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Terminal value problem for neutral fractional functional differential equations with Hilfer-Katugampola fractional derivative

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Abstract. In this paper, we establish the existence of solutions for a class of nonlinear neutral fractional differential equations with terminal condition and Hilfer-Katugampola fractional derivative. The arguments are based upon the Banach contraction principle, and Krasnoselskii's fixed point theorem. An example is included to show the applicability of our results.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order (non-integer); see the books [4, 5, 26, 28], the papers [9, 13, 15, 16] and the references therein. The study of fractional differential equations with delay has received great attention from many researchers, both from theoretical and practical point of view; we refer the reader to the monograph [4], and the papers [1, 2, 6–8, 11, 12, 14, 17].

The comparison principle for initial value problems (IVP for short) of ordinary differential equations is a very useful tool in the study of qualitative and quantitative theory. Terminal value problems (TVP for short) form an interesting and more challenging field of research than the theory of initial value problems [15, 18, 21, 29].

As acontinuation of the work [15], we establish in this paper existence and uniqueness results to the terminal value problem of the following Hilfer-Katugampola type fractional differential equation:

$$^{\rho}D^{\alpha,\beta}_{0^+}\left[y(t)-H(t,y_t)\right]=f\left(t,y_t\right), \quad t\in(0,b],\ 0<b<\infty, \tag{1}$$

$$y(b)=c\in\mathbb{R}, \tag{2}$$

$$y(t)=\varphi(t),\ t\in[-r,0],\ r>0, \tag{3}$$

where $^{\rho}D^{\alpha,\beta}_{0^+}, {}^{\rho}I^{1-\gamma}_{0^+}$ are the Hilfer-Katugampola fractional derivative of order $\alpha\in(0,1)$ and type $\beta\in[0,1]$ and Katugampola fractional integral of order $1-\gamma$, $(\gamma=\alpha+\beta-\alpha\beta)$ respectively, $f,H:(0,b]\times C([-r,0],\mathbb{R})\to\mathbb{R}$,

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are two given functions and $\varphi \in C([-r,0],\mathbb{R})$, with $\varphi(0) = H(0, y_0)$.

For each function y defined on $[-r,b]$ and for any $t \in [0,b]$, we denote by y_t the element of $C([-r,0],\mathbb{R})$ defined by:

$$y_t(\theta) = y(t+\theta), \quad \theta \in [-r,0],$$

The present paper is organized as follows. In Section 2, some notations are introduced and we recall some concepts of preliminaries about Hilfer-Katugampola fractional derivative and auxiliary results. In Section 3, two results for the problem (1)-(3) are presented: the first one is based on the Banach contraction principle, the second one on Krasnoselskii's fixed point theorem. Finally, in the last section, we give an example to illustrate the applicability of our main results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $b > 0, J = [0,b]$. By $C([-r,0],\mathbb{R})$, $C(J,\mathbb{R})$ we denote the Banach spaces of all continuous functions from $[-r,0]$ into $\mathbb{R}$ (resp from $[0,b]$ into $\mathbb{R}$) with the norms:

$$\|y\|_C = \{\sup |y(t)| : \ t \in [-r,0]\},$$

and

$$\|y\|_\infty = \{\sup |y(t)| : \ t \in [0,b]\},$$

respectively.

We consider the weighted spaces of continuous functions

$$C_{\gamma,\rho}(J) = \left\{ y : (0,b] \to \mathbb{R} : \left(\frac{t^\rho}{\rho}\right)^\gamma y(t) \in C(J,\mathbb{R}) \right\}, \ 0 \le \gamma < 1,$$

and

$$C^n_{\gamma,\rho}(J) = \left\{ y \in C^{n-1}(J) : y^{(n)} \in C_{\gamma,\rho}(J) \right\}, n \in \mathbb{N},$$
$$C^0_{\gamma,\rho}(J) = C_{\gamma,\rho}(J),$$

with the norms

$$\|y\|_{C_{\gamma,\rho}} = \sup_{t\in J} \left| \left(\frac{t^\rho}{\rho}\right)^\gamma y(t) \right|,$$

and

$$\|y\|_{C^n_{\gamma,\rho}} = \sum_{k=0}^{n-1} \|y^{(k)}\|_\infty + \|y^{(n)}\|_{C_{\gamma,\rho}}.$$

Consider the space $X^p_c(a,b)$, $(c \in \mathbb{R}, \ 1 \le p \le \infty)$ of those complex-valued Lebesgue measurable functions f on $[a,b]$ for which $\|f\|_{X^p_c} < \infty$, where the norm is defined by

$$\|f\|_{X^p_c} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}}, \ \ (1 \le p < \infty, c \in \mathbb{R}).$$

In particular, when $c = \frac{1}{p}$, the space $X^p_c(a,b)$ coincides with the $L_p(a,b)$ space: $X^p_{\frac{1}{p}}(a,b) = L_p(a,b)$.

Definition 2.1. *([24, 25]) (Katugampola fractional integral).*
Let $\alpha \in \mathbb{R}_+, c \in \mathbb{R}$ and $g \in X_c^p(a,b)$. The Katugampola fractional integral of order α is defined by

$$\left({}^\rho I_{a^+}^\alpha g\right)(t) = \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \frac{g(s)}{\Gamma(\alpha)} ds,\ t > a, \rho > 0,$$

where $\Gamma(\cdot)$ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt,\ \alpha > 0.$

Definition 2.2. *([24, 25])(Katugampola fractional derivative).*
Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ and $\rho > 0$. The Katugampola fractional derivative ${}^\rho D_{a^+}^\alpha$ of order α is defined by

$$\begin{aligned}\left({}^\rho D_{a^+}^\alpha g\right)(t) &= \delta_\rho^n({}^\rho I_{a^+}^{n-\alpha} g)(t) \\ &= \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t s^{\rho-1}\left(\frac{t^\rho - s^\rho}{\rho}\right)^{n-\alpha-1} \frac{g(s)}{\Gamma(n-\alpha)} ds,\ t > a, \rho > 0,\end{aligned}$$

where $n = [\alpha] + 1$ and $\delta_\rho^n = \left(t^{1-\rho}\frac{d}{dt}\right)^n$.

