < a < \min\{a_1, a_2\}$,
  (iii) $K < 0$, $S > 0$ and $\max\{a_2, a_3\} < a < a_1$,
(3) E_2 is a saddle fixed point if the following conditions hold:
  (i) $K > 0$, $S > 0$ and $a_3 < a < a_1$,
  (ii) $K < 0$, $S < 0$ and $\min\{a_1, a_3\} < a$,
  (iii) $K < 0$, $S > 0$ and $a > \max\{a_1, a_3\}$,
(4) The roots of Equation (2.2) are complex with modules one if and only if $a = a_2$, $a > a_3$, $-1 < \theta < 0$ and $0 < \beta < 1$, $K > 0$, $S > 0$,
(5) E_2 non-hyperbolic fixed point if the following conditions hold: $a = a_1$ and $a \neq \pm\frac{1}{S}$ where $a_1 = \frac{-3}{K}$, $a_2 = \frac{1}{S}$, $a_3 = \frac{1}{(1-\beta)(\beta-\theta)}$, $K = 3\beta - 5\beta^2 + 3\beta\theta - \theta$, $S = -3\beta^2 + 2\beta\theta - \theta + 2\beta$.

Example 1. For the parameter values $a = 3, \beta = 0.5, \theta = -0.3$ and initial condition $(X_0, Y_0) = (0.3, 0.17)$, the positive fixed point of the system (1.2) is obtained as $(X^*, Y^*) = (0.5, 0.2)$. Figure 1a is shown that the fixed point (X^*, Y^*) of the system

(1.2) is local asymptotically stable where $x(t)$ prey and $y(t)$ predator population represent, respectively. For showing the impact of the Allee effect on the stability of the model, Figure 1b is plotted for different values of θ. From Figure 1b, it is clearly that if $-1 < \theta < -0.5$, then the predator density of the model subject to Allee effect increases, otherwise $(-0.5 \leq \theta < 0)$ it decreases.



(A) A stable fixed point for the system (1.2).



(B) Time-population density graphs of systems with and without Allee effect for different values of θ.

FIGURE 1

3. BIFURCATION ANALYSIS

In this section, by choosing the parameter a as a bifurcation parameter, we analyze the existence conditions and directions of both Neimark-Sacker bifurcation and flip bifurcation at unique positive fixed point E_2 of the system (1.2).

3.1. *Neimark-Sacker Bifurcation at the Fixed Point E_2*

From Lemma 3 (4), we can write NSB_{E_2} as follow:

$$NSB_{E_2} = \{(a,\beta,\theta) : a > a_3,\ a_{NS} = a_2,\ -1 < \theta < 0 \text{ and } 0 < \beta < 1,\ K > 0,\ S > 0\}.$$

The two eigenvalues of the Jacobian matrix $J(E_2)$ are complex conjugate pairs with modulus one, if the parameters change in small neighborhood of NSB_{E_2}. This means that Neimark-Sacker bifurcation will occur at the fixed point E_2. The eigenvalues of the system (1.2) under these conditions are given by

$$\lambda,\bar{\lambda} = \frac{M \pm i\sqrt{Q}}{-2S},$$

where

$$M = 8\beta^2 - 5(1+\beta\theta) + 2\theta,$$
$$Q = -28\beta^4 + 32(1+\theta)\beta^3 + (-9 - 26\theta - 9\theta^2)\beta^2 + 4\theta(1+\theta)\beta.$$

One gets that $|\lambda| = \left|\bar{\lambda}\right| = 1$. The transversality condition imply that

$$\left.\frac{d\,|\lambda_i(a)|}{da}\right|_{a=a_{NS}} = -3\beta^2 + 2\beta\theta - \theta + 2\beta \neq 0 \quad i = 1,2.$$

The nonresonance condition $trJ_{E_2}(a_{NS}) \neq 0,-1$ namely

$$a_2 \neq 2R, 3R \tag{3.1}$$

where $R = \frac{1}{\beta(2\beta-\theta-1)}$. Then, we have $\lambda^k(a_{NS}) \neq 1$ for $k = 1,2,3,4$.

Assume that $q,p \in \mathbb{C}^2$ are two eigenvectors of $J(NSB_{E_2})$ and transposed matrix $J^T(NSB_{E_2})$ corresponding to λ and $\bar{\lambda}$, respectively. We have

$$q \sim \left(1, \frac{F-2G}{2\beta} - i\frac{\sqrt{Q}}{2\beta}\right)^T \text{ and } p \sim \left(-\frac{F-2G}{2\beta} + i\frac{\sqrt{Q}}{2\beta}, 1\right)^T.$$

where $F = a\beta(\theta+1)$, $G = a\beta^2$, $Q = 4(F - G + 1 + a\theta) - (F-2G)^2$. To achieve the normalization $< p,q >= 1$ where $<,>$ means the standard scalar product in C^2, we can take the normalized eigenvectors as

$$q = \left(1, \frac{F-2G}{2\beta} - i\frac{\sqrt{Q}}{2\beta}\right)^T,\ p = \left(\frac{1}{2} + \frac{i(F-2G)}{2\sqrt{Q}}, \frac{-i\beta}{\sqrt{Q}}\right)^T.$$

Let $x_t = X_t - x^*$, $y_t = Y_t - y^*$ and $J(E_*) = J(x^*,y^*)$. We transform the fixed point E_2 of the system (1.2) into the origin $(0,0)$. From Taylor expansion, the system (1.2) convert to

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} \to J(E_*)\begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} F_1(x_t,y_t) \\ F_2(x_t,y_t) \end{pmatrix}, \tag{3.2}$$

where $F_1(x_t,y_t) = -ax_t^3 + (a\theta - 3a\beta + a)x_t^2 - y_t x_t + O(X_t^4)$, $F_2(x_t,y_t) = \frac{x_t y_t}{\beta} + O(X_t^4)$, $X_t = (x_t, y_t)^T$.

