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Some Classification Theorems and Endomorphisms in New Classes

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Abstract: Let $\mho$ be a prime ring of char$(\mho) \neq 2$ with its center Z. This article introduces new classes of endomorphisms and investigates how they relate to antiautomorphisms of prime rings and the commutativity of prime rings. Additionally, we fully describe and classify some of these endomorphisms. We also give examples to prove the necessity of the numerous restrictions included in the hypotheses of our results.

Keywords: prime ring; endomorphism; automorphism; antiautomorphism; strong commutativity preserving (SCP); commutator

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1. Introduction

In this article, a ring with its center shall be denoted by the symbol $\mho$ and Z, respectively. The anticommutator $ba + ab$ and commutator $ba - ab$ are denoted by the symbols $b \circ a$ and $[b, a]$, where $b, a \in \mho$, respectively. A ring $\mho$ is called prime if $b\mho a = (0)$ for each $b, a \in \mho$ implies $b = 0$ or $a = 0$. An additive map $\eth : \mho \to \mho$ is called an endomorphism of $\mho$ if $\eth(ba) = \eth(b)\eth(a)$ holds for each $b, a \in \mho$. An additive bijective map $\theta : \mho \to \mho$ is called an antiautomorphism of $\mho$ if $\theta(ba) = \theta(a)\theta(b)$ holds for each $b, a \in \mho$; if $\theta^2(a) = a$, then $\theta = *$ is called an involution. An element b in a ring with involution $*$ is said to be (resp. skew) Hermitian if $b^* = b$ (resp. $b^* = -b$). In a ring $\mho$ of char$(\mho) \neq 2$, if the intersection of Z with the set of all skew-Hermitian is non-zero, the involution is said to be of the second kind; otherwise, it is said to be of the first kind.

Many authors have looked into the relationship between specific unique types of maps defined on commutativity of rings over the past thirty years. A simple Artinian ring is commutative if it has a commuting non-trivial automorphism, according to Divinsky [1], who provided the first result in this direction. A non-zero centralizing derivation on a prime ring necessitates the ring to be commutative, as Posner [2] proved two years later. Since then, many authors have improved and expanded these results in different directions (see [3–9], where further references can be found).

If $[\tau(b), \tau(a)] = 0$ whenever $[b, a] = 0$ for each $b, a \in \mho$, then a map from $\tau : \mho \to \mho$ preserves commutativity. Matrix theory, operator theory, and ring theory have all seen

active research in the study of commutativity-preserving maps (see [10,11] for references). On a subset S of $\mho$, a map τ is called strong commutativity preserving (SCP) if $[\tau(b), \tau(a)] = [b, a]$ for each b, a in S. In [12], Bell and Daif examined whether rings may admit an endomorphism or a derivation, which is SCP on a non-zero right ideal. In fact, they showed that $\mho$ is commutative ($J \subseteq Z$) if a (semi) prime ring $\mho$ admits a non-identity endomorphism (a derivation) δ fulfilling $[\delta(b), \delta(a)] = [b, a]$ for any b, a in a right ideal J of $\mho$. In particular, if $J = \mho$, then $\mho$ is commutative. Later, Deng and Ashraf [13] showed that a semiprime ring $\mho$ has a non-zero central ideal if it has a derivation δ and a map $\tau : J \to \mho$ defined on a non-zero ideal J, such that $[\tau(b), \delta(a)] = [b, a]$ for any b, a in J. In particular, they proved that if $J = \mho$, then $\mho$ is commutative. A map $\tau : \mho \to \mho$ is said to be strong skew-commutativity-preserving (Skew SCP) on a subset S of $\mho$ if $[\tau(b), \tau(a)] + [b, a] = 0$ for each $b, a \in S$. A non-zero central ideal is contained in $\mho$ if $\mho$ is a 2-torsion free semiprime ring and ξ is a derivation of $\mho$ that satisfies the Skew SCP on a non-zero ideal J of $\mho$, according to Ali and Huang [14]. The literature contains numerous generalizations of these results that are related (see, for instance, [15]).

The current paper draws its inspiration from the preceding outcomes, and in this context, we persistently advance along this trajectory of exploration. However, instead of focusing solely on involutions, we extend our inquiry to encompass new classes of mappings as (anti)automorphisms.

Example 1.

(i) *Any involution $*$ is an antiautomorphism θ, such that $\theta^2 = I_{id}$ and so $\theta = *$. In general, an antiautomorphism is not necessarily an involution as in (ii).*

(ii) *Let $\mathbb{H}$ be a real quaternion ring and a map θ from $\mathbb{H}$ to itself, such that $\theta(\beta) = (1 + i)\overline{\beta}(1 + i)^{-1}$, where $\overline{\beta}$ denotes the conjugate of β, i.e., if $\beta = r_1 + r_2 i + r_3 j + r_4 k$, where $r_m \in \mathbb{R}$, $m = 1, 2, 3, 4$, then $\overline{\beta} = r_1 - r_2 i - r_3 j - r_4 k$. However, θ has an order different from one and two, so it is not an involution.*

Theorem 1. *Let $\mho$ be a prime ring of char($\mho$) $\neq 2$ with an antiautomorphism θ, which is not Z-linear. Then, the following hold:*

(i) *$\mho$ possesses a non-trivial θ-SCP endomorphism if and only if $\mho$ is commutative.*

(ii) *$\mho$ possesses a non-trivial θ-Skew SCP endomorphism if and only if $\mho$ is either commutative or embeds in $M_2(\mathbb{F})$ for $\mathbb{F}$ a field.*

Theorem 2. *Let $\mho$ be a prime ring of char($\mho$) $\neq 2$ with an antiautomorphism θ, which is not Z-linear. Then, $\mho$ is commutative if any one of the following is satisfied:*

(i) *$\mho$ possesses a non-trivial θ-SACP endomorphism.*

(ii) *$\mho$ possesses a non-trivial θ-Skew SACP endomorphism.*

Theorem 3. *Let $\mho$ be a prime ring of char($\mho$) $\neq 2$ with an antiautomorphism θ, which is not Z-linear, and let $\eth$ be an endomorphism of $\mho$. Then, $\mho$ is commutative if any one of the following is satisfied:*

(i) *$\eth(b) \circ \eth(\theta(b)) = [b, \theta(b)]$ for each $b \in \mho$.*

(ii) *$\eth(b) \circ \eth(\theta(b)) = -[b, \theta(b)]$ for each $b \in \mho$.*

Theorem 4. *Let $\mho$ be a prime ring of char($\mho$) $\neq 2$ with an antiautomorphism θ, which is not Z-linear, and let $\eth$ be a non-identity endomorphism of $\mho$. Then, $\mho$ is commutative if any one of the following is satisfied.*

