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ORIGINAL RESEARCH

Note on Lie ideals with symmetric bi-derivations in semiprime rings

Emine Koç Sögütcü · Shuliang Huang

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Abstract Let R be a semiprime ring, U a square-closed Lie ideal of R and $D : R \times R \to R$ a symmetric bi-derivation and d be the trace of D. In the present paper, we prove that the R contains a nonzero central ideal if any one of the following holds: i) $d(x)y \pm xg(y) \in Z$, ii)$[d(x), y] = \pm[x, g(y)]$, iii) $d(x) \circ y = \pm x \circ g(y)$, iv) $[d(x), y] = \pm x \circ g(y)$, v) $d([x, y]) = [d(x), y] + [d(y), x]$, vi) $d(xy) \pm xy \in Z$, vii) $d(xy) \pm yx \in Z$, viii) $d(xy) \pm [x, y] \in Z$, ix) $d(xy) \pm x \circ y \in Z$, x) $g(xy) + d(x)d(y) \pm xy \in Z$, xi) $g(xy) + d(x)d(y) \pm yx \in Z$, xii) $g([x, y]) + [d(x), d(y)] \pm [x, y] \in Z$, xiii) $g(x \circ y) + d(x) \circ d(y) \pm x \circ y \in Z$, for all $x, y \in U$, where $G : R \times R \to R$ is symmetric bi-derivation such that g is the trace of G.

Keywords Semiprime ring · Lie ideal · Derivation · Bi-derivation · Symmetric bi-derivation.

Mathematics subject classification 16W25 · 16W10 · 16U80 · 16N60

1 Introduction

Throughout R will represent an associative ring with center Z. A ring R is said to be prime if $xRy = (0)$ implies that either $x = 0$ or $y = 0$ and semiprime if $xRx = (0)$ implies that $x = 0$, where $x, y \in R$. A prime ring is obviously semiprime. But the reverse is not always true. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $x \circ y$ stands for the anti-commutator $xy + yx$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$, for all $u \in U$, $r \in R$. U is called a square-closed Lie ideal of R if U is a Lie ideal and $u^2 \in U$ for all $u \in U$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. The concept of bi-derivation was introduced by Maksa in [7]. It is shown in [8] that symmetric bi-derivations are related to general solution of some functional equations. A mapping $D(., .) : R \times R \to R$ is said to be symmetric if $D(x, y) = D(y, x)$ for all $x, y \in R$. A mapping $d : R \to R$ is called the trace of $D(., .)$ if $d(x) = D(x, x)$ for all $x \in R$. It is obvious that if $D(., .)$ is bi-additive (i.e., additive in both arguments), then the trace d of $D(., .)$ satisfies the identity $d(x + y) = d(x) + d(y) + 2D(x, y)$, for all $x, y \in R$. If $D(., .)$ is bi-additive and satisfies the identities

$$D(xy, z) = D(x, z)y + xD(y, z)$$

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E. K. Sögütcü
Department of Mathematics, Faculty of Science, Cumhuriyet University, Sivas, Turkey
E-mail: eminekoc@cumhuriyet.edu.tr

S. Huang (✉)
School of Mathematics and Finance, Chuzhou University, Chuzhou 239000, China
E-mail: shulianghuang@163.com

and

$$D(x, yz) = D(x, y)z + yD(x, z)$$

for all $x, y, z \in R$. Then $D(., .)$ is called a symmetric bi-derivation.

Example 1 Suppose the ring $R = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$. Define map $D : R \times R \to R$ as follows:

$$D\left(\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}, \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ ac & 0 \end{pmatrix}.$$

Then it is easy to verify that D is a symmetric bi-derivation of R.