The system (3.2) can be expressed as

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = J(E_*)\begin{pmatrix} x_t \\ y_t \end{pmatrix} + \frac{1}{2}B(x_t,x_t) + \frac{1}{6}C(x_t,x_t,x_t) + O(X_t^4),$$

where $B(x,y) = \begin{pmatrix} B_1(x,y) \\ B_2(x,y) \end{pmatrix}$ and $C(x,y,u) = \begin{pmatrix} C_1(x,y,u) \\ C_2(x,y,u) \end{pmatrix}$ are symmetric multi-linear vector functions of $x,y,u \in R^2$. These functions are defined by as follows:

$$B_1(x,y) = \sum_{j,k=1}^{2} \frac{\partial^2 F_1}{\partial \xi_j \partial \xi_k}\bigg|_{\xi=0} x_j y_k = (2a\theta - 6a\beta + 2a)x_1y_1 - x_1y_2 - x_2y_1,$$

$$B_2(x,y) = \sum_{j,k=1}^{2} \frac{\partial^2 F_2}{\partial \xi_j \partial \xi_k}\bigg|_{\xi=0} x_j y_k = \frac{x_1y_2 + x_2y_1}{\beta},$$

$$C_1(x,y,u) = \sum_{j,k,l=1}^{2} \frac{\partial^3 F_1}{\partial \xi_j \partial \xi_k \partial \xi_l}\bigg|_{\xi=0} x_j y_k u_l = 6ax_1y_1u_1,$$

$$C_2(x,y,u) = \sum_{j,k,l=1}^{2} \frac{\partial^3 F_2}{\partial \xi_j \partial \xi_k \partial \xi_l}\bigg|_{\xi=0} x_j y_k u_l = 0.$$

Now, we decompose vector $X \in \mathbb{R}^2$ as $X = zq + \overline{zq}$, for r near to a_{NS} and $z \in \mathbb{C}$. The explicit formula of z is determined as $z = < p, X >$. The system (3.2) can be transformed for all sufficiently small $|a|$ into the form $z \to \lambda(a)z + g(z,\bar{z},a)$ where $\lambda(a) = (1+\varphi(a))e^{i\arctan(a)}$ with $\varphi(a_{NS}) = 0$ and $g(z,\bar{z},a)$ is smooth complex-valued function. After Taylor expression of g with respect to $(z,\bar{z})$, we obtain

$$g(z,\bar{z},a) = \sum_{k+l\geq 2} \frac{1}{k!l!} g_{kl}(a) z^k \bar{z}^l, \text{with } g_{kl} \in k,l = 0,1,\ldots.$$

By symmetric multi-linear vector functions, the Taylor coefficients g_{kl} can be expressed by the formulas

$$g_{20}(a_{NS}) = < p, B(q,q) >,$$
$$g_{11}(a_{NS}) = < p, B(q,\bar{q}) >,$$
$$g_{02}(a_{NS}) = < p, B(\bar{q},\bar{q}) >,$$
$$g_{21}(a_{NS}) = < p, C(q,q,\bar{q}) > .$$

The coefficient $\beta_2(a_{NS})$, which determines the direction of the appearance of the invariant curve in a generic system exhibiting the Neimark-Sacker bifurcation, can be calculated via

$$\beta_2(a_{NS}) = Re\left(\frac{e^{-i\theta(a_{NS})}g_{21}}{2}\right) - Re\left(\frac{\left(1-2e^{i\theta(a_{NS})}\right)e^{-2i\theta(a_{NS})}}{2\left(1-e^{i\theta(a_{NS})}\right)}g_{20}g_{11}\right)$$
$$-\frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2$$

where $e^{i\theta(a_{NS})} = \lambda(a_{NS})$.

One gets that the following result which gives parametric conditions for existence and direction of the Neimark-Sacker bifurcation for the positive fixed point E_2 of system (1.2):

Theorem 1. *Suppose that E_2 is a positive unique fixed point of the system* (1.2)*. If* (3.1) *holds, $\beta_2(a_{NS}) \neq 0$ and the parameter a changes its value in small vicinity of NSB_{E_2} then, the system* (1.2) *passes through a Neimark-Sacker bifurcation at only fixed point E_2. Moreover, if $\beta_2(a_{NS}) < 0$ (respectively $\beta_2(a_{NS}) > 0$), then the Neimark-Sacker bifurcation of the system* (1.2) *at $a = a_{NS}$ is supercritical (respectively subcritical) and there exists a unique closed invariant curve bifurcates from E_2 for $a = a_{NS}$, which is attracting (respectively repelling).*

3.2. *Flip Bifurcation at the Fixed Point E_2*

Let us consider the term of FB_{E_2} as follows:

$FB_{E_2} = \{(a,\beta,\theta) : a > a_3,\ a_F = a_1, -1 < \theta < 0 \text{ and } 0 < \beta < 1, K < 0, S < 0\}.$

We consider the system (1.2) at fixed point E_2 with parameters lie in FB_{E_2}. The eigenvalues of the system (1.2) under these conditions are obtained as $\lambda_1(a_F) = -1$, $\lambda_2(a_F) = \frac{3S}{K}$.

The condition $|\lambda_2(a_F)| \neq 1$ leads to

$$K \neq \pm 3R, \tag{3.3}$$

where $R = 3\beta^2 + \theta - 2\beta\theta - 2\beta$.