Theorem 2.3. *[24] Let $\alpha > 0, \beta > 0, 1 \le p \le \infty, a < b < \infty$ and $\rho, c \in \mathbb{R}, \rho \ge c$. Then, for $g \in X_c^p(a,b)$ the semigroup property is valid, i.e.*

$$\left({}^\rho I_{a^+}^\alpha\ {}^\rho I_{a^+}^\beta g\right)(t) = \left({}^\rho I_{a^+}^{\alpha+\beta} g\right)(t).$$

Lemma 2.4. *[24, 25, 27] Let $\alpha > 0$ and $0 \le \gamma < 1$. Then, ${}^\rho I_{a^+}^\alpha$ is bounded from $C_{\gamma,\rho}(J)$ into $C_{\gamma,\rho}(J)$, where $J = [a,b]$*

Lemma 2.5. *[27] Let $a < b < \infty, \alpha > 0, 0 \le \gamma < 1$ and $y \in C_{\gamma,\rho}(J)$. If $\alpha > \gamma$, then ${}^\rho I_{a^+}^\alpha y$ is continuous on J and*

$$\left({}^\rho I_{a^+}^\alpha y\right)(a) = \lim_{t\to a^+}\left({}^\rho I_{a^+}^\alpha y\right)(t) = 0.$$

Lemma 2.6. *[10] Let $x > a$. Then, for $\alpha \ge 0$ and $\beta > 0$, we have*

$$\begin{aligned}\left[{}^\rho I_{a^+}^\alpha\left(\frac{s^\rho - a^\rho}{\rho}\right)^{\beta-1}\right](t) &= \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta-1} \\ \left[{}^\rho D_{a^+}^\alpha\left(\frac{s^\rho - a^\rho}{\rho}\right)^{\alpha-1}\right](t) &= 0,\ \ 0 < \alpha < 1.\end{aligned}$$

Lemma 2.7. *[27] Let $\alpha > 0, 0 \le \gamma < 1$ and $g \in C_\gamma(J)$. Then,*

$$\left({}^\rho D_{a^+}^\alpha\ {}^\rho I_{a^+}^\alpha g\right)(t) = g(t), \quad \text{for all} \quad t \in (a,b].$$

Lemma 2.8. *[27] Let $0 < \alpha < 1, 0 \le \gamma < 1$. If $g \in C_{\gamma,\rho}(J)$ and ${}^\rho I_{a^+}^{1-\alpha} g \in C_{\gamma,\rho}^1(J)$, then*

$$\left({}^\rho I_{a^+}^\alpha\ {}^\rho D_{a^+}^\alpha g\right)(t) = g(t) - \frac{\left({}^\rho I_{a^+}^{1-\alpha} g\right)(a)}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1}, \ \text{for all} \quad t \in (a,b].$$

Definition 2.9. *([27]) Let order α and type β satisfy $n-1 < \alpha < n$ and $0 \le \beta \le 1$, with $n \in \mathbb{N}$. The Hilfer-Katugampola fractional derivative to t, with $\rho > 0$ of a function $g \in C_{1-\gamma,\rho}(J)$, is defined by*

$$\begin{aligned}\left({}^\rho D_{a^+}^{\alpha,\beta} g\right)(t) &= \left({}^\rho I_{a^+}^{\beta(n-\alpha)}\left(t^{\rho-1}\frac{d}{dt}\right)^n\ {}^\rho I_{a^+}^{(1-\beta)(n-\alpha)} g\right)(t) \\ &= \left({}^\rho I_{a^+}^{\beta(n-\alpha)}\delta_\rho^n\ {}^\rho I_{a^+}^{(1-\beta)(n-\alpha)} g\right)(t).\end{aligned}$$

In this paper we consider the case $n = 1$ only, because $0 < \alpha < 1$.

Property 2.10. *([27]) The operator* ${}^{\rho}D_{a^+}^{\alpha,\beta}$ *can be written as*

$$ {}^{\rho}D_{a^+}^{\alpha,\beta} = {}^{\rho}I_{a^+}^{\beta(1-\alpha)}\delta_{\rho}\, {}^{\rho}I_{a^+}^{1-\gamma} = {}^{\rho}I_{a^+}^{\beta(1-\alpha)}\, {}^{\rho}D_{a^+}^{\gamma}, \ \ \gamma = \alpha+\beta-\alpha\beta. $$

Property 2.11. *The fractional derivative* ${}^{\rho}D_{a^+}^{\alpha,\beta}$ *is an interpolator of the following fractional derivatives: Hilfer* ($\rho \to 1$) *[22], Hilfer–Hadamard* ($\rho \to 0^+$) *[25], generalized* ($\beta = 0$) *[24], Caputo–type* ($\beta = 1$) *[27], Riemann–Liouville* ($\beta = 0, \rho \to 1$) *[26], Hadamard* ($\beta = 0, \rho \to 0^+$) *[26], Caputo* ($\beta = 1, \rho \to 1$) *[26], Caputo–Hadamard* ($\beta = 1, \rho \to 0^+$) *[19], Liouville* ($\beta = 0, \rho \to 1, a = 0$) *[26] and Weyl* ($\beta = 0, \rho \to 1, a = -\infty$)*[23].*

Consider the following parameters α, β, γ satisfying

$$ \gamma = \alpha+\beta-\alpha\beta, \quad 0 < \alpha, \beta, \gamma < 1. $$

Thus, we define the spaces

$$ C_{1-\gamma,\rho}^{\alpha,\beta}(J) = \left\{ y \in C_{1-\gamma,\rho}(J),\ {}^{\rho}D_{a^+}^{\alpha,\beta} y \in C_{1-\gamma,\rho}(J) \right\} $$

and

$$ C_{1-\gamma,\rho}^{\gamma}(J) = \left\{ y \in C_{1-\gamma,\rho}(J),\ {}^{\rho}D_{a^+}^{\gamma} y \in C_{1-\gamma,\rho}(J) \right\}. $$

Since ${}^{\rho}D_{a^+}^{\alpha,\beta} y = {}^{\rho}I_{a^+}^{\gamma(1-\alpha)}\, {}^{\rho}D_{a^+}^{\gamma} y$, it follows from Lemma 2.4 that

$$ C_{1-\gamma,\rho}^{\gamma}(J) \subset C_{1-\gamma,\rho}^{\alpha,\beta}(J) \subset C_{1-\gamma,\rho}(J). $$