Ashraf and Rehman showed that R is prime ring with a nonzero ideal U of R and d is a derivation of R such that $d(xy) \pm xy \in Z$, for all $x, y \in U$, then R is commutative in [2]. Ashraf et al. proved this result for a generalized derivation of R in [3]. Further, Ali and Boua [1] proved the following result: Let R be a semiprime ring, I a non-zero ideal of R and F, G be two multiplicative generalized derivations of R satisfying $G(xy) + F(x)F(y) - xy = 0$ or $G(xy) + F(x)F(y) - yx = 0$ for all $x, y \in R$. In [6], Koç Sögütcü and Gölbaşı have proved that if R is a 2-torsion-free semiprime ring and $(F, d), (G, h) : R \to R$ are two generalized derivations on R such that $[F(u), v] = \pm[u, G(v)]$ or $F(u) \circ v = \pm u \circ G(v)$ or $[F(u), v] = \pm u \circ G(v)$ or $F([u, v]) = [F(u), v] + [d(v), u]$ for all $u, v \in U$, then h is commuting mapping.

We extend some well known results concerning Lie ideals in semiprime rings to a symmetric bi-derivations. Throughout the present paper, we shall make use of the following basic identities without any specific mention:
(i) $[x, yz] = y[x, z] + [x, y]z$
(ii) $[xy, z] = [x, z]y + x[y, z]$
(iii) $xyoz = (xoz)y + x[y, z] = x(yoz) - [x, z]y$
(iv) $xoyz = y(xoz) + [x, y]z = (xoy)z + y[z, x]$.

2 The results

Lemma 1 [5, Theorem 1.3] *Let R be a 2-torsion free semiprime ring and U a noncentral Lie ideal of R such that $u^2 \in U$ for all $u \in U$. Then there exist a nonzero ideal I of R such that $I \subseteq U$.*

Lemma 2 [4, Lemma 2 (b)] *If R is a semiprime ring, then the center of a nonzero ideal of R is contained in the center of R.*

Lemma 3 *Let R be a 2-torsion-free semiprime ring and I a nonzero ideal of R. If $[I, I] \subset Z$, then R contains a nonzero central ideal.*

Proof By the hypothesis, we get

$$[x, y] \in Z, \text{ for all } x, y \in I.$$

Replacing y by yx in above expression, we have

$$[x, y]x \in Z, \text{ for all } x, y \in I.$$

Commuting this term with $r, r \in R$, we obtain that

$$[[x, y]x, r] = 0, \text{ for all } x, y \in I, r \in R.$$

Using the hypothesis in the last expression, we get

$$[x, y][x, r] = 0, \text{ for all } x, y \in I, r \in R.$$

Replacing r by ry in the above equation and using this expression, we see that

$$[x, y]R[x, y] = (0), \text{ for all } x, y \in I.$$

Since R is semiprime ring, we get

$$[x, y] = 0, \text{ for all } x, y \in I.$$

That is, $[I, I] = (0)$. By Lemma 2, we get $I \subseteq Z$. Thus, R contains a nonzero central ideal. This completes the proof. □

Lemma 4 *Let R be a 2-torsion free semiprime ring and I a nonzero ideal of R. If $I \circ I \subset Z$, then R contains a nonzero central ideal.*

Proof We get

$$x \circ y \in Z, \text{ for all } x, y \in I.$$

Replacing y by yx in the last expression, we obtain that

$$(x \circ y)\, x \in Z, \text{ for all } x, y \in I.$$

This implies that

$$[(x \circ y)\, x, r] = 0, \text{ for all } x, y \in I,$$

and so

$$(x \circ y)[x, r] = 0, \text{ for all } x, y \in I, r \in R.$$

Replacing y by yt, $t \in R$ in the above expression and using this, we get

$$[x, y]t[x, r] = 0, \text{ for all } x, y \in I, r, t \in R.$$

Replacing r by y in this equation, we have

$$[x, y]R[x, y] = (0), \text{ for all } x, y \in I.$$

Since R is semiprime ring, we get

$$[x, y] = 0, \text{ for all } x, y \in I.$$

By Lemma 2, we have, $I \subseteq Z$. We conclude that R contains a nonzero central ideal. This completes the proof. □