Let $x_t = X_t - x^*$, $y_t = Y_t - y^*$ and $J(E_*) = J(x^*, y^*)$. We transform the fixed point E_2 of the system (1.2) into the origin $(0,0)$. By first-order Taylor expansion, the system (1.2) can be written

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} \to J(E_*)\begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} F_1(x_t, y_t, a) \\ F_2(x_t, y_t, a) \end{pmatrix}, \tag{3.4}$$

where $F_1(x_t, y_t) = -a_1x_t^3 + (a_1 - 3a_1\beta + a_1\theta)x_t^2 - y_tx_t + O\left(X_t^4\right)$, $F_2(x_t, y_t) = \frac{x_ty_t}{\beta} + O\left(X_t^4\right)$.

The system (3.4) can be expressed as

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = J(E_*)\begin{pmatrix} x_t \\ y_t \end{pmatrix} + \frac{1}{2}B(x_t, x_t) + \frac{1}{6}C(x_t, x_t, x_t) + O\left(x_t^4\right),$$

where $B(x,y)=\begin{pmatrix} B_1(x,y) \\ B_2(x,y)\end{pmatrix}$ and $C(x,y,u)=\begin{pmatrix} C_1(x,y,u) \\ C_2(x,y,u)\end{pmatrix}$ are symmetric multi-linear vector functions of $x,y,u\in R^2$. These functions are defined by as follows:

$$B_1(x,y)=\sum_{j,k=1}^{2}\left.\frac{\partial^2 F_1}{\partial\xi_j\partial\xi_k}\right|_{\xi=0}x_jy_k=(2a_1-6a_1\beta+2a_1\theta)x_1y_1-x_1y_2-x_2y_1,$$

$$B_2(x,y)=\sum_{j,k=1}^{2}\left.\frac{\partial^2 F_2}{\partial\xi_j\partial\xi_k}\right|_{\xi=0}x_jy_k=\frac{x_1y_2+x_2y_1}{\beta},$$

$$C_1(x,y,u)=\sum_{j,k,l=1}^{2}\left.\frac{\partial^3 F_1}{\partial\xi_j\partial\xi_k\partial\xi_l}\right|_{\xi=0}x_jy_ku_l=-6a_1x_1y_1u_1,$$

$$C_2(x,y,u)=\sum_{j,k,l=1}^{2}\left.\frac{\partial^3 F_2}{\partial\xi_j\partial\xi_k\partial\xi_l}\right|_{\xi=0}x_jy_ku_l=0.$$

Now, we assume that $q,p\in R^2$ are two eigenvectors of $J(FB_{E_2})$ for eigenvalue $\lambda_1(a_{FB})=-1$ such that $J(FB_{E_2})q=-q$ and $J^T(FB_{E_2})p=-p$. then by direct calculation we get

$$q\sim\left(\frac{(-3\beta+5-3\beta\theta+\theta)\beta}{-3\beta+2\theta+4\beta^2-3\beta\theta},1\right)^T \text{ and } p\sim\left(1,\frac{\beta}{2}\right)^T.$$

To achieve the normalization $<p,q>=1$ where $<,>$ means the standard scalar product in R^2, we can take the normalized vectors as

$$q=\left(\frac{(-3\beta+5-3\beta\theta+\theta)\beta}{-3\beta+2\theta+4\beta^2-3\beta\theta},1\right)^T,$$

$$p=\left(\frac{2(-3\beta+2\theta+4\beta^2-3\beta\theta)}{\beta(-9\beta+14\beta^2-9\beta\theta+4\theta)},\frac{-3\beta+2\theta+4\beta^2-3\beta\theta}{-9\beta+14\beta^2-9\beta\theta+4\theta}\right)^T.$$

We see that $<p,q>=1$. The direction of the flip bifurcation is determined by sign $c(a_F)$ and is computed by

$$c(a_F)=\frac{1}{6}\langle p,C(q,q,q)\rangle-\frac{1}{2}\langle p,B(q,(A-I)^{-1}B(q,q)\rangle. \tag{3.5}$$

From the above obtained results, one gets the following theorem for direction and existence of flip bifurcation for the positive fixed point E_2 of the system (1.2):

Theorem 2. *Suppose that E_2 is a positive unique fixed point of the system (1.2). If (3.3) holds, $c(a_F)\neq 0$ and the parameter a varies its value in a small vicinity of FB_{E_2}, the system (1.2) passes through a flip bifurcation at positive fixed point E_2. Moreover, if $c(a_F)<0$ (respectively $c(a_F)>0$), then there exists unstable (respectively stable) period-2 orbits bifurcate from E_2.*

4. Chaos control

The theory of bifurcation and chaos control have significant characteristics in biological species. So, in this section, we display how chaos can be ruled out from the irregular complex dynamics of the system (1.2) such as bifurcations and chaotic attractors. There is chaotic behavior in many areas like physics, biochemistry, econometrics, cardiology, communications, biology, and engineering. As a matter of fact, it is wished that the system be optimized with respect to some performance criterion and chaos be avoided in dynamical systems. The aim of chaos control is to make chaotic behavior more predible and stable. Chaos control is a technique of stabilization via the help of small perturbations. There are many techniques for chaos control in literature [10, 11]. The chaotic motion of the considered system (1.2) is controlled on the stable orbit through OGY control strategy, taking a as a control parameter. In order to apply OGY method introduced by Ott et al [4], we rewrite system (1.2) in the following form:

$$\begin{aligned} X_{t+1} &= aX_t(1-X_t)(X_t-\theta) - X_tY_t = f(X_t, Y_t, a), \\ Y_{t+1} &= \frac{1}{\beta}X_tY_t = g(X_t, Y_t, a), \end{aligned} \tag{4.1}$$

