Research Article

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Multiplicative (generalized)-reverse derivations in rings and Banach algebras

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Abstract: In this work, the subject of ideal in a semiprime ring with multiplicative (generalized)- reverse derivations studied is included. We give new essential results for researchers in this field and generalize some results found in the literature. Also, the application of continuous reverse derivations in Banach algebras is discussed for the first time.

Keywords: Semiprime ring, ideal, derivation, multiplicative (generalized)-reverse derivation, Banach algebras, radical

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1 Introduction

In what follows, unless stated otherwise, R will be an associative ring and Z will be the center of R. Recall that a ring R is prime if $xRy = (0)$ implies $x = 0$ or $y = 0$ and R is semiprime if $xRx = 0$ implies $x = 0$. Let A be a Banach algebra. The Jacobson radical $\operatorname{rad}(A)$ of A is the intersection of all primitive ideals. If the Jacobson radical reduces to the zero element, A is called semisimple. A Banach algebra A without a unity can be embedded into a unital Banach algebra $\tilde{A} = A \oplus C$ as an ideal of codimension one. We may identify A with the ideal $\{(x, 0) : x \in A\}$ in $\tilde{A}$ via the isometric isomorphism $x \to (x, 0)$.

An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. The mapping $d : R \to R$ is called a multiplicative derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. These maps are not additive. Inspired by the definition of a multiplicative derivation, the notion of multiplicative (generalized)-derivation was extended by Daif and Tamman El-Sayiad in [4] as follows: $F : R \to R$ is called a multiplicative (generalized)-derivation if there exists a map $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. The concept of reverse derivation was first time introduced by Herstein in [5]. An additive mapping d on R is said to be a reverse derivation if $d(xy) = d(y)x + yd(x)$ holds for all $x, y \in R$. Based on the definitions given above, the following definition of multiplicative (generalized)-reverse derivation can be given.

Definition 1.1. An additive mapping $F : R \to R$ is called a multiplicative right (generalized)-reverse derivation if there exists a map $d : R \to R$ such that

$$F(xy) = yF(x) + d(y)x \quad \text{for all } x, y \in R$$

and F is called a multiplicative left (generalized)-reverse derivation if there exists a map $d : R \to R$ such that

$$F(xy) = yd(x) + F(y)x \quad \text{for all } x, y \in R.$$

The mapping F is said to be a multiplicative (generalized)-reverse derivation with associated map d if it is both a multiplicative left and right (generalized)-reverse derivation with associated map d.

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Example 1.1. Suppose the ring

$$R = \left\{ \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & -b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

Define the maps $F, d : R \to R$ as follows:

$$F\left(\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & -b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & -a & -b & c \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$d\left(\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & -b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 & c^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then it is easy to verify that F is a multiplicative (generalized)-reverse derivation on R but not a multiplicative (generalized)-derivation on R.

In [9], Singer and Wermer proved that any continuous derivation in a commutative Banach algebra has the interval in the Jacobson radical of the algebra. Also, in the same paper, Singer and Wermer expressed the assumption that the continuity assumption can be removed. Based on this, this assumption was proved by Thomas in [10]. It is clear that the conclusion of Singer and Wermer is not valid for non-commutative Banach algebras. However, in this connection there arises the question of how to obtain a non-commutative version of the Singer–Wermer theorem. This question was addressed by Sinclair [8] who proved that every continuous derivation of a Banach algebra leaves the primitive ideals of the algebra invariant. After these studies, there appeared many papers without the condition of commutativity.

Let S be a nonempty subset of R. A mapping F from R to R is called commuting on S if $[F(x), x] = 0$ for all $x \in S$. In [2], the properties of commuting maps on the semiprime ring were investigated by Bell and Martindale. In addition, some conditions of commutativity in semiprime rings are considered and the results are generally given as the commuting map. In that study, better results were obtained in the multiplicative (generalized)-reverse derivation in semiprime rings. It was proved that the ring is commutative when the commuting map is defined. In [3], Daif and Bell proved that if R is a semiprime ring, U is a nonzero ideal of R and d is a derivation of R such that $d([x, y]) = \pm[x, y]$ for all $x, y \in U$, then $U \subset Z$. In [1], Bell and Kappe have proved that if d is a derivation of R which is either an homomorphism or anti-homomorphism in semiprime ring R or a nonzero right ideal of R, then $d = 0$. The above mentioned commutative conditions have been discussed by many researchers over years for various sets and derivations.

In the present paper, it is aimed to prove the results corresponding to the conditions mentioned above for multiplicative (generalized)-reverse derivations with ideals in semiprime and prime rings. We also apply our results on prime rings to the aforementioned conditions for continuous reverse derivations in non-commutative Banach algebras.