Theorem 1 *Let R be a 2-torsion free semiprime ring, U a square-closed Lie ideal of R and $D : R \times R \to R$, $G : R \times R \to R$ two symmetric bi-derivations where d is the trace of D and g is the trace of G where $UD(U, U) \neq (0)$. If $d(x)\, y \pm xg(y) \in Z$, for all $x, y \in U$, then R contains a nonzero central ideal.*

Proof By Lemma 1, there exist a nonzero ideal I of R such that $I \subseteq U$. By the hypothesis, we have

$$d(x)\, y \pm xg(y) \in Z, \text{ for all } x, y \in I.$$

Writing y by $y + z$, $z \in I$, we have

$$d(x)\, y + d(x)\, z \pm xg(y) \pm xg(z) \pm 2xG(y, z) \in Z.$$

Using the hypothesis, we get

$$2xG(y, z) \in Z.$$

Since R is 2-torsion free and replacing z by y, we have

$$xG(y, y) \in Z, \text{ for all } x, y \in I.$$

Thus,

$$xg(y) \in Z, \text{ for all } x, y \in I.$$

By the hypothesis, we get

$$d(x)\, y \in Z, \text{ for all } x, y \in I.$$

Commuting this term with r, $r \in R$, we obtain that

$$[d(x)\, y, r] = 0, \text{ for all } x, y \in I, r \in R,$$

and so

$$d(x)[y,r]+[d(x),r]y=0,\ \text{for all } x,y\in I,\ r\in R. \tag{2.1}$$

Replacing y by yz, $z\in I$, we have

$$d(x)[y,r]z+d(x)y[z,r]+[d(x),r]yz=0,\ \text{for all } x,y,z\in I,\ r\in R.$$

Using (2.1) equation, we have

$$d(x)y[z,r]=0,\ \text{for all } x,y,z\in I,\ r\in R.$$

Replacing r by $d(x)$ in this equation, we find that

$$d(x)y[z,d(x)]=0,\ \text{for all } x,y,z\in I. \tag{2.2}$$

Multiplying this equation on the left by z, we get

$$zd(x)y[z,d(x)]=0,\ \text{for all } x,y,z\in I.$$

Taking y by zy in equation (2.2), we find that

$$d(x)zy[z,d(x)]=0,\ \text{for all } x,y,z\in I.$$

Subctracting two last equations, we arrive at

$$[z,d(x)]y[z,d(x)]=0,\ \text{for all } x,y,z\in I.$$

That is,

$$[z,d(x)]yR[z,d(x)]y=(0),\ \text{for all } x,y,z\in I.$$

Since R is semiprime ring, we get

$$[z,d(x)]y=0,\ \text{for all } x,y,z\in I.$$

Replacing y by $r[z,d(x)]$, $r\in R$ in the last equation, we have

$$[z,d(x)]r[z,d(x)]=0,\ \text{for all } x,y,z\in I,\ r\in R.$$

Again, since R is semiprime ring, we have

$$[z,d(x)]=0,\ \text{for all } x,z\in I. \tag{2.3}$$

By Lemma 2, we have

$$d(x)\in Z,\ \text{for all } x\in I.$$

Replacing x by $x+y$, $y\in I$ in above expression, we get

$$d(x)+d(y)+2D(x,y)\in Z,\ \text{for all } x,y\in I.$$

Using $d(x)\in Z$, for all $x\in I$ in this equation, we have

$$2D(x,y)\in Z,\ \text{for all } x,y\in I.$$

Since R is 2-torsion free, we have

$$D(x,y)\in Z,\ \text{for all } x,y\in I. \tag{2.4}$$

Commuting this term with r, $r\in R$, we get

$$[D(x,y),r]=0,\ \text{for all } x,y\in I,\ r\in R.$$

Taking x by xs, $s \in R$ in the last equation, we obtain that

$$[sD(x, y) + D(s, y)x, r] = 0, \text{ for all } x, y \in I, \ r, s \in R.$$

Using equation (2.4), we get

$$[s, r]D(x, y) + D(s, y)[x, r] + [D(s, y), r]x = 0, \text{ for all } x, y \in I, \ r, s \in R.$$