Lemma 2.12. *[27] Let* $0 < \alpha < 1, 0 \le \beta \le 1$ *and* $\gamma = \alpha+\beta-\alpha\beta$*. If* $y \in C_{1-\gamma,\rho}^{\gamma}(J)$*, then*

$$ {}^{\rho}I_{a^+}^{\gamma}\, {}^{\rho}D_{a^+}^{\gamma} y = {}^{\rho}I_{a^+}^{\alpha}\, {}^{\rho}D_{a^+}^{\alpha,\beta} y $$

and

$$ {}^{\rho}D_{a^+}^{\gamma}\, {}^{\rho}I_{a^+}^{\alpha} y = {}^{\rho}D_{a^+}^{\beta(1-\alpha)} y. $$

Theorem 2.13. *[30]($C_{1-\gamma}$ type Arzela–Ascoli Theorem) Let* $A \subset C_{1-\gamma}(J, \mathbb{R})$*. A is relatively compact (i.e* $\overline{A}$ *is compact) if:*

1. *A is uniformly bounded i.e, there exists* $M > 0$ *such that*

   $\left| f(x) \right| < M$ *for every* $f \in A$ *and* $x \in J$*.*

2. *A is equicontinuous i.e, for every* $\epsilon > 0$*, there exists* $\delta > 0$ *such that for each* $x, \overline{x} \in J$*,* $\left| x - \overline{x} \right| \le \delta$ *implies* $\left| f(x) - f(\overline{x}) \right| \le \epsilon$*, for every* $f \in A$*.*

Theorem 2.14. *([20]) (**Banach's** fixed point theorem). Let C be a non-empty closed subset of a Banach space E, then any contraction mapping T of C into itself has a unique fixed point.*

Theorem 2.15. *([31]) (**Krasnoselskii's** fixed point theorem). Let C be a closed, convex, and nonempty subset of a Banach space X, and* A, B *the operators such that*

1) $Ax + By \in C$ *for all* $x, y \in C$*;*

2) *A is compact and continuous;*

3) *B is a contraction mapping.*

Then there exists $z \in C$ *such that* $z = Az + Bz$*.*

3. Existence of Solutions

Set

$$\Omega = \{y : [-r, b] \to \mathbb{R} : \ y|_{[-r,0]} \in C([-r, 0], \mathbb{R}) \text{ and } y|_{(0,b]} \in C^{\gamma}_{1-\gamma,\rho}(J)\}.$$

Ω is a Banach space with the norm

$$\|y\|_{\Omega} = \|y\|_{C} + \|y\|_{C_{1-\gamma,\rho}}.$$

We consider the following linear problem:

$$^{\rho}D^{\alpha,\beta}_{0^+}\left[y(t) - H(t, y_t)\right] = h(t), \quad t \in (0, b], \tag{4}$$

$$y(b) = c, \ c \in \mathbb{R}, \tag{5}$$

$$y(t) = \varphi(t), \ t \in [-r, 0]. \tag{6}$$

where $h(\cdot) \in C_{1-\gamma,\rho}(J)$ and $H(\cdot, y_\cdot) \in C^{\gamma}_{1-\gamma,\rho}(J)$, with $\varphi(0) = H(0, y_0)$. The following theorem shows that (4)–(6) is equivalent to the integral equation:

$$y(t) = \begin{cases} \varphi(t), & \text{if } t \in [-r, 0], \\ \left[c - H(b, y_b) - \frac{1}{\Gamma(\alpha)} \int_0^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} h(s) ds\right] \left(\frac{t}{b}\right)^{\rho(\gamma-1)} \\ +H(t, y_t) + \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} h(s) ds, \ if t \in (0, b]. \end{cases} \tag{7}$$

Theorem 3.1. *Let $\gamma = \alpha + \beta - \alpha\beta$, where $0 < \alpha < 1$ and $0 \le \beta \le 1$. If $h : (0, b] \to \mathbb{R}$ is a function such that $h(\cdot) \in C_{1-\gamma,\rho}(J)$, then $y \in \Omega$ satisfies the problem (4)–(6) if and only if it satisfies Equation (7).*

Proof. ($\Rightarrow$) Let $y \in \Omega$, if $t \in [-r, 0]$, then we have $y(t) = \varphi(t)$.
On the other hand, for $t \in (0, b]$, we have $y \in C^{\gamma}_{1-\gamma,\rho}(J)$ be a solution of Equations (4) and (5). We prove that y is also a solution of Equation (7). From the definition of $C^{\gamma}_{1-\gamma,\rho}(J)$, Lemma 2.4, and using Definition 2.2, we have:

$$^{\rho}I^{1-\gamma}_{0^+}\left[y(\cdot) - H(\cdot, y_\cdot)\right] \in C(J, \mathbb{R})$$

and

$$^{\rho}D^{\gamma}_{0^+}\left[y(\cdot) - H(\cdot, y_\cdot)\right] = \delta_\rho \, ^{\rho}I^{1-\gamma}_{0^+}\left[y(\cdot) - H(\cdot, y_\cdot)\right] \in C_{1-\gamma,\rho}(J). \tag{8}$$

By the Definition of the space $C^{n}_{1-\gamma,\rho}(J)$, it follows that

$$^{\rho}I^{1-\gamma}_{0^+}\left[y(\cdot) - H(\cdot, y_\cdot)\right] \in C^{1}_{1-\gamma,\rho}(J).$$