For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $x \circ y$ denotes the anti-commutator $xy + yx$. Given $x, y \in R$, we set

$$x \circ_0 y = x, \quad x \circ_1 y = x \circ y = xy + yx,$$

and, by induction,

$$x \circ_m y = (x \circ_{m-1} y) \circ y \quad \text{for } m > 1.$$

For each $n \geq 0$, set

$$[x, y]_0 = x, \quad [x, y]_1 = xy - yx.$$

Then an Engel condition is a polynomial

$$[x, y]_k = [[x, y]_{k-1}, y], \quad k = 1, 2, \ldots,$$

in a non-commuting indeterminate. Throughout the present paper, we shall make use of the following basic identities without any specific mention:
(i) $[x, yz] = y[x, z] + [x, y]z$,
(ii) $[xy, z] = [x, z]y + x[y, z]$,
(iii) $xy \circ z = (x \circ z)y + x[y, z] = x(y \circ z) - [x, z]y$,
(iv) $x \circ yz = y(x \circ z) + [x, y]z = (x \circ y)z + y[z, x]$.

2 The results in rings

Lemma 2.1 ([1, Theorem 3]). *Let R be a semiprime ring and let U be a nonzero left ideal. If R admits a derivation D which is nonzero on U and centralizing on U, then R contains a nonzero central ideal.*

Lemma 2.2 ([8, Theorem 2.2]). *Let A be a Banach algebra over the real or complex field and let d be a continuous derivation on A. Then d leaves the primitive ideals of A invariant.*

Lemma 2.3 ([6, Remark 4.3]). *Let F be a field and let A be a semisimple Banach algebra over F. If d is additive derivation on A, then d is continuous.*

Lemma 2.4 ([9, Singer–Wermer theorem]). *Let A be a commutative Banach algebra and let d be a derivation on A. If A is semisimple, then $d = 0$.*

Lemma 2.5 ([7, Theorem 5]). *Let d be a derivation of prime ring R with* char $R \neq 2$. *If $[x, d(x)] \in Z$ for all $x \in R$, then R must be commutative or $d = 0$.*

Lemma 2.6. *Let R be a semiprime ring and let I be a nonzero ideal of R. If F is a multiplicative (generalized)-reverse derivation associated with a nonzero reverse derivation d on I, then d is a derivation on I.*

Proof. By hypothesis, we get that F is a multiplicative (generalized) reverse derivation on I. We obtain

$$F(x^2y) = F(y)x^2 + yd(x^2) = F(y)x^2 + y(d(x)x + xd(x)),$$

and so,

$$F(x^2y) = F(y)x^2 + yd(x)x + yxd(x) \quad \text{for all } x, y \in I. \tag{2.1}$$

On the other hand,

$$F(x(xy)) = F(xy)x + xyd(x) = (F(y)x + yd(x))x + xyd(x).$$

This implies that

$$F(x(xy)) = F(y)x^2 + yd(x)x + xyd(x) \quad \text{for all } x, y \in I. \tag{2.2}$$