Replacing s by x in the last equation, we get

$$[x, r]D(x, y) + D(x, y)[x, r] + [D(x, y), r]x = 0, \text{ for all } x, y \in I, \ r, s \in R.$$

Appliying equation (2.4), we see that

$$2[x, r]D(x, y) = 0, \text{ for all } x, y \in I, \ r \in R.$$

Since R is 2-torsion free, we get

$$[x, r]D(x, y) = 0, \text{ for all } x, y \in I, \ r \in R.$$

Using $D(x, y) \in Z$, we have

$$[x, r]tD(x, y) = 0, \text{ for all } x, y \in I, \ r, t \in R.$$

Since R is semiprime, we must contain a family $\wp = \{P_\alpha | \alpha \in \Lambda\}$ of prime ideals such that $\cap P_\alpha = (0)$. If P is a typical member of $\wp$ and $x \in I$, we get

$$[x, R] \subseteq P \text{ or } D(x, y) \subseteq P, \text{ for all } y \in I$$

by Fact (ii). Define two additive subgroups $A = \{x \in I | [x, R] \subseteq P\}$ and $B = \{x \in I | D(x, y) \subseteq P$, for all $y \in I\}$. It is clear that $I = A \cup B$. Since a group cannot be a union of two of its subgroups, either $A = I$ or $B = I$, and so, we have $[I, R] \subseteq P$ or $D(I, I) \subseteq P$.Thus both cases together yield $[I, R]D(I, I) \subseteq P$, for any $P \in \wp$. Therefore $[I, R]D(I, I) \subseteq \cap P_\alpha = (0)$ and so $[I, R]D(I, I) = (0)$. That is, $[RID(I, I)R, R]D(I, I) = (0)$. This implies that $[J, R]RJ = (0)$ where $J = RID(I, I)R$ is a nonzero ideal of R by Theorem 1. Then $[J, R]R[J, R] = (0)$. By the semiprimeness of R, we get $[R, J] = (0)$, and so $J \subseteq Z$. We conclude that R contains a nonzero central ideal. □

Theorem 2 *Let R be a 2-torsion free semiprime ring and U a square-closed Lie ideal of R. Suppose that R admits two symmetric bi-derivations $D : R \times R \to R$, $G : R \times R \to R$ where d is the trace of D and g is the trace of G where $UD(U, U) \neq (0)$ such that*

(i) $[d(x), y] = \pm[x, g(y)]$, *for all $x, y \in U$, or*
(ii) $d(x) \circ y = \pm x \circ g(y)$, *for all $x, y \in U$, or*
(iii) $[d(x), y] = \pm x \circ g(y)$, *for all $x, y \in U$, or*
(iv) $d([x, y]) = [d(x), y] + [d(y), x]$, *for all $x, y \in U$. Then R contains a nonzero central ideal.*

Proof i) By Lemma 1, there exist a nonzero ideal I of R such that $I \subseteq U$. We have

$$[d(x), y] = \pm[x, g(y)], \text{ for all } x, y \in I.$$

Substituting $y + z$, $z \in I$ for y in the hypothesis, we obtain that

$$[d(x), y] + [d(x), z] = \pm[x, g(y)] \pm [x, g(z)] \pm 2[x, G(y, z)].$$

Using the hypothesis and 2-torsion freeness of R, we find that

$$[x, G(y, z)] = 0, \text{ for all } x, y, z \in I.$$

Writing z by y in last equation, we have

$$[x, g(y)] = 0, \text{ for all } x, y \in I.$$

Using the hypothesis in the last relation gives

$$[d(x), y] = 0, \text{ for all } x, y \in I.$$

Using the same arguments in the proof equation (2.3), we find that R contains a nonzero central ideal. We complete the proof.