(i) *$\eth(b)\eth(\theta(b)) = \pm[b, \theta(b)]$ for each $b \in \mho$.*

(ii) *$\eth(b)\eth(\theta(b)) = \pm b \circ \theta(b)$ for each $b \in \mho$.*

(iii) *$\eth(b)\eth(\theta(b)) = \pm b\theta(b)$ for each $b \in \mho$.*

(iv) *$\eth(b)\eth(\theta(b)) = \pm\theta(b)b$ for each $b \in \mho$.*

2. Results

Motivated by the concept of SCP derivations, the researchers in [16] explored a broader notion. This involved investigating the validity of the identity $[\delta(b), \delta(b^*)] = \pm[b, b^*]$. To be precise, they demonstrated in [16], Theorem 1, that a prime ring $\mho$ equipped with an involution of the second kind must possess commutative attributes if it accommodates a nontrivial derivation δ that adheres to the condition $[\delta(b), \delta(b^*)] = \pm[b, b^*]$ for all $b \in \mho$. This outcome has been extended by Nejjar et al. [17], who took the generalization a step further by contemplating the broader expression $[\delta(b), \delta(b^*)] \pm [b, b^*] \in Z$ for all $b \in \mho$ ($*$-SCP and $*$-Skew SCP derivations). Furthermore, they substantiated that the commutative nature of $\mho$ endures, even in the case when δ is zero mapping. Their objective was to probe the ramifications of this identity within the confines of a prime ring $\mho$, where char($\mho$) $\neq 2$, characterized by an involution $*$ of the second kind. In their work, as detailed in [17], Theorems 3.1, 3.5, and 3.8, they effectively demonstrated that when a non-zero derivation, denoted as μ, adheres to the condition $\delta(b^*)\delta(b) + p\delta(b)\delta(b^*) + q(b^*b + sbb^*) \in Z$ for each element $b \in \mho$, with p, q, and s taking values from $\{-1, 1\}$, then the ring must inevitably assume a commutative nature. This outcome has been further expanded upon by Rehman and Alnoghashi [18] (2022), who elevated the process of generalization by considering a broader expression: $[\Delta(b), \Delta(\theta(b))] \pm [b, \theta(b)] \in Z$ for all $b \in \mho$, where they replaced a derivation δ and an involution $*$ by a generalized derivation Δ and an antiautomorphism θ, respectively. In their noteworthy contribution, Mamouni et al. [19] (2021) made a substantial impact on the exploration of prime rings. They established a remarkable outcome that holds true for prime rings represented as $\mho$. As per their results, if such a prime ring with char($\mho$) $\neq 2$ and characterized by an involution $*$ of the second kind accommodates two derivations, denoted as δ and μ, satisfying $\delta(b^*)\mu(b) + p\mu(b)\delta(b^*) + q(b^* \circ b) \in Z$ for every $b \in \mho$, where p and q are chosen from $\{\pm 1\}$, it can be deduced that the ring $\mho$ must necessarily assume the characteristics of a commutative ring. In 2023, Alqarni et al. [20], Theorems 2 and 4, further expanded upon these results, taking the process of generalization a step ahead by considering the broader expression $\delta(\theta(b))\mu(b) + p\mu(b)\delta(\theta(b)) + q(b\theta(b) + s\theta(b)b) \in Z$ for every $b \in \mho$, where θ is an antiautomorphism of $\mho$, and $p, q, s \in \{-1, 0, 1\} \backslash \{q \neq 0\}$, it can be deduced that the ring $\mho$ must necessarily assume the characteristics of a commutative ring.

Mamouni et al. [21] (2020) introduced the concepts of $*$-SCP (resp. $*$-Skew SCP) as follows: a map $\tau : \mho \to \mho$ is said to be $*$-SCP (resp. $*$-Skew SCP) on a subset S of $\mho$ if $[\tau(b), \tau(b^*)] = [b, b^*]$ (Theorem 1(1)) (resp. $[\tau(b), \tau(b^*)] + [b, b^*] = 0$ (Theorem 1(2))) for each $b \in S$. They proved that a prime ring $\mho$ of char($\mho$) $\neq 2$ with involution $'*'$ of the second kind, which admits a non-trivial $*$-SCP (resp. $*$-Skew SCP) endomorphism if and only if $\mho$ is commutative. They also proved that $\mho$ is commutative if a prime ring $\mho$ of char($\mho$) $\neq 2$ with an involution $*$ of the second kind, which admits a non-trivial $*$-SACP endomorphism. However, they made an error in the proof of Theorem 1(2) in Equation (36) (as we will show in Example 4) by relying on a faulty result from [17] Corollary 3.6. As a result, we will provide a possible correction to the original result in [21], Theorem 1(2). The proposed correction includes the introduction of a new result (Corollary 1) that concludes that $\mho$ is either commutative or embeds in $M_2(\mathbb{F})$ for $\mathbb{F}$ a field, instead of stating that $\mho$ is commutative.

This paper is motivated by the preceding outcomes and builds upon the concepts of $*$-SCP ($*$-Skew SCP) and $*$-SACP ($*$-Skew SACP). Our objective is to extend the scope of previous results to encompass new classes of mappings rather than confining our focus solely to involutions. Specifically, our attention shifts toward a parallel scenario, wherein an involution denoted as $*$ is replaced with an antiautomorphism denoted as θ. In essence, we proceed to introduce the concepts of θ-SCP, θ-Skew SCP, θ-SACP, and θ-Skew SACP mappings, as outlined below.

Definition 1. *Let $\mho$ be a ring with an antiautomorphism θ, and let $\eth : \mho \to \mho$ be a map. Then, $\eth$ is called:*

(i) θ-SCP if $[\eth(b), \eth(\theta(b))] = [b, \theta(b)]$ *for each* $b \in \mho$.
(ii) θ-Skew SCP if $[\eth(b), \eth(\theta(b))] = -[b, \theta(b)]$ *for each* $b \in \mho$.
(iii) θ-SACP if $\eth(b) \circ \eth(\theta(b)) = b \circ \theta(b)$ *for each* $b \in \mho$.
(iv) θ-Skew SACP if $\eth(b) \circ \eth(\theta(b)) = -b \circ \theta(b)$ *for each* $b \in \mho$.

Example 2. *The examples that prove the notions of Definition 1 exist: For any commutative rings, antiautomorphisms θ, and maps $\eth$, we have Definition 1(i) and (ii). For any rings, antiautomorphisms θ, and maps $\eth = \pm I_{id}$, we obtain Definition 1(iii) and (iv). Furthermore, see examples on (p.73) and Examples 3–6 of [21] since any involution is an antiautomorphism.*

Remark 1.

(i) *Every $*$-SCP (resp. $*$-Skew SCP) is θ-SCP (resp. θ-Skew SCP).*
(ii) *Every $*$-SACP (resp. $*$-Skew SACP) is θ-SACP (resp. θ-Skew SACP).*

However, it should be noted that the converse is generally not true, as evidenced by Example 1(ii).