Combining (2.1) and (2.2), we get

$$[x, y]d(x) = 0 \quad \text{for all } x, y \in I.$$

Replacing y by ry, $r \in R$, in this equation, we see that

$$[x, r]yd(x) = 0 \quad \text{for all } x, y \in I,\ r \in R. \tag{2.3}$$

Replacing y by $d(x)t[x, r]$, $t \in R$, in (2.3), we find that

$$[x, r]d(x)t[x, r]d(x) = 0.$$

Thus

$$[x, r]d(x)R[x, r]d(x) = (0).$$

Since R is semiprime ring, we get

$$[x, r]d(x) = 0 \quad \text{for all } x \in I,\ r \in R.$$

Replacing x by $x + z$, $z \in I$, in this equation, we find that

$$[x, r]d(z) + [z, r]d(x) = 0.$$

Therefore,

$$[x,r]d(z) = -[z,r]d(x). \tag{2.4}$$

Replacing y by $d(z)t[z,r]$, $t \in R$, in equation (2.3), we get

$$[x,r]d(z)t[z,r]d(x) = 0.$$

Using equation (2.4) in this equation, we have

$$[x,r]d(z)t[x,r]d(z) = 0,$$

and so,

$$[x,r]d(z)R[x,r]d(z) = (0).$$

By the semiprimeness of R, we have

$$[x,r]d(z) = 0.$$

Replacing x by xy in this equation, we arrive at

$$[x,r]yd(z) = 0.$$

Replacing r by $d(z)$ in the above equation, we get

$$[x,d(z)]yd(z) = 0. \tag{2.5}$$

Replacing y by yx in this equation, we obtain

$$[x,d(z)]yxd(z) = 0. \tag{2.6}$$

Multiplying (2.5) on the right by x, we have

$$[x,d(z)]yd(z)x = 0. \tag{2.7}$$

Subtracting (2.6) from (2.7), we arrive at

$$[x,d(z)]I[x,d(z)] = (0).$$

Therefore,

$$[x,d(z)]IR[x,d(z)]I = (0) \quad \text{for all } x,z \in I.$$

By the semiprimeness of R, we get

$$[x,d(z)]I = (0) \quad \text{for all } x,z \in I,$$

and so,

$$[x,d(z)]RI = (0).$$

In particular, using $[x,d(z)] \in I$, we can write the last equation such as

$$[x,d(z)]R[x,d(z)] = (0).$$

Since R is semiprime ring, we have

$$[x,d(z)] = 0 \quad \text{for all } x,z \in I. \tag{2.8}$$

By hypothesis, we get

$$d(xz) = d(z)x + zd(x) \quad \text{for all } x,z \in I.$$

Using equation (2.8) in this equation, we obtain that

$$d(xz) = xd(z) + d(x)z = d(x)z + xd(z).$$

We conclude that d is derivation on I. □

Corollary 2.1. *Let R be a prime ring. If d is a multiplicative reverse derivation on R, then d is a derivation on R.*

Proof. Since d is a multiplicative (generalized)-reverse derivation associated with a nonzero reverse derivation d, we obtain that d is a derivation by Lemma 2.6. □

Theorem 2.1. *Let R be a semiprime ring and let I be a nonzero ideal of R. Suppose that R admits a multiplicative (generalized)-reverse derivation F associated with a nonzero reverse derivation d. If F acts as an anti-homomorphism or F acts as a homomorphism on I, then R contains a nonzero central ideal.*

Proof. Let F be an anti-homomorphism on I. We get

$$F(xy) = F(y)F(x) \quad \text{for all } x, y \in I.$$

Since F is a multiplicative (generalized)-reverse derivation on I, we have

$$F(y)x + yd(x) = F(y)F(x),$$

and so,

$$F(y)(x - F(x)) = -yd(x). \tag{2.9}$$

Replacing y by yr, $r \in R$, we obtain that

$$rF(y)(x - F(x)) + d(r)y(x - F(x)) = -yrd(x).$$

Using equation (2.9) in this equation, we get

$$-ryd(x) + d(r)y(x - F(x)) = -yrd(x).$$

Thus,

$$d(r)y(x - F(x)) = [r, y]d(x). \tag{2.10}$$

Replacing r by rt, $t \in R$, in this equation, we have

$$td(r)y(x - F(x)) + d(t)ry(x - F(x)) = r[t, y]d(x) + [r, y]td(x).$$

Using equation (2.10) in the last equation, we get

$$t[r, y]d(x) + d(t)ry(x - F(x)) = r[t, y]d(x) + [r, y]td(x).$$

Replacing r by $d(r)$ in this equation, we find that

$$t[d(r), y]d(x) + d(t)d(r)y(x - F(x)) = d(r)[t, y]d(x) + [d(r), y]td(x).$$

Using equation (2.10) in the above equation, we have

$$t[d(r), y]d(x) + d(t)[r, y]d(x) = d(r)[t, y]d(x) + [d(r), y]td(x).$$

Replacing t by r in this equation, we see that

$$r[d(r), y]d(x) + d(r)[r, y]d(x) = d(r)[r, y]d(x) + [d(r), y]rd(x),$$

and so,

$$r[d(r), y]d(x) = [d(r), y]rd(x) \quad \text{for all } r \in R,\ x, y \in I. \tag{2.11}$$

Replacing y by yt, $t \in R$, in this equation, we get

$$r[d(r), y]td(x) + ry[d(r), t]d(x) = y[d(r), t]rd(x) + [d(r), y]trd(x).$$

Replacing t by $d(r)$ in the above equation, we have

$$r[d(r), y]d(r)d(x) + ry[d(r), d(r)]d(x) = y[d(r), d(r)]rd(x) + [d(r), y]d(r)rd(x).$$

Consequently,

$$r[d(r), y]d(r)d(x) = [d(r), y]d(r)rd(x).$$

In particular, if $r \in I$ is taken in this equation, we can see

$$r[d(r), y]d(r)d(x) = [d(r), y]d(r)rd(x) \quad \text{for all } x, y, r \in I.$$

and use equation (2.11) in this equation, we see that

$$[d(r), y][r, d(r)]d(x) = 0.$$

Replacing y by yz, $z \in I$, in this equation, we find that

$$[d(r), y]z[r, d(r)]d(x) = 0 \quad \text{for all } x, y, z, r \in I.$$

Replacing x by rx in the last equation, we get

$$[d(r), y]z[r, d(r)]xd(r) = 0. \tag{2.12}$$

Replacing x by xr in this equation, we obtain that

$$[d(r), y]z[r, d(r)]xrd(r) = 0. \tag{2.13}$$

Multiplying (2.12) on the right by r, we have

$$[d(r), y]z[r, d(r)]xd(r)r = 0. \tag{2.14}$$

Subtracting (2.13) from (2.14), we arrive at

$$[d(r), y]z[r, d(r)]x[r, d(r)] = 0.$$

Replacing r by y in this equation, we get

$$[d(y), y]z[y, d(y)]x[y, d(y)] = 0 \qquad \text{for all } x, y, z \in I,$$
$$([d(y), y]I)^3 = (0) \quad \text{for all } y \in I.$$