(ii) By Lemma 1, there exist a nonzero ideal I of R such that $I \subseteq U$. By the hypothesis, we get

$$d(x) \circ y = \pm (x \circ g(y)), \text{ for all } x, y \in I.$$

Replacing y by $y + z$, $z \in I$, we obtain that

$$d(x) \circ y + d(x) \circ z = \pm (x \circ g(y)) \pm (x \circ g(z)) \pm 2(x \circ G(y, z)).$$

Using the hypothesis and 2-torsion freeness of R, we see that

$$(x \circ G(y, z)) = 0, \text{ for all } x, y, z \in I.$$

Taking z by y in this equation, we have

$$x \circ g(y) = 0, \text{ for all } x, y \in I.$$

By the hypothesis, we get

$$d(x) \circ y = 0, \text{ for all } x, y \in I.$$

Replacing y by yz, $z \in I$, we see that

$$y[z, d(x)] = 0, \text{ for all } x, y, z \in I.$$

Replacing y by $[z, d(x)]r$ in the last equation, we have

$$[z, d(x)]r[z, d(x)] = 0, \text{ for all } x, y, z \in I, \ r \in R.$$

Since R is semiprime ring, we have

$$[z, d(x)] = 0, \text{ for all } x, z \in I.$$

The rest of the proof is the same as equation (2.3). This completes proof.

(iii) By Lemma 1, there exist a nonzero ideal I of R such that $I \subseteq U$. We get

$$[d(x), y] = \pm x \circ g(y), \text{ for all } x, y \in I.$$

Taking y by $y + z$, $z \in I$, we see that

$$[d(x), y] + [d(x), z] = \pm x \circ g(y) \pm x \circ g(y) \pm 2x \circ G(y, z).$$

By the hypothesis, we see that

$$2(x \circ G(y, z)) = 0, \text{ for all } x, y, z \in I.$$

Since R is 2-torsion free, we have

$$x \circ G(y, z) = 0, \text{ for all } x, y, z \in I.$$

Replacing z by y in the above equation, we have

$$x \circ g(y) = 0, \text{ for all } x, y \in I.$$

Using the hypothesis, we get

$$[d(x), y] = 0, \text{ for all } x, y \in I.$$

This equation is the same as equation (2.3). Using the same arguments in the proof of Theorem 1, we find the required result.

iv) By Lemma 1, there exist a nonzero ideal I of R such that $I \subseteq U$. By the hypothesis, we get

$$d([x, y]) = [d(x), y] + [d(y), x], \text{ for all } x, y \in I.$$

Writing y by $y + z$, $z \in I$ in this equation, we obtain that

$$\begin{aligned} &d([x, y]) + d([x, z]) + 2D([x, y], [x, z]) \\ &= [d(x), y] + [d(x), z] + [d(y), x] + [d(z), x] + 2[D(y, z), x]. \end{aligned}$$

Using the hypothesis and 2-torsion freeness of R, we see that

$$D([x, y], [x, z]) = [D(y, z), x], \text{ for all } x, y, z \in I.$$

Replacing z by y in the last equation, we have

$$d([x, y]) = [d(y), x], \text{ for all } x, y \in I.$$

Using the hypothesis, we see that

$$[d(x), y] = 0, \text{ for all } x, y \in I.$$

The rest of the proof is the same as equation (2.3). This completes proof. □

Theorem 3 *Let R be a 2-torsion free semiprime ring and U a square-closed Lie ideal of R. Suppose that R admits a symmetric bi-derivation $D : R \times R \to R$ where d is the trace of D such that*