Using Lemma 2.8, with $\alpha = \gamma$, we obtain:

$$\begin{aligned} ^{\rho}I^{\gamma}_{0^+} \, ^{\rho}D^{\gamma}_{0^+}\left[y(s) - H(s, y_s))\right](t) &= y(t) - H(t, y_t) \\ &- \frac{\left(^{\rho}I^{1-\gamma}_{0^+}\left[y(s) - H(s, y_s)\right]\right)(0)}{\Gamma(\gamma)} \left(\frac{t^\rho}{\rho}\right)^{\gamma-1}, \end{aligned} \tag{9}$$

where $t \in (0, b]$. By hypothesis, $y \in C^{\gamma}_{1-\gamma,\rho}(J)$, using Lemma 2.12 with Equation (4), we have

$$\begin{aligned} \left(^{\rho}I^{\gamma}_{0^+} \, ^{\rho}D^{\gamma}_{0^+}\left[y(s) - H(s, y_s)\right]\right)(t) &= \left(^{\rho}I^{\alpha}_{0^+} \, ^{\rho}D^{\alpha,\beta}_{0^+}\left[y(s) - H(s, y_s)\right]\right)(t) \\ &= \left(^{\rho}I^{\alpha}_{0^+} h\right)(t). \end{aligned} \tag{10}$$

Comparing Equations (9) and (10), we see that

$$y(t)=\frac{\left({}^\rho I_{0^+}^{1-\gamma}\left[y(s)-H(s,y_s)\right]\right)(0)}{\Gamma(\gamma)}\left(\frac{t^\rho}{\rho}\right)^{\gamma-1}+H(t,y_t)+\left({}^\rho I_{0^+}^{\alpha}h\right)(t). \tag{11}$$

Using Equation (5) we obtain

$$\begin{aligned} y(t) &= \left[c-H(b,y_b)-\frac{1}{\Gamma(\alpha)}\int_0^b\left(\frac{b^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}h(s)ds\right]\left(\frac{t}{b}\right)^{\rho(\gamma-1)} \\ &+ H(t,y_t)+\frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}h(s)ds, \end{aligned}$$

with $t\in(0,b]$, that is $y(\cdot)$ satisfies Equation (7).

($\Leftarrow$) Let $y\in\Omega$, satisfying Equation (7). We show that y also satisfies the problem (4)−(6). If $t\in[-r,0]$, then $y(t)=\varphi(t)$, so the condition (6) is satisfied.
If $t\in(0,b]$, then $y\in C_{1-\gamma,\rho}^{\gamma}(J)$. Apply operator ${}^\rho D_{0^+}^{\gamma}$ on both sides of Equation (7). Then, from Lemmas 2.6 and 2.12 we get

$${}^\rho D_{0^+}^{\gamma}\left(y(s)-H(s,y_s)\right)(t)=\left({}^\rho D_{0^+}^{\beta(1-\alpha)}h\right)(t). \tag{12}$$

By Equation (8) we have ${}^\rho D_{0^+}^{\gamma}\left(y(\cdot)-H(\cdot,y_\cdot)\right)\in C_{1-\gamma,\rho}(J)$; then, Equation (12) implies

$${}^\rho D_{0^+}^{\gamma}\left(y(s)-H(s,y_s)\right)(t)=\left(\delta_\rho\,{}^\rho I_{0^+}^{1-\beta(1-\alpha)}h\right)(t)=\left({}^\rho D_{0^+}^{\beta(1-\alpha)}h\right)(t)\in C_{1-\gamma,\rho}(J). \tag{13}$$

As $h(\cdot)\in C_{1-\gamma,\rho}(J)$ and from Lemma 2.4, it follows

$$\left({}^\rho I_{0^+}^{1-\beta(1-\alpha)}h\right)\in C_{1-\gamma,\rho}(J). \tag{14}$$

From Equations (13) and (14) and by the definition of the space $C_{1-\gamma,\rho}^{n}(J)$, we obtain

$$\left({}^\rho I_{0^+}^{1-\beta(1-\alpha)}h\right)\in C_{1-\gamma,\rho}^{1}(J).$$

Applying operator ${}^\rho I_{0^+}^{\beta(1-\alpha)}$ to both sides of Equation (13) and using Lemmas 2.5 and 2.8, we have

$${}^\rho I_{0^+}^{\beta(1-\alpha)}\,{}^\rho D_{0^+}^{\gamma}\left(y(t)-H(t,y_t)\right) = h(t)+\frac{\left({}^\rho I_{0^+}^{1-\beta(1-\alpha)}h(t)\right)(0)}{\Gamma(\beta(1-\alpha))}\left(\frac{t^\rho}{\rho}\right)^{\beta(1-\alpha)-1},$$

which implies that

$${}^\rho D_{0^+}^{\alpha,\beta}\left(y(t)-H(t,y_t)\right)=h(t),$$

that is, Equation(4) holds. Clearly, if $y\in C_{1-\gamma,\rho}^{\gamma}(J)$ satisfies Equation (7), then it also satisfies Equation(5).

Suppose that the function $f:(0,b]\times C([-r,0],\mathbb{R})\to\mathbb{R}$ is continuous and satisfies the conditions

($B1$) The functions $f,H:(0,b]\times C([-r,0],\mathbb{R})\to\mathbb{R}$ are such that

$$f(\cdot,u(\cdot))\in C_{1-\gamma,\rho}^{\beta(1-\alpha)}(J)\ \text{ and } H(\cdot,u(\cdot))\in C_{1-\gamma,\rho}^{\gamma}(J)\ \text{ for any } u\in C_{1-\gamma,\rho}(J).$$

($B2$) There exist constants $K>0$ and $\overline{L}>0$ such that

$$|f(t,u)-f(t,\bar{u})|\le K\|u-\bar{u}\|_C$$

and

$$|H(t,w)-H(t,\bar{w})|\le \overline{L}\|w-\bar{w}\|_C$$

for any $u,w,\bar{u},\bar{w}\in C([-r,0],\mathbb{R})$ and $t\in(0,b]$.

As a consequence of Theorem 3.1, we have Theorem 3.2.

Theorem 3.2. *Let $\gamma = \alpha + \beta - \alpha\beta$ where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$; let $f : (0, b] \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ be a function such that $f(\cdot, y.) \in C^{\beta(1-\alpha)}_{1-\gamma,\rho}(J) \subset C_{1-\gamma,\rho}(J)$ and $H : (0, b] \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ where $H(\cdot, y.) \in C^{\gamma}_{1-\gamma,\rho}(J)$ for any $y \in C_{1-\gamma,\rho}(J)$.*
If $y \in \Omega$, then y satisfies the problem (1)−(3) if and only if y is the fixed point of the operator $N : \Omega \to \Omega$ defined by:

$$(Ny)(t) = \begin{cases} \varphi(t), & \text{if } t \in [-r, 0], \\ \left[c - H(b, y_b) - \frac{1}{\Gamma(\alpha)} \int_0^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g(s) ds\right] \left(\frac{t}{b}\right)^{\rho(\gamma-1)} \\ +H(t, y_t) + \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g(s) ds, \ t \in (0, b], \end{cases} \tag{15}$$

where $g : (0, b] \to \mathbb{R}$ is a function satisfying the functional equation

$$g(t) = f(t, y_t).$$

Proof. Clearly, $g \in C_{1-\gamma,\rho}(J)$.