Since R is a semiprime ring, we have $[d(y), y]I = (0)$ for all $y \in I$. That is, $[d(y), y]R[d(y), y] = (0)$ for all $y \in I$. By the semiprimeness of R, we have $[d(y), y] = 0$, for all $y \in I$. By Lemma 2.1 and Lemma 2.6, we get R contains a nonzero central ideal. This proof is completed.

Let F be a homomorphism on I. We have

$$F(xy) = F(x)F(y) \quad \text{for all } x, y \in I.$$

Since F is a multiplicative (generalized) reverse derivation on I, we have

$$F(y)x + yd(x) = F(x)F(y).$$

Replacing y by zy, $z \in I$, in this equation, we get

$$F(y)zx + yd(z)x + zyd(x) = F(x)F(y)z + F(x)yd(z),$$

and so,

$$F(y)zx + yd(z)x + zyd(x) - F(x)F(y)z - F(x)yd(z) = 0.$$

Using the hypothesis, we obtain that

$$F(y)zx + yd(z)x + zyd(x) - F(y)xz - yd(x)z - F(x)yd(z) = 0.$$

That is,

$$F(y)[z, x] + yd(z)x + [z, yd(x)] - F(x)yd(z) = 0. \tag{2.15}$$

Replacing y by zy in this equation, we get

$$F(y)z[z,x]+yd(z)[z,x]+zyd(z)x+z[z,yd(x)]-F(x)zyd(z)=0. \tag{2.16}$$

Replacing x by zx in equation (2.15), we have

$$F(y)z[z,x]+yd(z)zx+[z,yd(zx)]-F(x)zyd(z)-xd(z)yd(z)=0. \tag{2.17}$$

Subtracting (2.16) from (2.17), we arrive at

$$yd(z)[z,x]+zyd(z)x+z[z,yd(x)]-yd(z)zx-[z,yd(zx)]+xd(z)yd(z)=0. \tag{2.18}$$

Replacing y by zy in this equation, we get

$$zyd(z)[z,x]+z^2yd(z)x+z^2[z,yd(x)]-zyd(z)zx-z[z,yd(zx)]+xd(z)zyd(z)=0. \tag{2.19}$$

Multiplying (2.18) on the left by z, we have

$$zyd(z)[z,x]+z^2yd(z)x+z^2[z,yd(x)]-zyd(z)zx-z[z,yd(zx)]+zxd(z)yd(z)=0. \tag{2.20}$$

Subtracting (2.19) from (2.20), we arrive at

$$[xd(z),z]yd(z)=0.$$

That is,

$$x[d(z),z]yd(z)+[x,z]d(z)yd(z)=0.$$

Replacing x by rx, $r\in R$, in this equation, we get

$$[r,z]xd(z)yd(z)=0. \tag{2.21}$$

Taking y by yz in this equation, we see that

$$[r,z]xd(z)yzd(z)=0.$$

Multiplying (2.21) on the right by z, we have

$$[r,z]xd(z)yd(z)z=0.$$

If the last two equalities are subtracted, we get

$$[r,z]xd(z)y[d(z),z]=0.$$

Replacing r by $d(z)$, we find that

$$[d(z),z]xd(z)y[d(z),z]=0.$$

Thus

$$d(z)y[d(z),z]xRy[d(z),z]x=(0).$$

Since R is a semiprime ring, we have

$$d(z)y[d(z),z]x=0,$$

and so,

$$[d(z),z]xR[d(z),z]x=(0) \quad \text{for all } x,z\in I.$$

By the semiprimeness of R, we get $[d(z),z]x=0$ for all $x,z\in I$. Therefore, $[d(z),z]R[d(z),z]=(0)$ for all $z\in I$. By the semiprimeness of R, we have $[d(z),z]=0$ for all $z\in I$. By Lemma 2.1 and Lemma 2.6, we get R contains a nonzero central ideal. □

Corollary 2.2. *Let R be a prime ring. Suppose that R admits a nonzero multiplicative reverse derivation d. If d acts as an anti-homomorphism or d acts as a homomorphism on R, then R is a commutative ring.*