where a denotes parameter for chaos control. Furthermore, a is restricted to lie in some small interval $|a-a_0| < \mu$ with $\mu > 0$ and a_0 represents the nominal value belong to chaotic region. We apply the stabilizing feedback control strategy in order to move the trajectory towards the desired orbit. Suppose that (X^*, Y^*) be unstable fixed point of the system (1.2) in chaotic region produced by the emergence of Neimark-Sacker bifurcation, then the system (4.1) can be approximated in the neighbourhood of the unstable fixed point (X^*, Y^*) by the following linear map:

$$\begin{bmatrix} X_{t+1}-X^* \\ Y_{t+1}-Y^* \end{bmatrix} \approx A \begin{bmatrix} X_t - X^* \\ Y_t - Y^* \end{bmatrix} + B[a-a_0], \tag{4.2}$$

where

$$A = \begin{bmatrix} \frac{\partial f(X^*,Y^*,a_0)}{\partial X} & \frac{\partial f(X^*,Y^*,a_0)}{\partial Y} \\ \frac{\partial g(X^*,Y^*,a_0)}{\partial X} & \frac{\partial g(X^*,Y^*,a_0)}{\partial Y} \end{bmatrix} = \begin{bmatrix} a_0\beta - 2a_0\beta^2 + a_0\beta\theta + 1 & -\beta \\ -\frac{-a_0\beta + a_0\theta + a_0\beta^2 - a_0\beta\theta + 1}{\beta} & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} \frac{\partial f(X^*,Y^*,a_0)}{\partial a} \\ \frac{\partial g(X^*,Y^*,a_0)}{\partial a} \end{bmatrix} = \begin{bmatrix} \beta^2 - \beta\theta - \beta^3 + \beta^2\theta \\ 0 \end{bmatrix}.$$

In order to check that the system (4.1) is controllable, the following matrix is computed:

$$\begin{aligned} C &= [B : AB] \\ &= \begin{bmatrix} \beta^2 - \beta\theta - \beta^3 + \beta^2\theta & (a_0\beta - 2a_0\beta^2 + a_0\beta\theta + 1)(\beta^2 - \beta\theta - \beta^3 + \beta^2\theta) \\ 0 & -\frac{(a_0\beta + a_0\theta + a_0\beta^2 - a_0\beta\theta + 1)(\beta^2 - \beta\theta - \beta^3 + \beta^2\theta)}{\beta} \end{bmatrix}. \end{aligned}$$

C is of rank 2 at positive fixed point. We suppose that $[a - a_0] = -K \begin{bmatrix} X_t - X^* \\ Y_t - Y^* \end{bmatrix}$, where $K = [\rho_1 \; \rho_2]$, then the system (4.2) can be written as follows:

$$\begin{bmatrix} X_{t+1} - X^* \\ Y_{t+1} - Y^* \end{bmatrix} \approx [A - BK] \begin{bmatrix} X_t - X^* \\ Y_t - Y^* \end{bmatrix}.$$

The corresponding controlled system of (1.2) can be written as:

$$\begin{aligned} X_{t+1} &= (a_0 - \rho_1 (X_t - X^*) - \rho_2 (Y_t - Y^*)) X_t (1 - X_t)(X_t - \theta) - X_t Y_t, \\ Y_{t+1} &= \frac{1}{\beta} X_t Y_t. \end{aligned} \tag{4.3}$$

In addition, if the modulus the matrix $A - BK$'s eigenvalues is less than, the fixed point (X^*, Y^*) of the system (4.3) is locally asymptotically stable. The Jacobian matrix $A - BK$ of the controlled system (4.3) can be written as follows:

$$[A - BK] = \begin{bmatrix} a & -\beta - (\beta^2 - \beta\theta - \beta^3 + \beta^2\theta)\rho_2 \\ -\frac{-a_0\beta + a_0\theta + a_0\beta^2 - a_0\beta\theta + 1}{\beta} & 1 \end{bmatrix}$$

where

$$a = a_0\beta - 2a_0\beta^2 + a_0\beta\theta + 1 - (\beta^2 - \beta\theta - \beta^3 + \beta^2\theta)\rho_1.$$

The characteristic equation of the Jacobian matrix $A - BK$ is given by

$$P(\lambda) = \lambda^2 - (tr(A - BK))\lambda + \det(A - BK) = 0. \tag{4.4}$$

Let λ_1 and λ_2 are roots of characteristic equation (4.4), then

$$\begin{aligned} \lambda_1 + \lambda_2 &= (-\beta^2 + \beta\theta + \beta^3 - \beta^2\theta)\rho_1 + 2 + a_0\beta - 2a_0\beta^2 + a_0\beta\theta, \\ \lambda_1\lambda_2 &= ((\beta - \beta^2)\theta - \beta^2 + \beta^3)\rho_1 + ((a_0\beta^2 + a_0 - 2a_0\beta)\theta^2 \\ &\quad + (1 - 2a_0\beta^3 + 4a_0\beta^2 + (-1 - 2a_0)\beta)\theta + \beta^4 a_0 - 2a_0\beta^3 \\ &\quad + (a_0 + 1)\beta^2 - \beta)\rho_2 + (2a_0\beta - a_0)\theta + 2a_0\beta - 3a_0\beta^2 \end{aligned} \tag{4.5}$$

are valid. In order to obtain the lines of marginal stability we must solve equations $\lambda_1 = \pm 1$ and $\lambda_1\lambda_2 = 1$. These restrictions make sure that λ_1 and λ_2 have absolute less than 1. Assume that $\lambda_1\lambda_2 = 1$, then second part of Equation (4.5) implies that:

$$\begin{aligned} L_1 := &\left(\beta^3 + (-1 - \theta)\beta^2 + \beta\theta\right)\rho_1 + \left(\beta^4 a_0 + (2a_0(-1 - \theta)\beta^3 \right. \\ &\left. + (a_0 + 4a_0\theta + \theta^2 a_0 + 1)\beta^2 + \left(-1 - 2\theta^2 a_0 + (-1 - 2a_0)\theta\right)\beta + \theta + \theta^2 a_0\right)\rho_2 \\ &- 3a_0\beta^2 + (2a_0(1 + \theta))\beta - a_0 - 1 = 0 \end{aligned}$$