To establish our results, a collection of supplementary lemmas is necessary. From now on, we assume $\mho$ to be a prime ring with char$(\mho) \neq 2$, and θ to be an antiautomorphism that is not Z-linear. Let us initiate our endeavor with the following lemma:

Lemma 1. *Let $r_0 \in \mho$. If $[b\theta(b) \pm \theta(b)b, r_0] = 0$ for each $b \in \mho$, then $r_0 \in Z$.*

Proof. Assume that

$$[b\theta(b) \pm \theta(b)b, r_0] = 0 \tag{1}$$

for each $b \in \mho$. By linearizing (1), we have

$$[b\theta(a) + a\theta(b) \pm \theta(b)a \pm \theta(a)b, r_0] = 0.$$

Replacing b by sb in the above expression and using it, where $s \neq 0 \in Z$, we obtain

$$[-s(a\theta(b) \pm \theta(b)a) + \theta(s)(a\theta(b) \pm \theta(b)a), r_0] = 0.$$

That is, $[a\theta(b) \pm \theta(b)a, r_0](\theta(s) - s) = 0$. Since $\theta(s) \neq s$ for some $s \in Z$, we obtain $[a\theta(b) \pm \theta(b)a, r_0] = 0$. Putting b by $\theta^{-1}(b)$ in the previous relation, we see that $[ab \pm ba, r_0] = 0$. In this case, $[ab + ba, r_0] = 0$. Taking $a = s$ in the last expression, we find that $2[b, r_0] = 0$, and so $[b, r_0] = 0$. Hence, $r_0 \in Z$. In case, $[[a, b], r_0] = 0$. Taking b by ba in the previous expression and applying it, we infer that $[a, b][a, r_0] = 0$. Again, taking b by tb in the last equation and using it, where $t \in \mho$, conclude that $[a, t]b[a, r_0] = 0$. That is, $[a, t]\mho[a, r_0] = (0)$. Hence, $[a, t] = 0$ or $[a, r_0] = 0$, and so $\mho$ is commutative or $[a, r_0] = 0$. In both cases $r_0 \in Z$. $\square$

Lemma 2. *Let $\eth$ be a non-identity endomorphism of $\mho$. If*

$$\eth(b)\eth(a) + k\eth(a)\eth(b) + m(ba + nab) = 0$$

for each $b, a \in \mho$ and $k, m, n \in \{-1, 1\}$, then $\mho$ is commutative or $\eth(b) \circ \eth(a) + m(b \circ a) = 0$ for each $b, a \in \mho$.

Proof.

(1) Let $k = m = n = -1$; we infer that $[\eth(a), \eth(b)] - [a, b] = 0$. Since $\eth$ is non-identity, and by [12] Corollary 2, we conclude that $\mho$ is commutative.

(2) Let $k = n = -1$ and $m = 1$; we obtain $[\eth(b), \eth(a)] + [b, a] = 0$ for each $b, a \in \mho$. Replacing a by ba in the previous equation and applying it, we obtain $[b, a](\eth(b) - b) = 0$. One can see that $[b, a]\mho(\eth(b) - b) = (0)$ for each $b, a \in \mho$. By primeness of $\mho$, we obtain for

each for each $b \in \mho$ either $\eth(b) = b$ or $[b, a] = 0$ for each $a \in \mho$. Using Brauer's trick, we see both sets $\{b \in \mho : [b, a] = 0 \quad \text{for each } a \in \mho\}$ and $\{b \in \mho : \eth(b) - b = 0\}$ are additive subgroups of $\mho$. Therefore, either $[b, a] = 0$ for each $b, a \in \mho$ or $\eth(b) = b$ for each $b \in \mho$. Since $\eth$ is non-identity, we find that $[b, a] = 0$ for each $b, a \in \mho$. That is, $\mho$ is commutative.

(3) Let $k = -1 = -n$, and $m \in \{-1, 1\}$; we have $[\eth(a), \eth(b)] \pm b \circ a = 0$. By interchanging b and a in the last expression, we obtain $-[\eth(a), \eth(b)] \pm b \circ a = 0$. Comparing the two last relations, we obtain $\pm 2b \circ a = 0$. Hence, $b \circ a = 0$. By putting a by ar in the previous expression and using it, where $r \in \mho$, we see that $a[b, r] = 0$, and so $[b, r]\mho[b, r] = (0)$. Hence, $[b, r] = 0$ for each $b, r \in \mho$, and so $\mho$ is commutative. Now, since $\mho$ is commutative and from above, we have $b \circ a = 0$ for each $b, a \in \mho$, and so $2ba = 0$ for each $b, a \in \mho$, and hence $ba = 0$ for each $b, a \in \mho$, that is, $b\mho a = (0)$ for each $b, a \in \mho$. In particular, $b\mho b = (0)$ for each $b \in \mho$. Hence, $b = 0$ for each $b \in \mho$. Thus, $\mho = (0)$, a contradiction with our hypothesis that an antiautomorphism θ is not Z-linear. Therefore, this case is excluded.

(4) Let $k = 1 = -n$ and $m \in \{-1, 1\}$; we find that $\eth(a) \circ \eth(b) \pm [b, a] = 0$. By interchanging b and a in the last equation, we have $\eth(a) \circ \eth(b) \mp [b, a] = 0$. Comparing the two last relations, we infer that $\pm 2[b, a] = 0$. That is, $[b, a] = 0$ for each $b, a \in \mho$. Thus, $\mho$ is commutative.

(5) Let $k = n = 1$ and $m \in \{-1, 1\}$; we find that $\eth(a) \circ \eth(b) + m(b \circ a) = 0$, as desired. □

Lemma 3. *Let $\eth$ be a non-identity endomorphism of $\mho$. If*

$$\eth(b)\eth(\theta(b)) + k\eth(\theta(b))\eth(b) + m(b\theta(b) + n\theta(b)b) = 0$$

for each $b \in \mho$ and $k, m, n \in \{-1, 1\}$, then $\mho$ is either commutative or, embeds in $M_2(\mathbb{F})$ for $\mathbb{F}$ a field (when $k = n = -1 = -m$).

Proof. Assume that

$$\eth(b)\eth(\theta(b)) + k\eth(\theta(b))\eth(b) + m(b\theta(b) + n\theta(b)b) = 0 \tag{2}$$

for each $b \in \mho$ and $k, m, n \in \{-1, 1\}$. Let $s \neq 0 \in Z$, so $\theta(s) \in Z$. Putting $c \in \{s, \theta(s)\}$. From the definition of $\eth$, we have

$$\eth((b\theta(b) + k\theta(b)b)c) = \eth(b\theta(b) + k\theta(b)b)\eth(c)$$

and

$$\eth(c(b\theta(b) + k\theta(b)b)) = \eth(c)\eth(b\theta(b) + k\theta(b)b).$$

By comparing the two last relations, we infer that

$$[\eth(b\theta(b) + k\theta(b)b), \eth(c)] = 0. \tag{3}$$

However, from the definition of $\eth$ and (2), we obtain

$$\eth(b\theta(b) + k\theta(b)b) = \eth(b)\eth(\theta(b)) + k\eth(\theta(b))\eth(b) = -m(b\theta(b) + n\theta(b)b).$$

Applying the previous equation in (3), we arrive at

$$[-m(b\theta(b) + n\theta(b)b), \eth(c)] = 0.$$

Thus,

$$[b\theta(b) + n\theta(b)b, \eth(c)] = 0.$$

Using Lemma 1, we obtain $\eth(c) \in Z$. That is, $\eth(s)$ and $\eth(\theta(s))$ belong to Z. We will use the previous fact without mentioning it.