Proof. By Theorem 2.1, we obtain that R contains a nonzero central ideal. That is, I is an ideal of R such that $I \subset Z$. We get

$$[x, r] = 0 \quad \text{for all } x \in I,\ r \in R.$$

Replacing x by sx, $s \in R$, in this equation, we obtain that

$$[s, r]x = \quad \text{for all } x \in I,\ r, s \in R.$$

Thus, $[R, R]I = (0)$. Multiplying this equation on the right by $[R, R]$, we have $[R, R]I[R, R] = (0)$. By the primeness of R, we conclude that R is a commutative ring. □

Theorem 2.2. *Let R be a semiprime ring and let I be a nonzero ideal of R. Suppose that R admits a multiplicative (generalized)-reverse derivation F associated with a nonzero reverse derivation d such that one of the following holds:*
(i) $F([x, y]) = [x, y]$ *for all $x, y \in I$, or*
(ii) $F(x \circ y) = (x \circ y)$ *for all $x, y \in I$, or*
(iii) $F([x, y]) = (x \circ y)$ *for all $x, y \in I$, or*
(iv) $F(x \circ y) = [x, y]$ *for all $x, y \in I$.*
Then R contains a nonzero central ideal.

Proof. (i) By hypothesis, we have

$$F([x, y]) = [x, y] \quad \text{for all } x, y \in I.$$

Replacing y by yx in this equation, we get

$$F([x, yx]) = [x, yx].$$

That is,

$$F([x, y]x) = [x, y]x.$$

Since F is a multiplicative (generalized)-reverse derivation, we have

$$xF([x, y]) + d(x)[x, y] = [x, y]x.$$

By hypothesis, we get

$$x[x, y] + d(x)[x, y] = [x, y]x. \tag{2.22}$$

Replacing y by yr, $r \in R$,in the last equation, we obtain that

$$x[x, y]r + xy[x, r] + d(x)[x, y]r + d(x)y[x, r] = y[x, r]x + [x, y]rx.$$

Using (2.22) in this equation, we have

$$xy[x, r] + [x, y]xr + d(x)y[x, r] = y[x, r]x + [x, y]rx.$$

Thus

$$\begin{aligned} 0 &= xy[x, r] + [x, y][x, r] + d(x)y[x, r] - y[x, r]x \\ &= xyxr - xyrx + xy[x, r] - yx[x, r] + d(x)y[x, r] - yxrx + yrxx \\ &= xyxr - xyrx + xyxr - xyrx - yxxr + yxrx + d(x)y[x, r] - yxrx + yrxx \\ &= 2xyxr - 2xyrx - yxxr + yrxx + d(x)y[x, r] \\ &= 2xy[x, r] - yxxr + yrxx + d(x)y[x, r]. \end{aligned}$$

That is,

$$2xy[x, r] + d(x)y[x, r] = yxxr - yrxx \quad \text{for all } x, y \in I,\ r \in R. \tag{2.23}$$

Replacing y by yw, $w \in I$, in the last equation, we find that

$$2xyw[x, r] + d(x)yw[x, r] = ywxxr - ywrxx.$$

Using equation (2.23) in the above equation, we get

$$2xyw[x, r] + d(x)yw[x, r] = 2yxw[x, r] + yd(x)w[x, r].$$

Hence

$$2[x, y]w[x, r] + [d(x), y]w[x, r] = 0 \quad \text{for all } x, y, w \in I, \ r \in R.$$

Replacing y by x in this equation, we get

$$[d(x), x]w[x, r] = 0.$$

Replacing r by $d(x)$ in this equation, we get

$$[d(x), x]w[x, d(x)] = 0 \quad \text{for all } x, w \in I.$$

Thus

$$[d(x), x]IR[d(x), x]I = (0) \quad \text{for all } x \in I.$$

Since R is semiprime, we have

$$[d(x), x]I = (0) \quad \text{for all } x \in I.$$

Therefore,

$$[d(x), x]R[d(x), x] = (0) \quad \text{for all } x \in I.$$

By the semiprimeness of R, we have $[d(x), x] = 0$ for all $x \in I$. By Lemma 2.1 and Lemma 2.6, we get R contains a nonzero central ideal.