Moreover, we suppose that $\lambda_1 = 1$, then using the equations (4.5) yield that:

$$L_2 := \left(\beta^4 a_0 + (-2a_0 - 2a_0\theta)\beta^3 + (a_0 + 4a_0\theta + \theta^2 a_0 + 1)\beta^2 \right.$$
$$+ (-1 - 2\theta^2 a_0 + (-1 - 2a_0)\theta)\,\beta + \theta + \theta^2 a_0)\rho_2 + \theta + \theta^2 a_0\big)\,\rho_2\beta^2$$
$$+ a_0(1+\theta)\beta - a_0\theta - 1 = 0$$

Finally, taking $\lambda_1 = -1$ and using equations (4.5) we get

$$L_3 := \left(\left(-2\beta^2 + 2\beta\right)\theta - 2\beta^2 + 2\beta^3\right)\rho_1 + (a_0\beta^2 + a_0 - 2a_0\beta)\theta^2$$
$$+ \left(1 - 2a_0\beta^3 + 4a_0\beta^2 + (-1 - 2a_0)\beta\right)\theta + \beta^4 a_0\beta^3 - 2a_0\beta^3\beta^3$$
$$+ (a+1)\beta^2 - \beta)\rho_2 + (-a_0 + 3a_0\beta)\theta + 3 - 5a_0\beta^2 + 3a_0\beta = 0$$

Then, stable eigenvalues lie within the triangular region in $\rho_1\rho_2$ plane bounded by the straight lines L_1, L_2, L_3 for particular parametric values.

5. NUMERICAL ANALYSIS

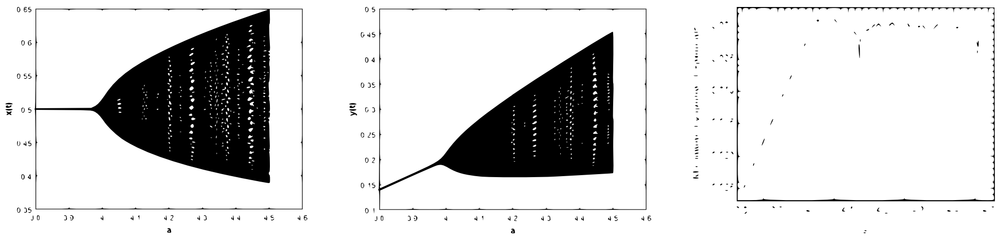
In this section, numerical simulations including bifurcation diagrams, phase portraits, and maximum Lyapunov exponents are presented both to illustrate the results of theoretical analyses and to exhibit complex and new dynamical behaviours.

Example 2. By taking the parameters values $(\beta, \theta) = (0.5, -0.1)$, the critical value of Neimark-Sacker bifurcation point is $a_{NS} = 4$. Also, the positive fixed point of the system (1.2) is evaluated as $E_2 = (0.5, 0.2)$. Using these parameter values, we get the Jacobian matrix as $J_{E_2}(a_{NS}) = \begin{bmatrix} 0.8 & -0.5 \\ 0.4 & 1 \end{bmatrix}$. Also, we have

$$\lambda_{1,2} = 0.9 \pm 0.4358898944i,$$
$$g_{20}(a_{NS}) = 0.276346573 + 0.795433216i,$$
$$g_{11}(a_{NS}) = 2.774882688 + 4.072599295i,$$
$$g_{02}(a_{NS}) = 7.749765376 + 4.635245515i,$$
$$g_{21}(a_{NS}) = -15.000000 - 23.93149824i,$$
$$\beta_2(a_{NS}) = -41.52149634 < 0.$$

Therefore, Neimark-Sacker bifurcation is supercritical and it shows the correctness of Theorem 1. The bifurcation diagram, maximum Lyapunov exponents and the phase portraits of the system (1.2) are shown in Figure 2 and Figure 3, respectively.

The bifurcation diagrams shown in Figure 2a and Figure 2b show that the stability of E_2 happens for $a < 4$ and loses its stability at $a = 4$ and an attracting invariant curve appears if $r > 4$. We compute the maximum Lyapunov exponents for detecting the presence of chaos in the model. The existence of chaotic regions in the parameter space is clearly visible in Figure 2c.



(A) (B) (C)

FIGURE 2. Bifurcation diagrams and MLE for the system (1.2) with values of $\beta = 0.5, \theta = -0.1$ and $a \in [3.8, 4.6]$ and initial value $(x_0, y_0) = (0.4, 0.1)$.