Here, by linearizing (2), we obtain

$$\eth(b)\eth(\theta(a)) + k\eth(\theta(a))\eth(b) + \eth(a)\eth(\theta(b)) + k\eth(\theta(b))\eth(a) \tag{4}$$
$$+ m(b\theta(a) + n\theta(a)b) + m(a\theta(b) + n\theta(b)a) = 0$$

for each $b, a \in \mho$ and $k, m, n \in \{-1, 1\}$. Replacing a by as in (4), we obtain

$$\{\eth(b)\eth(\theta(a)) + k\eth(\theta(a))\eth(b)\}\eth(\theta(s)) + \{\eth(a)\eth(\theta(b)) + k\eth(\theta(b))\eth(a)\}\eth(s) \tag{5}$$
$$+ \{m(b\theta(a) + n\theta(a)b)\}\theta(s) + \{m(a\theta(b) + n\theta(b)a)\}s = 0$$

for each $b, a \in \mho$. Again, replacing b by bs in (5), we obtain

$$\{\eth(b)\eth(\theta(a)) + k\eth(\theta(a))\eth(b)\}\eth(\theta(s))\eth(s) \tag{6}$$
$$+ \{\eth(a)\eth(\theta(b)) + k\eth(\theta(b))\eth(a)\}\eth(\theta(s))\eth(s)$$
$$+ \{m(b\theta(a) + n\theta(a)b)\}\theta(s)s + \{m(a\theta(b) + n\theta(b)a)\}\theta(s)s = 0$$

for each $b, a \in \mho$. Multiplying (4) by $\eth(\theta(s))\eth(s)$, we see that

$$\{\eth(b)\eth(\theta(a)) + k\eth(\theta(a))\eth(b)\}\eth(\theta(s))\eth(s) \tag{7}$$
$$+ \{\eth(a)\eth(\theta(b)) + k\eth(\theta(b))\eth(a)\}\eth(\theta(s))\eth(s)$$
$$+ \{m(b\theta(a) + n\theta(a)b)\}\eth(\theta(s))\eth(s) + \{m(a\theta(b) + n\theta(b)a)\}\eth(\theta(s))\eth(s) = 0$$

for each $b, a \in \mho$. Comparing (6) and (7) gives

$$m\{(b\theta(a) + n\theta(a)b) + (a\theta(b) + n\theta(b)a)\}\{\eth(\theta(s))\eth(s) - \theta(s)s\} = 0.$$

That is,

$$\{(b\theta(a) + n\theta(a)b) + (a\theta(b) + n\theta(b)a)\}\mho\{\eth(\theta(s))\eth(s) - \theta(s)s\} = (0).$$

Hence, $(b\theta(a) + n\theta(a)b) + (a\theta(b) + n\theta(b)a) = 0$ or $\eth(\theta(s))\eth(s) - \theta(s)s = 0$. In this case,

$$(b\theta(a) + n\theta(a)b) + (a\theta(b) + n\theta(b)a) = 0.$$

By putting $a = b$ in the above expression, we find that $(b\theta(b) + n\theta(b)b) = 0$. Using [18] Lemmas 2.3 and 2.4, we conclude that $\mho$ is commutative. Here, if

$$\eth(\theta(s))\eth(s) - \theta(s)s = 0. \tag{8}$$

Taking $a = b$ in (5) and using (2), we arrive at

$$(b\theta(b) + n\theta(b)b)\mho(\eth(\theta(s)) + \eth(s) - \theta(s) - s) = (0).$$

Hence, $b\theta(b) + n\theta(b)b = 0$ or $\eth(\theta(s)) + \eth(s) - \theta(s) - s = 0$. In this case, $b\theta(b) + n\theta(b)b = 0$, and, hence, using a similar approach as the above. Here, suppose that

$$\eth(\theta(s)) + \eth(s) - \theta(s) - s = 0. \tag{9}$$

Multiplying (9) by $\eth(s)$, we infer that

$$\eth(\theta(s))\eth(s) + \eth(s)^2 - \theta(s)\eth(s) - s\eth(s) = 0.$$

Applying (8) in the above relation, we have

$$\theta(s)s + \eth(s)^2 - \theta(s)\eth(s) - s\eth(s) = 0.$$

That is,

$$(\eth(s) - s)\mho(\eth(s) - \theta(s)) = (0).$$

Thus, $\eth(s) - s = 0$ or $\eth(s) - \theta(s) = 0$.

Case (I): Suppose that

$$\eth(s) = s. \tag{10}$$

Using (10) in (9), we obtain

$$\eth(\theta(s)) = \theta(s). \tag{11}$$

Applying (10) and (11) in (5), we obtain

$$\{\eth(b)\eth(\theta(a)) + k\eth(\theta(a))\eth(b) + m(b\theta(a) + n\theta(a)b)\}\theta(s)$$
$$+\{\eth(a)\eth(\theta(b)) + k\eth(\theta(b))\eth(a) + m(a\theta(b) + n\theta(b)a)\}s = 0.$$

Multiplying (4) by s and then using it in the above equation, we conclude that

$$\{\eth(b)\eth(\theta(a)) + k\eth(\theta(a))\eth(b) + m(b\theta(a) + n\theta(a)b)\}\mho(\theta(s) - s) = (0).$$

Putting a by $\theta^{-1}(a)$ in the previous relation, we see that

$$\{\eth(b)\eth(a) + k\eth(a)\eth(b) + m(ba + nab)\}\mho(\theta(s) - s) = (0).$$

Since $\theta(s) \neq s$ for some $0 \neq s \in Z$, we see that

$$\eth(b)\eth(a) + k\eth(a)\eth(b) + m(ba + nab) = 0.$$

Using Lemma 2, we obtain that $\mho$ is commutative or $\eth(a) \circ \eth(b) + m(b \circ a) = 0$ for each $b, a \in \mho$. Here, in this case,

$$\eth(a) \circ \eth(b) + m(b \circ a) = 0. \tag{12}$$

Taking $a = s$ in (12) and applying (10), we obtain $2s\eth(b) + 2msb = 0$; thus, $\eth(b) = -mb$ for each $b \in \mho$. Since $\eth$ is non-identity, we see that $\eth(b) = -b$ for each $b \in \mho$. However, $\eth$ is an endomorphism and $ba = \eth(b)\eth(a) = \eth(ba) = -ba$; hence, $2ba = 0$, and $ba = 0$, a contradiction.