(ii) We assume that

$$F(x \circ y) = x \circ y \quad \text{for all } x, y \in I.$$

Replacing y by yx in this equation, we obtain

$$F(x \circ yx) = x \circ yx \quad \text{for all } x, y \in I.$$

Using $x \circ yx = (x \circ y)x$, we have

$$F((x \circ y)x) = (x \circ y)x \quad \text{for all } x, y \in I.$$

Consequently,

$$xF(x \circ y) + d(x)(x \circ y) = (x \circ y)x.$$

Using the hypothesis, we obtain

$$x(x \circ y) + d(x)(x \circ y) = (x \circ y)x. \tag{2.24}$$

Replacing y by yr, $r \in R$, in this equation, we have

$$x(x \circ y)r + xy[r, x] + d(x)(x \circ y)r + d(x)y[r, x] = (x \circ y)rx + y[r, x]x.$$

Using (2.24) in this equation, we have

$$xy[r, x] + (x \circ y)xr + d(x)y[r, x] = (x \circ y)rx + y[r, x]x.$$

Thus

$$\begin{aligned} 0 &= xy[r, x] + (x \circ y)[x, r] + d(x)y[r, x] - y[r, x]x \\ &= -xyxr + xyrx + xy[x, r] + yx[x, r] + d(x)y[r, x] - yrxx + yxrx \\ &= -xyxr + xyrx + xyxr - xyrx + yxxr - yxrx + d(x)y[r, x] - yrxx + yxrx \\ &= yxxr - yrxx + d(x)y[r, x]. \end{aligned}$$

Hence

$$d(x)y[r, x] = -yxxr + yrxx \quad \text{for all } x, y \in I, \ r \in R. \tag{2.25}$$

Replacing y by yw, $w \in I$, in the last equation, we find that

$$d(x)yw[r, x] = -ywxxr + ywrxx.$$

Using equation (2.25), we have

$$d(x)yw[r, x] = yd(x)w[r, x],$$

and so,

$$[d(x),y]w[r,x] = 0 \quad \text{for all } x,y,w \in I,\ r \in R.$$

Replacing r by $d(x)$ and y by x in this equation, we get

$$[d(x),x]w[x,d(x)] = 0.$$

That is,

$$[d(x),x]IR[d(x),x]I = (0) \quad \text{for all } x \in I.$$

Since R is a semiprime ring, we get $[d(x),x]I = (0)$ for all $x \in I$. That is, $[d(x),x]R[d(x),x] = (0)$ for all $x \in I$. By the semiprimeness of R, we have $[d(x),x] = 0$ for all $x \in I$. By Lemma 2.1 and Lemma 2.6, we get that R contains a nonzero central ideal.

(iii) By hypothesis, we have

$$F([x,y]) = x \circ y \quad \text{for all } x,y \in I.$$

Replacing y by yx in this equation, we get

$$F([x,y]x) = (x \circ y)x.$$

Since F is a multiplicative (generalized)-reverse derivation, we have

$$xF([x,y]) + d(x)[x,y] = (x \circ y)x.$$

By hypothesis, we get

$$x(x \circ y) + d(x)[x,y] = (x \circ y)x. \tag{2.26}$$

Replacing y by yr, $r \in R$, in the last equation, we obtain that

$$x(x \circ y)r + xy[r,x] + d(x)[x,y]r + d(x)y[x,r] = (x \circ y)rx + y[r,x]x.$$

Using (2.26) in this equation, we have

$$(x \circ y)xr + xy[r,x] + d(x)y[x,r] = (x \circ y)rx + y[r,x]x.$$

Thus

$$xyxr + yxxr + xyrx - xyxr + d(x)y[x,r] = xyrx + yxrx + yrxx - yxrx,$$

and so,

$$yxxr + d(x)y[x,r] = yrxx.$$

Using the same arguments as above, after equation (2.25) in the proof of Theorem 2.2 (ii), we get that R contains a nonzero central ideal.

(iv) We assume that

$$F(x \circ y) = [x,y] \quad \text{for all } x,y \in I.$$

Replacing y by yx in this equation, we obtain

$$F((x \circ y)x) = [x,y]x \quad \text{for all } x,y \in I.$$

Thus

$$xF(x \circ y) + d(x)(x \circ y) = [x,y]x.$$

Using the hypothesis, we obtain

$$x[x,y] + d(x)(x \circ y) = [x,y]x. \tag{2.27}$$

Replacing y by yr, $r \in R$ in this equation, we have

$$x[x,y]r + xy[x,r] + d(x)(x \circ y)r + d(x)y[r,x] = [x,y]rx + y[x,r]x.$$

Using (2.27) in this equation, we have

$$[x,y]xr + xy[x,r] + d(x)y[r,x] = [x,y]rx + y[x,r]x.$$

Hence

$$\begin{aligned}0 &= xyxr - yxxr + xyxr - xyrx + d(x)y[r,x] - xyrx + yxrx - yxrx + yrxx\\ &= 2xyxr - 2xyrx - yxxr + yrxx + d(x)y[r,x]\\ &= 2xyxr - 2xyrx + d(x)y[r,x] - yxxr + yrxx.\end{aligned}$$

Therefore,

$$2xy[x,r] + d(x)y[r,x] = yxxr - yrxx \quad \text{for all } x,y \in I,\ r \in R. \tag{2.28}$$