We show that if for any $y \in \Omega$ then $Ny \in \Omega$ (the operator is well defined).
Let $y \in \Omega$, if $t \in [-r, 0]$ then $(Ny)(t) = \varphi(t)$.
If $t \in (0, b]$ then we have:

$$\begin{aligned} (Ny)(t) &= \left[c - H(b, y_b) - \frac{1}{\Gamma(\alpha)} \int_0^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g(s) ds\right] \left(\frac{t}{b}\right)^{\rho(\gamma-1)} \\ &+ H(t, y_t) + \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} g(s) ds. \end{aligned}$$

Applying ${}^\rho D^\gamma_{0^+}$ to both sides and by Lemmas 2.6 and 2.12, we have

$$\begin{aligned} {}^\rho D^\gamma_{0^+} (Ny)(t) &= {}^\rho D^\gamma_{0^+} H(t, y_t) + \left({}^\rho D^\gamma_{0^+} \, {}^\rho I^\alpha_{0^+} f(s, y_s)\right)(t) \\ &= {}^\rho D^\gamma_{0^+} H(t, y_t) + \left({}^\rho D^{\beta(1-\alpha)}_{0^+} f(s, y_s)\right)(t). \end{aligned}$$

Since $\gamma \geq \alpha$, and $f(\cdot, y.) \in C^{\beta(1-\alpha)}_{1-\gamma,\rho}(J)$, the right hand side is in $C_{1-\gamma,\rho}(J)$ and thus ${}^\rho D^\gamma_{0^+} Ny \in C_{1-\gamma,\rho}(J)$, which implies that $Ny \in C^\gamma_{1-\gamma,\rho}(J)$. We can conclude that $Ny \in \Omega$.

Now, we state and prove our existence result for the problem (1)−(3) based on Banach's fixed point.

Theorem 3.3. *Assume* (*B*1) *and* (*B*2) *hold. If*

$$\max\left\{\overline{L} + \frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \left(\frac{b^\rho}{\rho}\right)^\alpha ; \overline{L} \left(\frac{b^\rho}{\rho}\right)^{1-\gamma} + \frac{K}{\Gamma(\alpha+1)} \left(\frac{b^\rho}{\rho}\right)^{1-\gamma+\alpha}\right\} < \frac{1}{2}, \tag{16}$$

then the problem (1)−(3) has unique solution in Ω.

Proof. We show that the operator N defined in Equation (15) has a unique fixed point y^* in Ω. Let $y, u \in \Omega$. If $t \in [-r, 0]$, then

$$|(Ny)(t) - Nu(t)| = 0.$$

For $t \in (0, b]$, we have

$$\begin{aligned} |(Ny)(t) - (Nu)(t)| &\leq \frac{1}{\Gamma(\alpha)} \left(\frac{t}{b}\right)^{\rho(\gamma-1)} \int_0^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} |g(s) - h(s)| ds \\ &+ |H(t, y_t) - H(t, u_t)| + \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} |g(s) - h(s)| ds \\ &+ \left(\frac{t}{b}\right)^{\rho(\gamma-1)} |H(b, y_b) - H(b, u_b)|, \end{aligned}$$

where $g, h \in C_{1-\gamma,\rho}(J)$ such that

$$g(t) = f(t, y_t),$$
$$h(t) = f(t, u_t).$$

By (B2), we have

$$\begin{aligned} |g(t) - h(t)| &= |f(t, y_t) - f(t, u_t)| \\ &\leq K\|y_t - u_t\|_C. \end{aligned}$$

On the other hand, we have $\|y_t - u_t\|_C = \sup\{|y_t(\theta) - u_t(\theta)|,\ \theta \in [-r, 0]\}$, then at least one $\theta^* \in [-r, 0]$ such that

$$\|y_t - u_t\|_C = |y_t(\theta^*) - u_t(\theta_*)| = |y(t + \theta^*) - u(t + \theta^*)|.$$

If $t + \theta^* \in [-r, 0]$, then

$$\|y_t - u_t\|_C \leq \|y - u\|_C = \|y - u\|_\Omega.$$

This implies that for each $t \in (0, b]$

$$\begin{aligned} |(Ny)(t) - (Nu)(t)| &\leq \frac{K}{\Gamma(\alpha)}\left(\frac{t}{b}\right)^{\rho(\gamma-1)}\int_0^b\left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\|y_s - u_s\|_C ds \\ &+ \overline{L}\|y_t - u_t\|_C + \frac{K}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\|y_s - u_s\|_C ds \\ &+ \left(\frac{t}{b}\right)^{\rho(\gamma-1)}\overline{L}\|y_b - u_b\|_C \\ &\leq \frac{K}{\Gamma(\alpha+1)}\left(\frac{t}{b}\right)^{\rho(\gamma-1)}\left(\frac{b^\rho}{\rho}\right)^{\alpha}\|y - u\|_\Omega \\ &+ \overline{L}\|y - u\|_\Omega + \frac{K}{\Gamma(\alpha+1)}\left(\frac{t^\rho}{\rho}\right)^{\alpha}\|y - u\|_\Omega \\ &+ \left(\frac{t}{b}\right)^{\rho(\gamma-1)}\overline{L}\|y - u\|_\Omega. \end{aligned}$$