Case (II): Suppose that

$$\eth(s) = \theta(s). \tag{13}$$

Using (13) in (9), we conclude that

$$\eth(\theta(s)) = s. \tag{14}$$

Applying (13) and (14) in (5), we infer that

$$\{\eth(a)\eth(\theta(b)) + k\eth(\theta(b))\eth(a) + m(b\theta(a) + n\theta(a)b)\}\theta(s)$$
$$+\{\eth(b)\eth(\theta(a)) + k\eth(\theta(a))\eth(b) + m(a\theta(b) + n\theta(b)a)\}s = 0.$$

Multiplying (4) by s and then using it in the previous expression, we obtain

$$\{\eth(a)\eth(\theta(b)) + k\eth(\theta(b))\eth(a) + m(b\theta(a) + n\theta(a)b)\}\mho(\theta(s) - s) = (0).$$

Hence,

$$\eth(a)\eth(\theta(b)) + k\eth(\theta(b))\eth(a) + m(b\theta(a) + n\theta(a)b) = 0. \tag{15}$$

Here, we have four cases:

$(1)'$ Let $k = n = -1$ and $m \in \{-1, 1\}$ in (15), we obtain

$$[\eth(a), \eth(\theta(b))] \pm [b, \theta(a)] = 0 \tag{16}$$

for each $b, a \in \mho$. Putting a by $\theta(b)$, we have

$$[b, \theta^2(b)] = 0. \tag{17}$$

Suppose that $\theta^2 \neq I_{id}$, indicating that $\alpha = \theta^2$ is a non-identity automorphism. By applying (17), we obtain $[b, \alpha(b)] = 0$. According to the theorem of [22], we conclude that $\mho$ is commutative. Here, consider the case where $\theta^2 = I_{id}$. In this case, we have $\theta = *$ as an involution on $\mho$ of the second kind. By setting $a = b$ in (16), we obtain a $*$-SCP and $*$-Skew SCP endomorphism. For the case of a $*$-SCP endomorphism, we can apply the proof of Theorem 1(1) of [21] to conclude that $\mho$ is commutative. For the case of a $*$-Skew SCP endomorphism, we can apply part of the proof of Theorem 1(2) of [21] ([21], from Equation (19) to Equation (36)). This yields $d_b(c) \circ d_b(a) \in Z$ for each $b, c, a \in \mho$, where $d_b(\mho) = [b, \mho]$. Here, if $d_b = 0$ for each $b \in \mho$, then $\mho$ is commutative; otherwise, putting $a = c$, we obtain $2d_b(c)^2 \in Z$ for each $b, c \in \mho$, and $d_b(c)^2 \in Z$ for each $b, c \in \mho$. By using [23] (Theorem 4), we obtain that $\mho$ satisfies s_4, the standard identity in four variables. By applying [24] (Lemma 2.1), we conclude that $\mho$ is either commutative or embeds in $M_2(\mathbb{F})$ for $\mathbb{F}$ a field.

$(2)'$ Let $k = -1 = -n$ and $m \in \{-1, 1\}$ in (15), we have $[\eth(a), \eth(\theta(b))] + m(b \circ \theta(a)) = 0$. Putting $a = s$ in the last relation, we obtain $2m(b\theta(s)) = 0$, so $b\theta(s) = 0$, and since $s \neq 0$ and θ is an antiautomorphism, we obtain $\theta(s) \neq 0$, so $b = 0$ for each $b \in \mho$, a contradiction.

$(3)'$ Let $-k = -1 = n$ and $m \in \{-1, 1\}$ in (15), we infer that

$$\eth(a) \circ \eth(\theta(b)) + m[b, \theta(a)] = 0. \tag{18}$$

Putting $b = s$ in the above expression, we obtain $2\eth(a)\eth(\theta(s)) = 0$, and hence $\eth(a)\eth(\theta(s)) = 0$, by using (14) in the last equation, we obtain $\eth(a) = 0$ for each $a \in \mho$. Applying the previous expression in (18), we see that $[b, \theta(a)] = 0$. Using [18] (Lemma 2.3), we see that $\mho$ is commutative.

$(4)'$ Let $k = 1 = n$ and $m \in \{-1, 1\}$ in (15), we conclude that

$$\eth(a) \circ \eth(\theta(b)) + m(b \circ \theta(a)) = 0. \tag{19}$$

Putting $b = s$ in (19) and using (14) and the fact that $\theta(s) \neq 0$, we have $\eth(a) + m\theta(a) = 0$ for each $a \in \mho$. That is, $\eth(a) = -m\theta(a)$ for each $a \in \mho$. That is, $\eth(ba) = -m\theta(ba)$, and from the definitions of $\eth$ and θ, we infer that $\eth(b)\eth(a) = -m\theta(a)\theta(b)$. Thus, $m^2\theta(b)\theta(a) = \theta(a)\theta(b)$, and since $m \in \{-1, 1\}$, we obtain $m^2 = 1$. That is, $\theta(b)\theta(a) = \theta(a)\theta(b)$, this means, $[\theta(b), \theta(a)] = 0$. Hence, $[b, a] = 0$ for each $b, a \in \mho$, so $\mho$ is commutative. $\square$

Using Lemma 3, we obtain the proofs of Theorems 1–3.

Lemma 4. *Let $\eth$ be an endomorphism of $\mho$. If*

$$\eth(b)\eth(\theta(b)) + k(mb\theta(b) + n\theta(b)b) = 0$$

for each $b \in \mho$ and $k \in \{-1, 1\}$, $m, n \in \{-1, 0, 1\}$, then $\mho$ is commutative or $\eth$ is identity or $(\eth(b)\eth(\theta(b) = 0$ for each $b \in \mho$ and $m = 0 = n$).

Proof. Suppose that $\eth$ is non-identity and $\eth(b)\eth(\theta(b)) \neq 0$ for some $b \in \mho$ and (either m or n is not zero). Thus, from the previous assumptions, we obtain $m, n \in \{-1, 0, 1\}\backslash\{m = n = 0\}$. From our hypothesis, we have

$$\eth(b)\eth(\theta(b)) + k(mb\theta(b) + n\theta(b)b) = 0 \tag{20}$$

for each $b \in \mho$ and $k \in \{-1, 1\}$, and $m, n \in \{-1, 0, 1\}\backslash\{m = n = 0\}$. By linearizing (20), we obtain

$$\eth(a)\eth(\theta(b)) + \eth(b)\eth(\theta(a)) + k(ma\theta(b) + mb\theta(a) + n\theta(a)b + n\theta(b)a) = 0 \tag{21}$$

for each $b, a \in \mho$. Replacing b by bs in (21), where $s \in Z$ and $\theta(s) \neq s$, we obtain

$$\eth(a)\eth(\theta(b))\eth(\theta(s)) + \eth(b)\eth(\theta(a))\eth(s) + k(ma\theta(b)\theta(s) + mb\theta(a)s \tag{22}$$
$$+ n\theta(a)bs + n\theta(b)a\theta(s)) = 0.$$

Case (I): Suppose that $\eth(\theta(s)) \neq 0$. Right multiplying (21) by $\eth(\theta(s))$ and using it in (22), we see that

$$\eth(b)\eth(\theta(a))(\eth(s) - \eth(\theta(s))) + k((ma\theta(b) + n\theta(b)a)(\theta(s) - \eth(\theta(s))) \tag{23}$$
$$+ (mb\theta(a) + n\theta(a)b)(s - \eth(\theta(s)))) = 0.$$