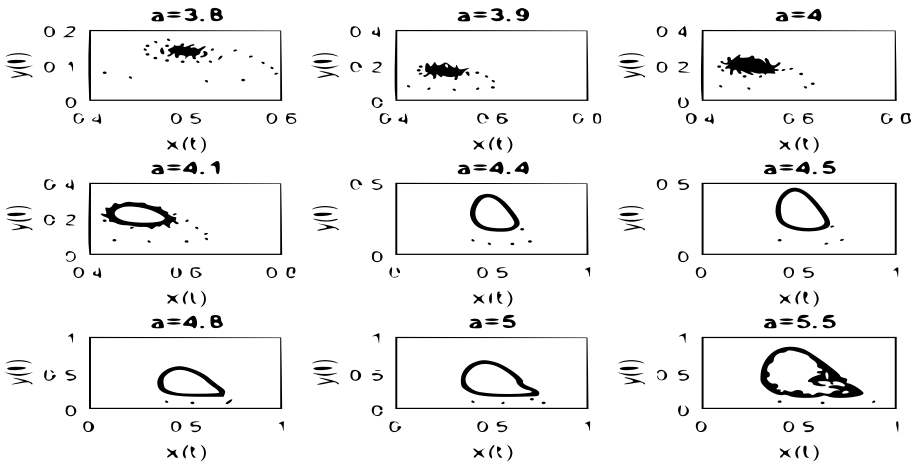
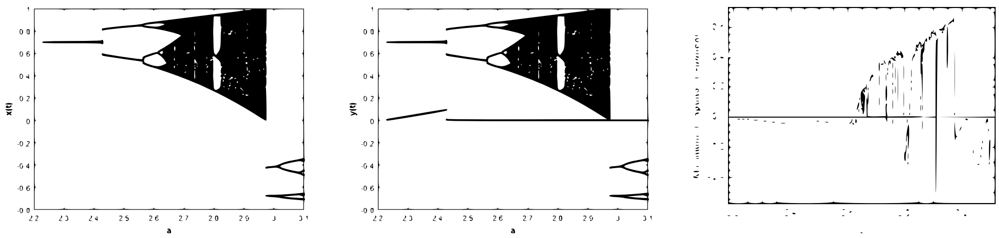


FIGURE 3. Phase portraits of the system (1.2) for different values of a.

The phase portraits for different values of a are presented in Figure 3, which clearly depicts the process of how a smooth invariant circle bifurcates from the stable fixed point $E_2 = (0.5, 0.2)$. When a exceeds 4 there appears a circular curve enclosing the fixed point E_2, and its radius becomes larger with respect to the growth of a.

Example 3. For the parameters values $\theta = -0.8$, $\beta = 0.7$, the critical value of flip bifurcation point is obtained as $a_F = 2.439024390$ and the positive fixed point of the system (1.2) is evaluated as $E_2 = (0.6999999998, 0.09756097611)$. The Jacobian matrix of the system (1.2) is $J_{E_2}(a_F) = \begin{bmatrix} -1.048780486 & -0.6999999998 \\ 0.1393728231 & 1.000000000 \end{bmatrix}$.



(A) (B) (C)

FIGURE 4. Bifurcation diagrams and MLE for the system (1.2) with values of $\theta = -0.8, \beta = 0.7, a_F = 2.439024390$ and initial value $(x_0, y_0) = (0.67, 0.08)$.

The eigenvalues are evaluated as $\lambda_1 = -1$, $\lambda_2 = 0.9512195120$. Direct computation shows

$$q \sim (-0.9975807099, 0.06951781980)^T,$$
$$p \sim (-0.9438583564, -0.3303504248)^T.$$

To obtain the normalization $\langle p, q \rangle = 1$, we can take normalized vectors as

$$q = (-0.9975807099, 0.06951781980)^T$$
$$p = (-1.027485787, -0.3596200254)^T.$$

Then, symmetric multi-linear vector functions are obtained as follows

$$B_1(x,y) = -9.268292682\, x_1 y_1 - x_1 y_2 - x_2 y_1,$$
$$B_2(x,y) = 1.428571429\, x_1 y_2 + 1.428571429\, x_2 y_1,$$
$$C_1(x,y,u) = -14.63414634\, x_1 y_1 u_1,$$
$$C_2(x,y,u) = 0.$$

From (3.5), the critical part is obtained as $c(a_F) = -1.091637569 < 0$. Therefore, the flip bifurcation is unstable period-2 orbits bifurcate from E_2 and it shows the correctness of Theorem 2. The bifurcations diagram, maximum Lyapunov exponents and the phase portraits of the system (1.2) are shown in Figure 4.

From Figures 4a and 4b, we see that for $a < 2.439024390$ the fixed point E_2 is stable, and loses its stability at the flip bifurcation parameter value $a = 2.439024390$. We also observe that if let $a > 2.439024390$, the system (1.2) with Allee effect displays chaotic dynamics through flip bifurcation in Figure 4a and 4b. After the stable fixed point, periodic oscillations are observed with periods 2,4,8 and eventually leading to chaos with gradually increasing the parameter a. Moreover, the maximum Lyapunov exponents are computed and the existence of chaotic regions in the parameter space is clearly visible in Figure 4c.

Example 4. In order to discuss the OGY feedback control method for system (1.2), we take $a_0 = 4.2$ and $(\beta, \theta) = (0.5, -0.1)$. In this case system (1.2) has unique positive fixed point $(X^*, Y^*) = (0.5, 0.260)$ which is unstable. Then corresponding controlled system is given by:

$$\begin{aligned} X_{t+1} &= (4.2 - \rho_1(X_t - 0.5) - \rho_2(Y_t - 0.260))X_t(1 - X_t)(X_t + 0.1) - X_tY_t \\ Y_{t+1} &= \frac{1}{\beta}X_tY_t \end{aligned} \tag{5.1}$$

where $K = [\rho_1\ \rho_2]$ be gain matrix and $(X^*, Y^*) = (0.5, 0.26)$ is unstable fixed point of the system (1.2). We have

$$A = \begin{bmatrix} 0.790 & -0.5 \\ 0.520 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.150 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0.150 & 0.11850 \\ 0 & 0.07800 \end{bmatrix}.$$

Then it is easy to check that rank of C matrix is 2. Therefore, the system (5.1) is controllable. Then, Jacobian matrix $A - BK$ of the controlled system (5.1) is given by

$$[A - BK] = \begin{bmatrix} 0.790 - 0.150\rho_1 & -0.5 - 0.150\rho_2 \\ 0.520000 & 1 \end{bmatrix}$$

Moreover, the lines L_1, L_2 and L_3 of marginal stability are given by:

$$\begin{aligned} L_1 &= 0.05 - 0.15\rho_1 + 0.0780\rho_2 = 0, \\ L_2 &= 0.26 + 0.0780\rho_2 = 0, \\ L_3 &= 3.84 - 0.3\rho_1 + 0.078\rho_2 = 0. \end{aligned}$$

Then, the stable triangular region bounded by marginal lines L_1, L_2 and L_3 for the controlled system (5.1) is shown in Figure 5.