Hence,

$$\begin{aligned} \left|\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}(Ny)(t) - (Nu)(t))\right| &\leq \left[\frac{K}{\Gamma(\alpha+1)}\left(\frac{b^\rho}{\rho}\right)^{1-\gamma+\alpha} + \overline{L}\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}\right. \\ &+ \left.\frac{K}{\Gamma(\alpha+1)}\left(\frac{t^\rho}{\rho}\right)^{1-\gamma+\alpha} + \left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\overline{L}\right]\|y - u\|_\Omega \\ &\leq 2\left[\overline{L}\left(\frac{b^\rho}{\rho}\right)^{1-\gamma} + \frac{K}{\Gamma(\alpha+1)}\left(\frac{b^\rho}{\rho}\right)^{1-\gamma+\alpha}\right]\|y - u\|_\Omega, \end{aligned}$$

which implies that

$$\begin{aligned} \|Ny - Nu\|_{C_{1-\gamma,\rho}} &= \|Ny - Nu\|_\Omega \\ &\leq 2\left[\overline{L}\left(\frac{b^\rho}{\rho}\right)^{1-\gamma} + \frac{K}{\Gamma(\alpha+1)}\left(\frac{b^\rho}{\rho}\right)^{1-\gamma+\alpha}\right]\|y - u\|_\Omega. \end{aligned}$$

If $t + \theta^* \in [0, b]$, then we have

$$\begin{aligned} \|y_t - u_t\|_C &= \left(\frac{t^\rho}{\rho}\right)^{\gamma-1}\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}\|y_t - u_t\|_C \\ &\leq \left(\frac{t^\rho}{\rho}\right)^{\gamma-1}\|y - u\|_{C_{1-\gamma,\rho}} \\ &\leq \left(\frac{t^\rho}{\rho}\right)^{\gamma-1}\|y - u\|_\Omega. \end{aligned}$$

Hence, for each $t \in (0, b]$

$$\begin{aligned}
|(Ny)(t) - (Nu)(t)| &\leq \frac{K}{\Gamma(\alpha)}\left(\frac{t}{b}\right)^{\rho(\gamma-1)} \int_0^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\|y_s - u_s\|_C ds \\
&+ \overline{L}\|y_t - u_t\|_C + \frac{K}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\|y_s - u_s\|_C ds \\
&+ \left(\frac{t}{b}\right)^{\rho(\gamma-1)} \overline{L}\|y_b - u_b\|_C \\
&\leq K\left(\frac{t}{b}\right)^{\rho(\gamma-1)} \|y - u\|_{C_{1-\gamma,\rho}} \left({}^\rho I_{0^+}^\alpha \left(\frac{s^\rho}{\rho}\right)^{\gamma-1}\right)(b) \\
&+ 2\overline{L}\left(\frac{t^\rho}{\rho}\right)^{\gamma-1} \|y - u\|_{C_{1-\gamma,\rho}} + K\left(I_{0^+}^\alpha \left(\frac{s^\rho}{\rho}\right)^{\gamma-1}\right)(t)\|y - u\|_{C_{1-\gamma,\rho}}.
\end{aligned}$$

By Lemma 2.6, we have

$$\begin{aligned}
|(Ny)(t) - (Nu)(t)| &\leq \left[\frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho}{\rho}\right)^\alpha \left(\frac{t^\rho}{\rho}\right)^{\gamma-1}\right. \\
&+ \left. 2\overline{L}\left(\frac{t^\rho}{\rho}\right)^{\gamma-1} + \frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{t^\rho}{\rho}\right)^{\alpha+\gamma-1}\right]\|y - u\|_{C_{1-\gamma,\rho}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left|\left(\frac{t^\rho}{\rho}\right)^{1-\gamma} ((Ny)(t) - (Nu)(t))\right| &\leq \left[\frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho}{\rho}\right)^\alpha + 2\overline{L} + \frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{t^\rho}{\rho}\right)^\alpha\right]\|y - u\|_{C_{1-\gamma,\rho}} \\
&\leq 2\left[\overline{L} + \frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho}{\rho}\right)^\alpha\right]\|y - u\|_\Omega,
\end{aligned}$$

which implies that

$$\|Ny - Nu\|_{C_{1-\gamma,\rho}} = \|Ny - Nu\|_\Omega \leq 2\left[\overline{L} + \frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho}{\rho}\right)^\alpha\right]\|y - u\|_\Omega.$$

By Equations (16), the operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point $y^* \in \Omega$. As a consequence of Theorem 3.2, we can conclude that the problem (1)−(3) has a unique solution in Ω.

We present now the second result, which is based on Krasnoselskii fixed point theorem.

Theorem 3.4. *Assume (B1) and (B2) hold. If*

$$\overline{L}\left[1 + \left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\right] + \frac{2K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left[\left(\frac{b^\rho}{\rho}\right)^\alpha + \left(\frac{b^\rho}{\rho}\right)^{\alpha+1-\gamma}\right] < 1,$$

and

$$\max\left\{\frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho}{\rho}\right)^\alpha + 2\overline{L}; \frac{K}{\Gamma(\alpha+1)}\left(\frac{b^\rho}{\rho}\right)^{\alpha+1-\gamma} + 2\overline{L}\left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\right\} := \max(\lambda_2; \lambda_1) < 1, \tag{17}$$

then the problem (1)−(3) has at least one solution.

Proof. Consider the set

$$B_{\eta^*} = \{y \in \Omega : \|y\|_\Omega \le \eta^*\},$$

where

$$\eta^* \ge \max\left\{ \frac{\left(\frac{b^\rho}{\rho}\right)^{1-\gamma}|c| + 2H^* + \frac{2\Gamma(\gamma)f^*}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho}{\rho}\right)^{\alpha}}{1-\left[\overline{L}\left[1+\left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\right]+\frac{2K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left[\left(\frac{b^\rho}{\rho}\right)^{\alpha}+\left(\frac{b^\rho}{\rho}\right)^{\alpha+1-\gamma}\right]\right]}; \sup_{t\in[-r,0]}|\varphi(t)| \right\},$$

$$f^* = \sup_{t\in J}\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}|f(t,0)| \text{ and } H^* = \sup_{t\in J}\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}|H(t,0)|.$$