Taking a by as in (23), we find that

$$\eth(b)\eth(\theta(a))(\eth(s) - \eth(\theta(s)))\eth(\theta(s)) + k((ma\theta(b) + n\theta(b)a)(\theta(s) - \eth(\theta(s)))s \tag{24}$$
$$+ (mb\theta(a) + n\theta(a)b)(s - \eth(\theta(s)))\theta(s)) = 0.$$

Right multiplying (23) by $\eth(\theta(s))$ and using (24), we arrive at

$$k((ma\theta(b) + n\theta(b)a)(\theta(s) - \eth(\theta(s)))(s - \eth(\theta(s)))$$
$$+ (mb\theta(a) + n\theta(a)b)(s - \eth(\theta(s)))\theta(s)(\theta(s) - \eth(\theta(s)))) = 0.$$

That is,

$$(ma\theta(b) + n\theta(b)a + mb\theta(a) + n\theta(a)b)(\theta(s) - \eth(\theta(s)))(s - \eth(\theta(s))) = 0. \tag{25}$$

Putting b by bs in (25) and then multiplying (25) by s and then comparing them, we have

$$(ma\theta(b) + n\theta(b)a)(\theta(s) - s)(\theta(s) - \eth(\theta(s)))(s - \eth(\theta(s))) = 0.$$

Using our hypothesis, we obtain

$$(ma\theta(b) + n\theta(b)a)(\theta(s) - \eth(\theta(s)))(s - \eth(\theta(s))) = 0.$$

That is,

$$(mab + nba)(\theta(s) - \eth(\theta(s)))(s - \eth(\theta(s))) = 0. \tag{26}$$

Putting $b = s$ in (26), we obtain

$$(m + n)a(\theta(s) - \eth(\theta(s)))(s - \eth(\theta(s))) = 0.$$

Replacing a by ab in the last relation, where $b \in \mho$, we see that

$$(m + n)ab(\theta(s) - \eth(\theta(s)))(s - \eth(\theta(s))) = 0.$$

That is,

$$(m + n)a\mho(\theta(s) - \eth(\theta(s)))(s - \eth(\theta(s))) = (0).$$

From the primeness of $\mho$, we find that $(m+n)a = 0$ or $(\theta(s) - \eth(\theta(s)))(s - \eth(\theta(s))) = 0$.

Subcase (1): Suppose that $(\theta(s) - \eth(\theta(s)))(s - \eth(\theta(s))) \neq 0$. Thus, $(m+n)a = 0$, so $ma = -na$, and from (26), we obtain

$$(-nab + nba)(\theta(s) - \eth(\theta(s)))(s - \eth(\theta(s))) = 0.$$

That is,

$$n[b,a](\theta(s) - \eth(\theta(s)))(s - \eth(\theta(s))) = 0.$$

Replacing a by ca in the last relation and using it, where $c \in \mho$, we see that $n[b,c]a(\theta(s) - \eth(\theta(s)))(s - \eth(\theta(s))) = 0$. From Subcase 1 and the primeness of $\mho$, we find that $n[b,c] = 0$ for each $b,c \in \mho$, so $n = 0$ or $\mho$ is commutative. In case $n = 0$, we obtain $m = 0$, a contradiction.

Subcase (2): Suppose that

$$(\theta(s) - \eth(\theta(s)))(s - \eth(\theta(s))) = 0. \tag{27}$$

Replacing b by as in (21), we obtain

$$\eth(a)\eth(\theta(a))(\eth(\theta(s)) + \eth(s)) + k(ma\theta(a) + n\theta(a)a)(\theta(s) + s) = 0.$$

This implies that

$$\eth(a)\eth(\theta(a))\eth(\theta(s) + s) + k(ma\theta(a) + n\theta(a)a)(\theta(s) + s) = 0.$$

Using (20) in the last relation, we see that

$$(ma\theta(a) + n\theta(a)a)(\eth(\theta(s) + s) - (\theta(s) + s)) = 0.$$

By linearizing the last relation, we obtain

$$(ma\theta(b) + mb\theta(a) + n\theta(a)b + n\theta(b)a)(\eth(\theta(s) + s) - (\theta(s) + s)) = 0$$

for each $a,b \in \mho$. Replacing b by bs in the last relation and using it, we obtain

$$(ma\theta(b) + n\theta(b)a)(\theta(s) - s)(\eth(\theta(s) + s) - (\theta(s) + s)) = 0.$$

That is,

$$(mab + nba)(\eth(\theta(s) + s) - (\theta(s) + s)) = 0. \tag{28}$$

In particular, putting $b = s$ and a by ac, where $c \in \mho$, we see that $(m+n)ac(\eth(\theta(s) + s) - (\theta(s) + s)) = 0$. That is, $(m+n)a\mho(\eth(\theta(s) + s) - (\theta(s) + s)) = (0)$. By the primeness of $\mho$, we find that $(m+n)a = 0$ for each $a \in \mho$ or $\eth(\theta(s) + s) - (\theta(s) + s) = 0$. Here, if $\eth(\theta(s) + s) \neq (\theta(s) + s)$, then $(m+n)a = 0$ for each $a \in \mho$, and using the same technique as in Subcase (1), we obtain that $\mho$ is commutative. Suppose that

$$\eth(\theta(s) + s) = \theta(s) + s. \tag{29}$$

Subsubcase (1): Suppose that $\theta(s) + s = 0$. That is,

$$\theta(s) = -s. \tag{30}$$

By the definition of $\eth$, we obtain

$$\eth(\theta(s)) = -\eth(s). \tag{31}$$

Putting $b = s$ in (21) and using (30) and (31), we see that

$$\eth(\theta(a) - a)\eth(s) + k(m + n)(\theta(a) - a)s = 0. \tag{32}$$

Replacing a by as in (32) and using (30), we find that

$$\eth(\theta(a) + a)\eth(s)^2 + k(m + n)(\theta(a) + a)s^2 = 0. \tag{33}$$

Taking a by s in (32) and using (30), we have

$$\eth(s)^2 = -k(m + n)s^2. \tag{34}$$

Using (30) and (31) in (27)