(i) $d(xy) \pm xy \in Z$, *for all* $x, y \in U$, *or*
(ii) $d(xy) \pm yx \in Z$, *for all* $x, y \in U$, *or*
(iii) $d(xy) \pm [x, y] \in Z$, *for all* $x, y \in U$, *or*
(iv) $d(xy) \pm x \circ y \in Z$, *for all* $x, y \in U$. *Then R contains a nonzero central ideal.*

Proof (i) By Lemma 1, there exist a nonzero ideal I of R such that $I \subseteq U$. By the hypothesis, we get

$$d(xy) \pm xy \in Z, \text{ for all } x, y \in I.$$

Taking y by $y + z$, $z \in I$ in the hypothesis, we see that

$$d(xy) + d(xz) + 2D(xy, xz) \pm xy \pm xz \in Z.$$

Since R is 2-torsion free and using the hypothesis, we arrive at

$$D(xy, xz) \in Z, \text{ for all } x, y \in I.$$

Substituting y for z in the last equation, we have

$$d(xy) \in Z, \text{ for all } x, y \in I.$$

Using the hypothesis, we find

$$xy \in Z, \text{ for all } x, y \in I. \tag{2.5}$$

Commuting this term with r, $r \in R$, we get

$$[xy, r] = 0, \text{ for all } x, y \in I, r \in R,$$

and so

$$[x, r]y + x[y, r] = 0, \text{ for all } x, y \in I, r \in R. \tag{2.6}$$

Replacing y by yz in this equation and using equation (2.6), we have

$$xy[z, r] = 0, \text{ for all } x, y, z \in I, r \in R.$$

Writting x by $[z, r]$, we arrive at

$$[z, r]y[z, r] = 0, \text{ for all } y, z \in I, r \in R.$$

That is,

$$[z, r]yR[z, r]y = (0), \text{ for all } y, z \in I, r \in R.$$

Since R is semiprime ring, we have

$$[z, r]y = 0, \text{ for all } y, z \in I, r \in R.$$

Taking y by $t[z, r]$, $t \in R$, we see that

$$[z, r]R[z, r] = (0), \text{ for all } y, z \in I, r \in R.$$

By the semiprime of R, we conclude that $I \subset Z$. Thus, R contains a nonzero central ideal. This completes the proof.

(ii) By Lemma 1, there exists a nonzero ideal I of R such that $I \subseteq U$. We get

$$d(xy) \pm yx \in Z, \text{ for all } x, y \in I.$$

Taking y by $y + z$, $z \in I$, we have

$$d(xy) + d(xz) + 2D(xy, xz) \pm (y + z)x \in Z, \text{ for all } x, y, z \in I.$$

Using the hypothesis, we obtain

$$2D(xy, xz) \in Z, \text{ for all } x, y, z \in I.$$

Since R is 2-torsion free, we have

$$D(xy, xz) \in Z, \text{ for all } x, y, z \in I.$$

Replacing z by y in the last expression, we get

$$d(xy) \in Z, \text{ for all } x, y \in I.$$

By the hypothesis, we get

$$yx \in Z, \text{ for all } x, y \in I.$$

Using the same arguments after (2.5) in the proof of Theorem 3 (i), we get the required results.

(iii) By Lemma 1, there exists a nonzero ideal I of R such that $I \subseteq U$. We get

$$d(xy) \pm [x, y] \in Z, \text{ for all } x, y \in I.$$

Replacing y by $y + z$, $z \in I$, we have

$$d(xy) + d(xz) + 2D(xy, xz) \pm [x, y] \pm [x, z] \in Z, \text{ for all } x, y, z \in I.$$

By the hypothesis, we obtain that

$$2D(xy, xz) \in Z, \text{ for all } x, y, z \in I.$$

Since R is 2-torsion free, we have

$$D(xy, xz) \in Z, \text{ for all } x, y, z \in I.$$

Replacing z by y in the last expression, we get

$$d(xy) \in Z, \text{ for all } x, y \in I.$$

Again by the hypothesis, we get

$$[x, y] \in Z, \text{ for all } x, y \in I.$$

That is, $[I, I] \subset Z$. By Lemma 3, we obtain that R contains a nonzero central ideal.