We define the operators P and Q on B_{η^*} by:

$$Py(t) = \begin{cases} \frac{1}{2}\varphi(t), & \text{if } t\in[-r,0] \\ \left[c - H(b,y_b) - \frac{1}{\Gamma(\alpha)}\int_0^b \left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}g(s)ds\right]\left(\frac{t}{b}\right)^{\rho(\gamma-1)} \\ +H(t,y_t),\ t\in(0,b]. \end{cases} \tag{18}$$

$$Qy(t) = \begin{cases} \frac{1}{2}\varphi(t), & \text{if } t\in[-r,0] \\ \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}g(s)ds,\ t\in(0,b]. \end{cases} \tag{19}$$

Then the fractional integral Equation (15) can be written as the operator equation

$$(Ny)(t) = Py(t) + Qy(t),\ \ t\in[-r,b].$$

The proof will be given in several steps:

Step 1: We prove that $Py + Qu \in B_{\eta^*}$ for any $y, u \in B_{\eta^*}$.
If $t\in[-r,0]$ then

$$|Py(t)| \le \frac{1}{2}\sup_{t\in[-r,0]}|\varphi(t)|,$$

and

$$|Qu(t)| \le \frac{1}{2}\sup_{t\in[-r,0]}|\varphi(t)|,$$

which implies that

$$\begin{aligned} |Py + Qu|_\Omega &\le \frac{1}{2}\sup_{t\in[-r,0]}|\varphi(t)| + \frac{1}{2}\sup_{t\in[-r,0]}|\varphi(t)| \\ &= \sup_{t\in[-r,0]}|\varphi(t)| \\ &\le \eta^*. \end{aligned}$$

If $t\in(0,b]$ then multiplying both sides of Equation (18) by $\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}$, we have

$$\left(\frac{t^\rho}{\rho}\right)^{1-\gamma} Py(t) = \left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\left[c - H(b,y_b) - \frac{1}{\Gamma(\alpha)}\int_0^b\left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}g(s)ds\right]$$

$$+H(t,y_t)\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}.$$

Then,

$$\begin{aligned}\left|\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}Py(t)\right| &\leq \left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\left[\frac{1}{\Gamma(\alpha)}\int_0^b\left(\frac{b^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}|g(s)|ds\right.\\ &\quad \left.+|c|+|H(b,y_b)|\right]+\left[|H(t,y_t)-H(t,0)|+|H(t,0)|\right]\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}\\ &\leq \left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\left[\frac{1}{\Gamma(\alpha)}\int_0^b\left(\frac{b^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}|g(s)|ds+|c|\right]\\ &+ H^*+\left[\overline{L}\|y_t\|_C+H^*\left(\frac{t^\rho}{\rho}\right)^{\gamma-1}\right]\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}.\end{aligned}$$

Thus,

$$\begin{aligned}\left|\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}Py(t)\right| &\leq \left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\left[|c|+\frac{1}{\Gamma(\alpha)}\int_0^b\left(\frac{b^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}|g(s)|ds\right]\\ &+ 2H^*+\overline{L}\|y_t\|_C\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}.\end{aligned} \tag{20}$$

By (B2), we have for each $t\in(0,b]$

$$\begin{aligned}|g(t)| &= |f(t,y_t)-f(t,0)+f(t,0)|\\ &\leq |f(t,y_t)-f(t,0)|+|f(t,0)|\\ &\leq K\|y_t\|_C+\left(\frac{t^\rho}{\rho}\right)^{\gamma-1}f^*.\end{aligned}$$

Multiplying both sides of the above inequality by $\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}$, we get

$$\left|\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}g(t)\right| \leq f^*+K\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}\|y_t\|_C.$$

If $t+\theta^*\in[-r,0]$ then $\|y_t\|_C=\|y\|_C=\|y\|_\Omega$, which implies that for each $t\in(0,b]$, we have

$$\left|\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}g(t)\right|\leq f^*+K\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}\|y\|_\Omega\leq f^*+K\left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\eta^*:=M. \tag{21}$$

By replacing (21) in the inequality (20) and using Lemma 2.6, we have

$$\begin{aligned}\left|\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}Py(t)\right| &\leq \left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\left[|c|+\frac{M\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho}{\rho}\right)^{\alpha+\gamma-1}+\overline{L}\eta^*\right]\\ &+ 2H^*.\end{aligned}$$

This gives

$$\|Py\|_{C_{1-\gamma,\rho}} \leq \left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\left[|c|+\frac{M\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho}{\rho}\right)^{\alpha+\gamma-1}+\overline{L}\eta^*\right]+2H^*:=R_1$$

If $t+\theta^* \in [0,b]$ then

$$\|y_t\|_C = \left(\frac{t^\rho}{\rho}\right)^{\gamma-1}\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}\|y_t\|_C \leq \left(\frac{t^\rho}{\rho}\right)^{\gamma-1}\|y\|_\Omega.$$

Then, for each $t \in (0,b]$, we have

$$\left|\left(\frac{t^\rho}{\rho}\right)^{1-\gamma} g(t)\right| \leq f^* + K\eta^* := M_1. \tag{22}$$

By replacing (22) in the inequality (20) and using Lemma 2.6, we have

$$\begin{aligned}\left|\left(\frac{t^\rho}{\rho}\right)^{1-\gamma} Py(t)\right| &\leq \left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\left[|c| + \frac{M_1\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho}{\rho}\right)^{\alpha+\gamma-1}\right] \\ &+ \overline{L}\eta^* + 2H^*.\end{aligned}$$

This gives

$$\|Py\|_{C_{1-\gamma,\rho}} \leq \left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\left[|c| + \frac{M_1\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho}{\rho}\right)^{\alpha+\gamma-1}\right] + \overline{L}\eta^* + 2H^* := R_2.$$

For operator Q. If $t+\theta^* \in [-r,0]$ with $t \in (0,b]$ then using Equation (21) and Lemma 2.6, we have:

$$|Q(u)(t)| \leq \left[\frac{\Gamma(\gamma)f^*}{\Gamma(\alpha+\gamma)} + \frac{K\Gamma(\gamma)\eta^*}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\right]\left(\frac{t^\rho}{\rho}\right)^{\alpha+\gamma-1}.$$