6. Discussions

In this paper, we extend the system (1.1) by introducing weak Allee effect and investigate the dynamical behaviors of the modified the system (1.2) around coexistence fixed point. The system (1.2) obtained by modified from of the system (1.1) has different dynamics properties compared to the system (1.1). While the system (1.1) has two fixed points, model (1.2) has three fixed points. One of them is the same as in the system (1.1) without Allee effect, E_0 and the others have a new expressions E_1 and E_2. Because of its biological significance, we focused on the coexistence fixed point E_2 and analyze the topological classifications of this fixed point of the discrete-time predator-prey system with weak Allee effect. In Figure 1b, we have compared the local stability analysis of the coexistence fixed point of the predator-prey model with and without Allee effect. One can see that the system with Allee effect reaches the fixed solution more slowly than in the system without the Allee effect. Moreover, if $-1 < \theta < -0.5$, then the predator density of the model subject to Allee effect increases, otherwise $(-0.5 \leq \theta < 0)$ it decreases.

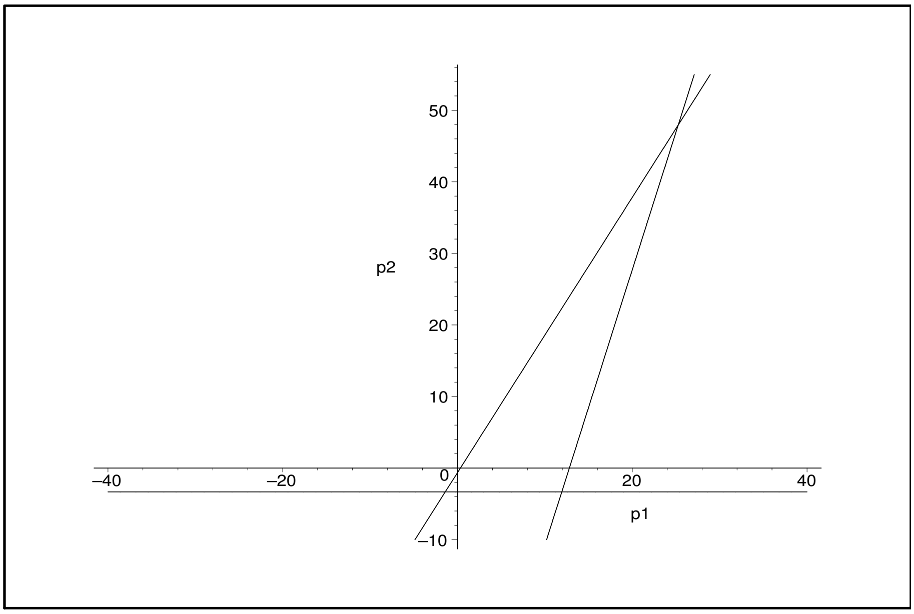


FIGURE 5. Triangular stability region bounded by L_1, L_2 and L_3 for the controlled system (5.1).

We display that flip and Neimark-Sacker bifurcations happen at certain bifurcation parameter a and some conditions on parameters θ and β. The directions of both flip bifurcation and Neimark-Sacker bifurcation are given. In order to support to obtained theoretical results and show the complex dynamical behaviors of the system (1.2), we present bifurcation diagrams, phase portraits and maximum Lyapunov exponents. Maximum Lyapunov exponents exhibit the existences of periodic orbits and chaos as parameter a increases.

In study [3], while the system (1.1) undergoes Neimark- Sacker bifurcation around the coexistence fixed point, the system (1.2) undergoes both flip and Neimark- Sacker bifurcations. Under the influence of Neimark-Sacker bifurcation unstable invariant closed curves are produced. Moreover, when the system (1.1) undergoes flip bifurcation, we can observe the existence many attractors such as steady state, period-2 orbit, period-4 orbit, period-8 orbit, chaos etc as parameter a varies. These results show that the system (1.2) has far richer dynamics compared to the system (1.1).

On the other hand, Neimark-Sacker bifurcation is successfully controlled with OGY control strategy. From our numerical investigation, it is clear that OGY method based on feedback control strategy can restore the stability. This controlling method is effective in order to advance or completely vanish the chaos due to emergence of Neimark-Sacker bifurcation.

REFERENCES

[1] A. Atabaigi., "Multiple bifurcations and dynamics of a discrete-time predator-prey system with group defense and non-monotonic functional response." *Differ Equ Dyn Syst.*, vol. 21, no. 2, pp. 423–441, 2016, doi: 10.1007/s12591-016-0305.

[2] A. J. Lotka, *Elements of Mathematical Biology.* Baltimore: Williams & Wilkins, 1925.

[3] A. Q. Khan, "Neimark-Sacker bifurcation of a two-dimensional discrete-time predator-prey model." *Springer Plus.*, vol. 5, no. 1, p. 126, 2016, doi: 10.1186/s40064-015-1618.