(iv) By Lemma 1, there exists a nonzero ideal I of R such that $I \subseteq U$. We see that

$$d(xy) \pm x \circ y \in Z, \text{ for all } x, y \in I.$$

Taking y by $y + z$, $z \in I$, we have

$$d(xy) + d(xz) + 2D(xy, xz) \pm x \circ y \pm x \circ z \in Z, \text{ for all } x, y, z \in I.$$

By the hypothesis, we obtain

$$2D(xy, xz) \in Z, \text{ for all } x, y, z \in I.$$

Since R is 2-torsion free, we have

$$D(xy, xz) \in Z, \text{ for all } x, y, z \in I.$$

Writting z by y in the last expression, we get

$$d(xy) \in Z, \text{ for all } x, y \in I.$$

By the hypothesis, we get

$$x \circ y \in Z, \text{ for all } x, y \in I.$$

That is, $I \circ I \subset Z$. We conclude that R contains a nonzero central ideal. by Lemma 4. The proof is completed. □

Theorem 4 *Let R be a 2-torsion free semiprime ring and U a square-closed Lie ideal of R. Suppose that R admits two symmetric bi-derivations $D : R \times R \to R$, $G : R \times R \to R$ where d is the trace of D and g is the trace of G such that*

(i) $g(xy) + d(x)d(y) \pm xy \in Z$, *for all* $x, y \in U$, *or*
(ii) $g(xy) + d(x)d(y) \pm yx \in Z$, *for all* $x, y \in U$, *or*
(iii) $g([x, y]) + [d(x), d(y)] \pm [x, y] \in Z$, *for all* $x, y \in U$, *or*
(iv) $g(x \circ y) + d(x) \circ d(y) \pm x \circ y \in Z$, *for all* $x, y \in U$. *Then R contains a nonzero central ideal.*

Proof (i) By Lemma 1, there exist a nonzero ideal I of R such that $I \subseteq U$. By the hypothesis, we get

$$g(xy) + d(x)d(y) \pm xy \in Z, \text{ for all } x, y \in I.$$

Replacing y by $y + z$, $z \in I$, we arrive at

$$g(xy) + g(xz) + 2G(xy, xz) + d(x)(d(y) + 2D(y, z) + d(z)) \pm xy \pm xz \in Z.$$

Using the hypothesis and 2-torsion freeness of R, we have

$$G(xy, xz) + d(x)D(y, z) \in Z, \text{ for all } x, y \in I.$$

Writing z by y in this expression, we have

$$g(xy) + d(x)d(y) \in Z, \text{ for all } x, y \in I.$$

Using the hypothesis, we get

$$xy \in Z, \text{ for all } x, y \in I.$$

Using the same arguments after (2.5) in the proof of Theorem 3 (i), we get the required results.

(ii) By Lemma 1, there exists a nonzero ideal I of R such that $I \subseteq U$. We get

$$g(xy) + d(x)d(y) \pm yx \in Z, \text{ for all } x, y \in I.$$

Taking y by $y+z$, $z \in I$, we have

$$g(xy) + g(xz) + 2G(xy, xz) + d(x)(d(y) + 2D(y, z) + d(z)) \pm (y+z)x \in Z.$$

Using the hypothesis, we obtain

$$2G(xy, xz) + 2d(x)D(y, z) \in Z, \text{ for all } x, y, z \in I.$$

Since R is 2-torsion free, we have

$$G(xy, xz) + d(x)D(y, z) \in Z, \text{ for all } x, y, z \in I.$$

Replacing z by y in the last expression, we get

$$g(xy) + d(x)d(y) \in Z, \text{ for all } x, y \in I.$$

By the hypothesis, we get

$$yx \in Z, \text{ for all } x, y \in I.$$

Using the same techniques in the proof of Theorem 3 (i), we can prove that R contains a nonzero central ideal.