$$(-s + \eth(s))(s + \eth(s)) = 0. \tag{35}$$

That is,

$$\eth(s)^2 = s^2. \tag{36}$$

Using (36) in (34), we obtain

$$k(m + n) = -1. \tag{37}$$

Using (36) and (37) in (33), we see that

$$\eth(\theta(a) + a) = \theta(a) + a. \tag{38}$$

Using (37) in (32), we find that

$$\eth(\theta(a) - a)\eth(s) = (\theta(a) - a)s. \tag{39}$$

Right multiplying (39) by $\eth(s)$ and using (36), we obtain

$$\eth(\theta(a) - a)s^2 = (\theta(a) - a)s\eth(s).$$

That is,

$$\eth(\theta(a) - a)s = (\theta(a) - a)\eth(s). \tag{40}$$

Left multiplying (40) by $\eth(s)$, we obtain

$$\eth(s)\eth(\theta(a) - a)s = \eth(s)(\theta(a) - a)\eth(s).$$

This implies that

$$\eth(\theta(a) - a\eth(s)s = \eth(s)(\theta(a) - a)\eth(s).$$

Using (39) in the last relation, we see that

$$(\theta(a) - a)s^2 = \eth(s)(\theta(a) - a)\eth(s).$$

Left multiplying the last relation by $\eth(s)$ and using (36), we find that

$$\eth(s)(\theta(a) - a)s^2 = s^2(\theta(a) - a)\eth(s).$$

Hence,

$$\eth(s)(\theta(a) - a) = (\theta(a) - a)\eth(s).$$

That is,

$$[\eth(s), \theta(a) - a] = 0. \tag{41}$$

Note that $[s, \theta(a) + a] = 0$. This implies that $\eth([s, \theta(a) + a]) = 0$. Thus, $[\eth(s), \eth(\theta(a) + a)]) = 0$. Using (38) in the last relation, we infer that $[\eth(s), \theta(a) + a] = 0$. Comparing the last relation and (41), we obtain $[\eth(s), a] = 0$ for each $a \in \mho$. That is, $\eth(s) \in Z$. Using the last relation in (35), we see that $(\eth(s) - s)\mho(s + \eth(s)) = (0)$. From the primeness of $\mho$, we obtain $\eth(s) - s = 0$ or $\eth(s) + s = 0$.

Firstly: In case $\eth(s) = s$, using it in (40), we obtain $\eth(\theta(a) - a) = \theta(a) - a$. Using the last relation in (38), we find that

$$\eth(a) = a \tag{42}$$

for each $a \in \mho$. Thus, $\eth$ is a non-identity map, a contradiction.

Secondly: Here, in case $\eth(s) = -s$, using it in (40), we obtain $\eth(\theta(a) - a) = -(\theta(a) - a)$. Using the last relation in (38), we see that $\eth(a) = \theta(a)$. Taking a by ab in the last relation, where $b \in \mho$, we arrive at $\eth(a)\eth(b) = \theta(b)\theta(a)$. Using the fact that $\eth(a) = \theta(a)$ in the last relation, we see that $\theta(a)\theta(b) = \theta(b)\theta(a)$. Hence, $ab = ba$ for each $a, b \in \mho$. Thus, $\mho$ is commutative.

Subsubcase (2): Suppose that $\theta(s) \neq -s$. That is,

$$\theta(s) + s \neq 0. \tag{43}$$

Replacing b by $b(\theta(s) + s)$ in (23) and using (29) and then multiplying it by $\theta(s) + s$ and then comparing them, we obtain

$$k(ma\theta(b) + n\theta(b)a)(\theta(s) - \eth(\theta(s)))(\theta(\theta(s) + s)) = 0.$$

Hence,

$$(ma\theta(b) + n\theta(b)a)(\theta(s) - \eth(\theta(s))) = 0.$$

This implies that

$$(ma\theta(b) + n\theta(b)a)(\theta(s) - \eth(\theta(s) + s - s)) = 0.$$

Using (29) in the last relation, we obtain $(ma\theta(b) + n\theta(b)a)(\eth(s) - s) = 0$. That is,

$$(mab + nba)(\eth(s) - s) = 0. \tag{44}$$

Putting $b = s$ in the last relation, we see that $(m + n)a(\eth(s) - s) = 0$. Taking a by ab in the last relation, where $b \in \mho$, we have $(m + n)ab(\eth(s) - s) = 0$. That is, $(m + n)a\mho(\eth(s) - s) = (0)$. Thus, $(m + n)a = 0$ or $\eth(s) - s = 0$.

Firstly: Suppose that $\eth(s) \neq s$. Thus, $(m + n)a = 0$, so $ma = -na$, and from (44), we obtain $(-nab + nba)(\eth(s) - s) = 0$. That is, $n[b, a](\eth(s) - s) = 0$. Using similar arguments as the above, we obtain $n[b, a]\mho(\eth(s) - s) = (0)$. Hence, $n[b, a] = 0$ and so $n = 0$ or $\mho$ is commutative. In case $n = 0$, we obtain $m = 0$, a contradiction.

Secondly: Suppose that $\eth(s) = s$. That is,

$$\eth(s) - s = 0. \tag{45}$$

Using (45) in (29), we obtain

$$\eth(\theta(s)) - \theta(s) = 0. \tag{46}$$

Using (45) and (46) in (23), we have

$$\eth(b)\eth(\theta(a))(s - \theta(s)) + (mb\theta(a) + n\theta(a)b)(s - \theta(s)) = 0.$$

That is,

$$\eth(b)\eth(\theta(a)) + k(mb\theta(a) + n\theta(a)b) = 0.$$

Hence,

$$\eth(b)\eth(a) + k(mba + nab) = 0.$$

Putting $b = s$ in the last relation and using (45), we see that

$$\eth(a) + k(m+n)a = 0. \tag{47}$$

Taking $a = s$ in (47) and using (45), we arrive at $k(m+n) = -1$. Using the last relation in (47), we obtain $\eth(a) = a$, a contradiction.

Case(II): Suppose that $\eth(\theta(s)) = 0$. Putting $b = s$ (20), we obtain $(m+n)s\theta(s) = 0$ and so $(m+n)\mho = (0)$. In particular, $ma = -na$. Using it in (22), we see that

$$\eth(b)\eth(\theta(a))\eth(s) - kn(a\theta(b)\theta(s) + b\theta(a)s - \theta(a)bs - \theta(b)a\theta(s)) = 0.$$

Replacing a by as in the last relation, we obtain

$$n(a\theta(b)\theta(s)s + b\theta(a)s\theta(s) - \theta(a)bs\theta(s) - \theta(b)a\theta(s)s) = 0.$$

That is,

$$n(a\theta(b) + b\theta(a) - \theta(a)b - \theta(b)a) = 0.$$

In particular, $n[a, \theta(a)] = 0$. This means $n\mho = (0)$ or $[a, \theta(a)] = 0$. In case $[a, \theta(a)] = 0$ for each $a \in \mho$, and using [18] (Lemma 2.3), we obtain that $\mho$ is commutative. In case $n\mho = (0)$, we obtain that $m\mho = (0)$, a contradiction. $\square$

3. Applications

The following two corollaries are obtained from Theorem 1. On the other hand, we present a possible correction to the original result in [21] (Theorem 1(2)). The proposed correction includes the introduction of the following result, which concludes that $\mho$ is either commutative or embeds in $M_2(\mathbb{F})$ for $\mathbb{F}$ a field, instead of stating that $\mho$ is commutative. Furthermore, we support our correction by presenting Example 4, which serves as a counterexample to the original claim [21] (Theorem 1(2)) and demonstrates the necessity of our proposed correction.

Corollary 1 ([21] Theorem 1). *Let $\mho$ be a prime ring of char($\mho$) $\neq$ 2 with involution $*$ of second kind. Then, the following hold:*

(i) $\mho$ possesses a non-trivial $*$-SCP endomorphism if and only if $\mho$ is commutative.
(ii) $\mho$ possesses a non-trivial $*$-Skew SCP endomorphism if and only if $\mho$ is either commutative or embeds in $M_2(\mathbb{F})$ for $\mathbb{F}$ a field.