Replacing y by yw, $w \in I$, in the last equation, we find that

$$2xyw[x,r] + d(x)yw[r,x] = ywxxr - ywrxx.$$

Using equation (2.28) in the above equation, we get

$$2xyw[x,r] + d(x)yw[r,x] = 2yxw[x,r] + yd(x)w[r,x].$$

This implies that

$$2[x,y]w[x,r] + [d(x),y]w[r,x] = 0.$$

Replacing y by x in this equation, we get

$$[d(x),x]w[r,x] = 0 \quad \text{for all } x,w \in I,\ r \in R.$$

Replacing r by $d(x)$ in this equation, we get

$$[d(x),x]w[x,d(x)] = 0 \quad \text{for all } x,w \in I.$$

That is,

$$[d(x),x]IR[d(x),x]I = (0) \quad \text{for all } x \in I.$$

By the semiprimeness of R, we have $[d(x),x]I = (0)$ for all $x \in I$. That is, $[d(x),x]R[d(x),x] = (0)$ for all $x \in I$. Since R is semiprime, we get $[x,d(x)] = 0$ for all $x \in I$. By Lemma 2.1 and Lemma 2.6, we get that R contains a nonzero central ideal. □

Corollary 2.3. *Let R be a prime ring. Suppose that R admits a nonzero multiplicative reverse derivation d such that one of the following holds:*
(i) $d([x,y]) = [x,y]$ *for all* $x,y \in R$, *or*
(ii) $d(x \circ y) = (x \circ y)$ *for all* $x,y \in R$, *or*
(iii) $d([x,y]) = (x \circ y)$ *for all* $x,y \in R$, *or*
(iv) $d(x \circ y) = [x,y]$ *for all* $x,y \in R$.
Then R a is commutative ring.

3 Applications on reverse derivation in Banach algebras

In this section, we consider the conditions in which every continuous reverse derivation in a Banach algebra maps to a radical, the purely algebraic results given in the preceding section. As we have mentioned earlier, Thomas generalized the Singer–Wermer theorem by proving that any derivation in the commutative Banach algebra maps the algebra to the radical. In this context, there arises a question whether the theorem can be proven without any commutativity property.

Moreover, in this section, the subject of continuous reverse derivations is discussed on Banach algebras for the first time.

Theorem 3.1. *Let A be a non-commutative Banach algebra,* $\operatorname{rad}(A)$ *be a Jacobson radical. Suppose that R admits a nonzero continuous reverse derivation d such that for all $x,y \in A$, one of the following holds:*
(i) $d(xy) \pm d(x)d(y) \in \operatorname{rad}(A)$, *or*
(ii) $d(xy) \pm d(y)d(x) \in \operatorname{rad}(A)$, *or*
(iii) $d([x,v]) \pm [x,y] \in \operatorname{rad}(A)$, *or*

(iv) $d(x \circ y) \pm (x \circ y) \in \operatorname{rad}(A)$, *or*
(v) $d([x, y]) \pm (x \circ y) \in \operatorname{rad}(A)$, *or*
(vi) $d(x \circ y) \pm [x, y] \in \operatorname{rad}(A)$.
Then $d(A) \subseteq \operatorname{rad}(A)$.

Proof. (i) Let $\mathbb{J}$ be a primitive ideal of A. By Corollary 2.1, we obtain that d is a continuous derivation on A. By Lemma 2.2, we see that d leaves the primitive ideals invariant. Therefore, we obtain that $d(\mathbb{J}) \subseteq \mathbb{J}$. Since the Jacobson radical is the intersection of all primitive ideals, we have $d(\operatorname{rad}(\mathbb{J})) \subseteq \operatorname{rad}(\mathbb{J})$ and there is no loss of generality in assuming that A is semisimple. Denote $A/\mathbb{J}$ and a nonzero derivation such that

$$d_{\mathbb{J}} : A/\mathbb{J} \to A/\mathbb{J}, \quad d_{\mathbb{J}}(\overline{y}) = d_{\mathbb{J}}(y + \mathbb{J}) = d(y) + \mathbb{J}$$

for all $y \in A$ and $\overline{y} = y + \mathbb{J}$, where $A/\mathbb{J}$ is a factor Banach algebra. By Lemma 2.3, we obtain that every derivation on a semisimple Banach algebra is continuous. Let $A/\mathbb{J}$ be commutative. By Lemma 2.4, we see that $d_{\mathbb{J}} = 0$, since $A/\mathbb{J}$ is semisimple. If we show that $d_{\mathbb{J}} = 0$ even when $A/\mathbb{J}$ is not commutative, we generally get $d_{\mathbb{J}} = 0$. Since $\mathbb{J}$ is a primitive ideal, the factor algebra $A/\mathbb{J}$ is primitive and so it is prime. By hypothesis, we get

$$d(xy) \pm d(x)d(y) \in \operatorname{rad}(A) \quad \text{for all } x, y \in A.$$