(iii) By Lemma 1, there exists a nonzero ideal I of R such that $I \subseteq U$. We get

$$g([x, y]) + [d(x), d(y)] \pm [x, y] \in Z, \text{ for all } x, y \in I.$$

Writing y by $y+z$, $z \in I$ in the hypothesis, we have

$$\begin{aligned}&g([x, y]) + g([x, y]) + 2G([x, y], [x, z]) + [d(x), d(y)]\\&+2[d(x), D(y, z)] + [d(x), d(z)] \pm [x, y] \pm [x, z] \in Z.\end{aligned}$$

Using the hypothesis, we obtain that

$$2G([x, y], [x, z]) + 2[d(x), D(y, z)] \in Z, \text{ for all } x, y, z \in I.$$

Since R is 2-torsion free, we have

$$G([x, y], [x, z]) + [d(x), D(y, z)] \in Z, \text{ for all } x, y, z \in I.$$

Replacing z by y in the above expression, we get

$$g([x, y]) + [d(x), d(y)] \in Z, \text{ for all } x, y \in I.$$

By the hypothesis, we get

$$[x, y] \in Z, \text{ for all } x, y \in I.$$

By Lemma 3, we obtain that R contains a nonzero central ideal.

iv) By Lemma 1, there exists a nonzero ideal I of R such that $I \subseteq U$. We see that

$$g(x \circ y) + d(x) \circ d(y) \pm x \circ y \in Z, \text{ for all } x, y \in I.$$

Taking y by $y+z$, $z \in I$, we have

$$\begin{aligned}&g(x \circ y) + g(x \circ z) + 2G(x \circ y, x \circ z) + d(x) \circ d(y)\\&+d(x) \circ d(z) + 2(d(x) \circ D(y, z)) \pm x \circ y \pm x \circ z \in Z.\end{aligned}$$

By the hypothesis, we obtain

$$2G(x \circ y, x \circ z) + 2(d(x) \circ D(y, z)) \in Z, \text{ for all } x, y, z \in I.$$

Since R is 2-torsion free, we have

$$G(x \circ y, x \circ z) + (d(x) \circ D(y, z)) \in Z, \text{ for all } x, y, z \in I.$$

Writing z by y in the last equation, we get

$$g(x \circ y) + d(x) \circ d(y) \in Z, \text{ for all } x, y \in I.$$

By the hypothesis, we get

$$x \circ y \in Z, \text{ for all } x, y \in I.$$

We conclude that R contains a nonzero central ideal by Lemma 4. The proof is completed. □

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References

1. A. Ali, A. Bano, Identities with multiplicative (generalized) (α, β)-derivations in semiprime rings, Int. J. Math. And Appl., 2018, 6(4), 195-202.
2. M. Ashraf, N. Rehman, On derivations and commutativity in prime rings, East-West J. Math., 2001, 3(1), 87-91.
3. M. Ashraf, A. Asma and A. Shakir, Some commutativiy theorems for rings with generalized derivations, Southeast Asain Bull. of Math., 2007, 31, 415-421.
4. M. N. Daif, H. E. Bell, Remarks on derivations on semiprime rings, Internat J. Math. Math. Sci., 1992, 15(1), 205-206.
5. Z. Bedir, Ö. Gölbaşı, Some commutativity on Lie ideals on semiprime rings, Adıyaman University Journal of Science, 2020, 10(2), 548-556.
6. E. Koç Sögütcü, Ö. Gölbaşı, Lie ideals of semiprime rings with generalized derivations, Adıyaman University Journal of Science, 2018, 8(1), 1-12.
7. G. Maksa, A remark on symmetric biadditive functions having non-negative diagonalization, Glasnik. Mat., 1980, 15 (35), 279–282.
8. G. Maksa, On the trace of symmetric biderivations , C. R. Math. Rep. Acad. Sci. Canada, 1987, 9, 303–307.