Corollary 2. *Let $\mho$ be a prime ring of char($\mho$) $\neq$ 2 with involution $*$ of the second kind. Then, the following hold:*

(i) $\mho$ possesses a non-trivial SCP endomorphism if and only if $\mho$ is commutative.
(ii) $\mho$ possesses a non-trivial Skew SCP endomorphism if and only if $\mho$ is either commutative or embeds in $M_2(\mathbb{F})$ for $\mathbb{F}$ a field.

We obtain the following two corollaries from Theorem 2:

Corollary 3 ([21] Theorem 2). *Let $\mho$ be a prime ring of char($\mho$) $\neq$ 2 with involution $*$ of the second kind. Then, $\mho$ is commutative if any one of the following is satisfied:*

(i) $\mho$ possesses a non-trivial $*$-SACP endomorphism;
(ii) $\mho$ possesses a non-trivial $*$-Skew SACP endomorphism.

Corollary 4. *Let $\mho$ be a prime ring of char($\mho$) $\neq 2$ with involution $*$ of the second kind. Then, the following hold:*

(i) $\mho$ *is commutative if $\mho$ possesses a non-trivial SACP endomorphism.*

(ii) *If $\mho$ possesses a non-trivial Skew SACP endomorphism $\eth$, then it is an identity map.*

The following corollaries follow from Theorem 3:

Corollary 5. *Let $\mho$ be a prime ring of char($\mho$) $\neq 2$ with involution $*$ of the second kind and let $\eth$ be an endomorphism of $\mho$. Then, $\mho$ is commutative if any one of the following is satisfied:*

(i) $\eth(b) \circ \eth(b^*) = \pm[b, b^*]$ *for each $b \in \mho$.*

(ii) $\eth(b) \circ \eth(a) = \pm[b, a]$ *for each $b, a \in \mho$.*

Corollary 6. *Let $\mho$ be a prime ring of char($\mho$) $\neq 2$ with involution $*$ of the second kind and let $\eth$ be a non-identity endomorphism of $\mho$. Then, $\mho$ is commutative if any one of the following is satisfied:*

(i) $\eth(b)\eth(b^*) = \mp b \circ b^*$ *for each $b \in \mho$,*

(ii) $\eth(b)\eth(b^*) = \mp[b, b^*]$ *for each $b \in \mho$,*

(iii) $\eth(b)\eth(b^*) = \pm bb^*$ *for each $b \in \mho$,*

(iv) $\eth(b)\eth(b^*) = \mp b^*b$ *for each $b \in \mho$.*

Corollary 7. *Let $\mho$ be a prime ring of char($\mho$) $\neq 2$ with involution $*$ of the second kind and let $\eth$ be a non-identity endomorphism of $\mho$. Then, $\mho$ is commutative if any one of the following is satisfying.*

(i) $\eth(b)\eth(a) = \mp b \circ a$ *for each $b, a \in \mho$,*

(ii) $\eth(b)\eth(a) = \mp[b, a]$ *for each $b, a \in \mho$,*

(iii) $\eth(b)\eth(a) = \pm ba$ *for each $b, a \in \mho$,*

(iv) $\eth(b)\eth(a) = \pm ab$ *for each $b, a \in \mho$.*

Example 3.

(i) *The example that proves that the condition "θ is not Z-linear" is essential in our Theorems 1(i), 3, and 4(i): take $\eth = 0$ and θ corresponds any element in the prime ring of real quaternions to its conjugate.*

(ii) *The example that shows that the hypothesis "the primeness of $\mho$" is essential in our Theorems 1, 3, and 4(i): Let $\mho = \mathbb{Z} \times M_2(\mathbb{Z}) \times \mathbb{Z} \times$, $\theta(X, S, A) = (A, adj(S), X)$, and $\eth(X, S, A) = (X, 0, A)$ in Theorem 1 for each $(X, S, A) \in \mho$, where $\mathbb{Z}$ is the set of all integers, and $\eth = 0$ in Theorems 3 and 4(i). Then, θ is an antiautomorphism, and it is not Z-linear, and $\eth$ is an endomorphism on $\mho$, and $\mho$ is non-commutative.*

Example 4. *Let $\mho = M_2(\mathbb{C})$ and consider the involution $'*'$ of the second kind from $\mho$ to itself given by*
$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \mapsto \begin{pmatrix} \overline{b_{22}} & \overline{b_{12}} \\ \overline{b_{21}} & \overline{b_{11}} \end{pmatrix}, \text{ where } \overline{b_{ij}} \text{ is the complex conjugate of } b_{ij} \text{ in } \mathbb{C}, \text{ where } i, j \in \{1, 2\},$$
and the endomorphism $\eth : \mho \to \mho$ given by $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \mapsto \begin{pmatrix} \overline{b_{11}} & -\overline{b_{12}} \\ -\overline{b_{21}} & \overline{b_{22}} \end{pmatrix}$, *which is a $*$-Skew SCP endomorphism. It satisfies the assumptions of [21] (Theorem 1(2)), but $\mho$ is not commutative. We present this example to demonstrate that the claim made in [21] (Theorem 1(2)) is not correct.*

Remark 2.

(i) *When θ is taken to be an automorphism, all the findings outlined within this paper persist in their validity.*

(ii) *It is of significance to underscore that the conclusions drawn in this exposition maintain their validity even when the various assumptions are posited as accurate concerning a nontrivial ideal, as opposed to the entirety of the ring $\mho$.*

4. Future Research

In the realm of future exploration, two primary avenues beckon for the expansion of the current findings: First and foremost, delving into the realm of semiprime rings Ʊ instead of prime rings Ʊ within the context of the theorems can be a fertile trajectory. Examining the dynamics surrounding endomorphism and specialized mappings within semiprime rings can unveil novel perspectives regarding their commutative properties. Secondly, considering scenarios where functional identities are not necessarily equal to zero but belong to the center Z of Ʊ in the theorems could potentially unlock fresh opportunities for comprehending the intricate interplay between mappings and the underlying structural facets of rings.

5. Conclusions

In their groundbreaking research, Mamouni et al. [21] (2020) introduced the concepts of $*$-SCP, $*$-Skew SCP, $*$-SACP, and $*$-Skew SACP as maps on Ʊ. They demonstrated that a prime ring Ʊ with an involution of the second kind, denoted by $*$, admitted a non-trivial $*$-SCP or $*$-SACP endomorphism only if Ʊ is commutative. Building upon these significant findings, our paper was motivated to explore further possibilities by extending the concepts of $*$-SCP, $*$-Skew SCP, $*$-SACP, and $*$-Skew SACP. Instead of solely focusing on involutions denoted by $*$, we aimed to broaden the scope by considering antiautomorphisms denoted by θ. In this pursuit, we introduced the concepts of θ-SCP, θ-Skew SCP, θ-SACP, and θ-Skew SACP mappings, which enabled us to delve deeper into this parallel scenario. Moreover, we characterized prime rings under these concepts in Theorems 1 and 2, while Theorem 3 presented a well-known mixture of these concepts. Additionally, we examined famous functional identities in Theorem 4, which is parallel to the works of many researchers, such as [25–29].

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