[4] E. Ott, C. Grebogi, J.A. Yorke, "Controlling choas. ," *Physical review letters*, vol. 64, no. 11, pp. 1196–1199, 1990, doi: 10.1103/PhysRevLett.64.1196.

[5] F. Kangalgil, "Neimark–Sacker bifurcation and stability analysis of a discrete-time prey–predator model with Allee effect in prey." , vol:92, 2019, doi:-y," *Adv. Differ Equ.*, vol. 2019, no. 1, pp. 1–12, 2019, doi: 10.1186/s13662-019-2039.

[6] F. Kangalgil, N. Topsakal, N. Ozturk, "Analyzing bifurcation, stability and chaos control for a discrete-time prey-predator model with Allee effect," *Turk. J. Math.*, vol. 46, no. 6, pp. 2047–2068, 2022, doi: 10.55730/1300-0098.3253.

[7] Kuznetsov, *Elements of applied bifurcation theory.* Springer Science & Business Media, 2013.

[8] L. Cheng, H. Cao, " Bifurcation analysis of a discrete-time ratio-dependent predator-prey model with Allee effect." *Commun Nonlinear Sci Numer Simulat.*, vol. 38, pp. 288–302, 2016, doi: 10.1016/j.cnsns.2016.02.038.

[9] P. Feng, and Y. Kang, "Dynamics of a modified Leslie-Gower model with double Allee effects," *Nonlinear Dyn .*, vol. 80, no. 1-2, pp. 1051–1062, 2015, doi: 10.1007/s11071-015-1927-2.

[10] Q. Din, "Neimark-Sacker bifurcation and chaos control in Hassel-Varley model," *J. Differ. Equ. Appl.*, vol. 23, no. 4, pp. 741–762, 2016, doi: 10.1080/10236198.20161277213.

[11] Q. Din, "Complexity and choas control in a discrete-time prey-predator model," *Commun Nonlinear Sci. Numer. Simulat.*, vol. 49, pp. 113–134, 2017, doi: 10.1016/j.cnsns.2017.01.025.

[12] S. Isik., "A study of stability and bifurcation analysis in discrete-time predator–prey system involving the Allee effect." *Int. J.Biomath.*, vol. 12, no. 1, p. 1950011, 2019, doi: 10.1142/S1793524519500116.

[13] S. Kartal, "Mathematical modeling and analysis of tumor-immune system interastion by using Lotka-Volterra Predator-prey like model with piecewise constant arguments," *Periodicals of Engineering and Natural Scie.*, vol. 2, no. 1, pp. 7–12, 2014, doi: 10.21533/pen.v2i1.36.

[14] S. Kartal, "Dynamics of a plant-herbivore model with differential-difference equations," *Cogent Math Statistic.*, vol. 3, no. 1, p. 1136198, 2016, doi: 10.1080/23311835.2015.1136198.

[15] S. Kartal, "Flip and Neimark–Sacker bifurcation in a differential equation with piecewise constant arguments model," *J. Differ Equ. Appl.*, vol. 23, no. 4, pp. 763–778, 2017, doi: 10.1080/10236198.2016.1277214.

[16] S. Kartal, F. Gurcan, "Global behaviour of a predator–prey like model with piecewise constant arguments," *J. Biol. Dyn.*, vol. 9, no. 1, pp. 159–171, 2015, doi: 10.1080/17531758.2015.1049225.

[17] S. M. Rana, "Bifurcation and complex dynamics of a discrete-time predator-prey system." *Comput. Eco.Software.*, vol. 5, no. 2, pp. 187–200, 2015, doi: 10.1155/2017/9705985.

[18] S. M. Rana, "Bifurcation and complex dynamics of a discrete-time predator-prey system involving group defense." *Computational Ecology and Software.*, vol. 5, no. 3, pp. 222–238, 2015.

[19] S. M. Rana and U. Kulsum, "Bifurcation analysis and chaos control in a discrete-time predator-prey system of Leslie type with simplified holling type IV functional response," *Discrete Dyn. Nat. Soc.*, vol. 2017, pp. –, 2017, doi: 10.1155/2017/9705985.

[20] S. Pal., S. K. Sasmal, N. Pal, "Chaos control in a discrete-time predator-prey model with weak Allee effect." *Int. J. Biomath..*, vol. 11, no. 7, p. 1850089, 2018, doi: 10.1142/S1793524518500894.

[21] S. Zhou, T. Liu, and G. Wang, "The stability of predator-prey systems subject to the Allee effects," *Theor. Popul. Biol.*, vol. 67, no. 1, pp. 23–31, 2005, doi: 10.1016/j.tpb.2004.06.007.

[22] V. Volterra, " Variazioni e fluttuazioni del numero di' individui in specie animali conviventi," *Mem. R. Accad. Naz. Dei Lincei.*, no. 2, pp. 31–113, 1926.

[23] X. Lai, S. Liu, R. Lin , "Rich dynamical behaviours for predator-prey model with weak Allee." *Applicable Anal.*, vol. 89, no. 8, pp. 1271–1292, 2010, doi: 10.1080/00036811.2010.483557.

[24] Z. He, X. Lai, "Bifurcation and chaotic behavior of a discrete-time predator-prey system," *Nonlinear Anal.:Real World Appl.*, vol. 12, no. 1, pp. 403–417, 2011, doi: 10.1016/j.nonrwa.2010.06.026.

Authors' addresses

Figen Kangalgil

(**Corresponding author**) Dokuz Eylul University, Bergama Vocational School, 35700, Izmir, Turkey

E-mail address: `figen.kangalgil@deu.edu.tr`

Seval Işık

Sivas Cumhuriyet University, Faculty of Education, Department of Mathematics and Science Education, 58140, Sivas, Turkey

E-mail address: `skaracan@cumhuriyet.edu.tr`