From the above expression we obtain the following equation:

$$d_{\mathbb{J}}(\overline{xy}) \pm d_{\mathbb{J}}(\overline{x})d_{\mathbb{J}}(\overline{y}) = \overline{0} \quad \text{for all } \overline{x}, \overline{y} \in A/\mathbb{J}.$$

Using the same arguments as in the proof of Theorem 2.1, we arrive at

$$[d_{\mathbb{J}}(\overline{y}), \overline{y}](A/\mathbb{J})[\overline{y}, d_{\mathbb{J}}(\overline{y})] = (\overline{0}) \quad \text{for all } \overline{y} \in A/\mathbb{J}.$$

Since $A/\mathbb{J}$ is prime, we get

$$[\overline{y}, d_{\mathbb{J}}(\overline{y})] = \overline{0} \quad \text{for all } \overline{y} \in A/\mathbb{J}.$$

By Lemma 2.5, we conclude that $d_{\mathbb{J}} = 0$ on $A/\mathbb{J}$. Therefore, in both cases $d_{\mathbb{J}} = 0$ is obtained. Thus, we conclude that $d(A) \subseteq A$ for any primitive ideal $\mathbb{J}$. Since $\operatorname{rad}(A)$ is the intersection of all primitive ideals in A, we find that $d(A) \subseteq \operatorname{rad}(A)$. This completes the proof.

(ii)–(vi) Using Theorem 3.1 (i) in other conditions, the following is obtained: for all $\overline{x}, \overline{y} \in A/\mathbb{J}$,

$$\begin{aligned}
d_{\mathbb{J}}(\overline{xy}) \pm d_{\mathbb{J}}(\overline{y})d_{\mathbb{J}}(\overline{x}) &\in \operatorname{rad}(A),\\
d_{\mathbb{J}}([\overline{x}, \overline{y}]) \pm [\overline{x}, \overline{y}] &\in \operatorname{rad}(A),\\
d_{\mathbb{J}}(\overline{x} \circ \overline{y}) \pm (\overline{x} \circ \overline{y}) &\in \operatorname{rad}(A),\\
d_{\mathbb{J}}([\overline{x}, \overline{y}]) \pm (\overline{x} \circ \overline{y}) &\in \operatorname{rad}(A),\\
d_{\mathbb{J}}(\overline{x} \circ \overline{y}) \pm [\overline{x}, \overline{y}] &\in \operatorname{rad}(A).
\end{aligned}$$

By Theorem 2.1 and Theorem 2.2, we get

$$[\overline{y}, d(\overline{y})] = \overline{0} \quad \text{for all } \overline{y} \in A/\mathbb{J}.$$

By Lemma 2.5, we conclude that $d_{\mathbb{J}} = 0$ on $\tilde{A}/\mathbb{J}$. Therefore, in both cases $d_{\mathbb{J}} = 0$ is obtained. Thus, we conclude that $d(A) \subseteq A$ for any primitive ideal $\mathbb{J}$. Since $\operatorname{rad}(A)$ is the intersection of all primitive ideals in A, we find that $d(A) \subseteq \operatorname{rad}(A)$. □

Example 3.1. Suppose the ring

$$R = \left\{ \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix} : a, b, c, d, e \in \mathbb{R} \right\}.$$

Define maps $F, d : R \to R$ as follows:

$$F\left(\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 & b+e \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$d\left(\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 0 & b-d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then it is easy to verify that F is a multiplicative (generalized)-reverse derivation on R. It is obvious that R is not a semiprime ring. Let

$$I = \left\{ \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

Then I is a nonzero right ideal of R and
(i) F acts as an anti-homomorphism,
(ii) F acts as an homomorphism,
(iii) $F([x, y]) = [x, y]$,
(iv) $F(x \circ y) = (x \circ y)$,
(v) $F([x, y]) = (x \circ y)$,
(vi) $F(x \circ y) = [x, y]$ for all $x, y \in I$.
However, R is not commutative.

In conclusion, it is tempting to conjecture as follows.

Conjecture 3.1. Let A be a non-commutative Banach algebra, $\mathrm{rad}(A)$ a Jacobson radical, m, n be fixed positive integers. Suppose that R admits a nonzero continuous reverse derivation d such that for all $x, y \in A$, one of the following holds:
(i) $d([x, y]_m) \pm [x, y]_n \in \mathrm{rad}(A)$, or
(iv) $d(x \circ_m y) \pm (x \circ_n y) \in \mathrm{rad}(A)$, or
(v) $d([x, y]_m) \pm (x \circ y)_n \in \mathrm{rad}(A)$, or
(vi) $d(x \circ_m y) \pm [x, y]_n \in \mathrm{rad}(A)$.
Then $d(A) \subseteq \mathrm{rad}(A)$.

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