Therefore

$$\begin{aligned}\left|\left(\frac{t^\rho}{\rho}\right)^{1-\gamma} Qu(t)\right| &\leq \left[\frac{\Gamma(\gamma)f^*}{\Gamma(\alpha+\gamma)} + \frac{K\Gamma(\gamma)\eta^*}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\right]\left(\frac{t^\rho}{\rho}\right)^{\alpha}, \\ &\leq \left[\frac{\Gamma(\gamma)f^*}{\Gamma(\alpha+\gamma)} + \frac{K\Gamma(\gamma)\eta^*}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\right]\left(\frac{b^\rho}{\rho}\right)^{\alpha}.\end{aligned}$$

Thus,

$$\|Qu\|_{C_{1-\gamma,\rho}} \leq \left[\frac{\Gamma(\gamma)f^*}{\Gamma(\alpha+\gamma)} + \frac{K\Gamma(\gamma)\eta^*}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\right]\left(\frac{b^\rho}{\rho}\right)^{\alpha} := T_1.$$

If $t+\theta^* \in [0,b]$ then for each $t \in (0,b]$ and by using Equation (22) and Lemma 2.6, we have

$$|Q(u)(t)| \leq \left[\frac{\Gamma(\gamma)f^*}{\Gamma(\alpha+\gamma)} + \frac{K\Gamma(\gamma)\eta^*}{\Gamma(\alpha+\gamma)}\right]\left(\frac{t^\rho}{\rho}\right)^{\alpha+\gamma-1}.$$

Therefore

$$\begin{aligned}\left|\left(\frac{t^\rho}{\rho}\right)^{1-\gamma} Qu(t)\right| &\leq \left[\frac{\Gamma(\gamma)f^*}{\Gamma(\alpha+\gamma)} + \frac{K\Gamma(\gamma)\eta^*}{\Gamma(\alpha+\gamma)}\right]\left(\frac{t^\rho}{\rho}\right)^{\alpha}, \\ &\leq \left[\frac{\Gamma(\gamma)f^*}{\Gamma(\alpha+\gamma)} + \frac{K\Gamma(\gamma)\eta^*}{\Gamma(\alpha+\gamma)}\right]\left(\frac{b^\rho}{\rho}\right)^{\alpha}.\end{aligned}$$

Thus,

$$\|Qu\|_{C_{1-\gamma,\rho}} \leq \left[\frac{\Gamma(\gamma)f^*}{\Gamma(\alpha+\gamma)} + \frac{K\Gamma(\gamma)\eta^*}{\Gamma(\alpha+\gamma)}\right]\left(\frac{b^\rho}{\rho}\right)^{\alpha} := T_2.$$

Which implies that for every $y, u \in B_{\eta^*}$ we obtain

$$\begin{aligned}
\|Py+Qu\|_{C_{1-\gamma,\rho}} &\leq \|Py\|_{C_{1-\gamma,\rho}} + \|Qu\|_{C_{1-\gamma,\rho}} \\
&\leq \max\{R_1;R_2\} + \max\{T_1;T_2\} \\
&\leq \frac{2\Gamma(\gamma)f^*}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho}{\rho}\right)^{\alpha} + \frac{2K\Gamma(\gamma)\eta^*}{\Gamma(\alpha+\gamma)}\left[\left(\frac{b^\rho}{\rho}\right)^{\alpha} + \left(\frac{b^\rho}{\rho}\right)^{\alpha+1-\gamma}\right] \\
&+ \left(\frac{b^\rho}{\rho}\right)^{1-\gamma}|c| + 2H^* + \overline{L}\eta^*\left[1+\left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\right].
\end{aligned}$$

Since

$$\eta^* \geq \frac{\left(\frac{b^\rho}{\rho}\right)^{1-\gamma}|c| + 2H^* + \frac{2\Gamma(\gamma)f^*}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho}{\rho}\right)^{\alpha}}{1-\left[\overline{L}\left[1+\left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\right] + \frac{2K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left[\left(\frac{b^\rho}{\rho}\right)^{\alpha} + \left(\frac{b^\rho}{\rho}\right)^{\alpha+1-\gamma}\right]\right]}$$

we have

$$\|Py+Qu\|_{C_{1-\gamma,\rho}} = \|Py+Qu\|_{\Omega} \leq \eta^*,$$

which implies that for each $t \in [-r, b]$ we have

$$\|Py+Qu\|_{\Omega} \leq \eta^*,$$

which infers that $Py + Qu \in B_{\eta^*}$.

Step 2: P is a contraction.
Let $y, u \in \Omega$. If $t \in [-r, 0]$; then

$$|Py(t) - Pu(t)| = 0.$$

For $t \in (0, b]$, we have

$$\begin{aligned}
|Py(t)-Pu(t)| &\leq \frac{1}{\Gamma(\alpha)}\left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t^\rho}{\rho}\right)^{\gamma-1}\int_0^b\left(\frac{b^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}|g(s)-h(s)|ds \\
&+ |H(t,y_t)-H(t,u_t)| + \left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t^\rho}{\rho}\right)^{\gamma-1}|H(b,y_b)-H(b,u_b)| \\
&\leq \frac{1}{\Gamma(\alpha)}\left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t^\rho}{\rho}\right)^{\gamma-1}\int_0^b\left(\frac{b^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}|g(s)-h(s)|ds \\
&+ \overline{L}\|y_t-u_t\|_C + \left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t^\rho}{\rho}\right)^{\gamma-1}\overline{L}\|y_b-u_b\|_C,
\end{aligned}$$

where $g, h \in C_{1-\gamma,\rho}(J)$ such that

$$g(t) = f(t, y_t),$$
$$h(t) = f(t, u_t).$$

By (B2), we have

$$\begin{aligned}
|g(t)-h(t)| &= |f(t,y_t)-f(t,u_t)| \\
&\leq K\|y_t-u_t\|_C.
\end{aligned}$$

If $t + \theta^* \in [-r, 0]$ then

$$\|y_t - u_t\|_C \leq \|y-u\|_C = \|y-u\|_{\Omega}.$$

Which implies that for each $t \in (0, b]$

$$\begin{aligned}
& |Py(t) - Pu(t)| \\
\leq\ & \frac{K}{\Gamma(\alpha)}\left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t^\rho}{\rho}\right)^{\gamma-1}\int_0^b\left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\|y_s - u_s\|_C ds \\
+\ & \overline{L}\|y_t - u_t\|_C + \left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t^\rho}{\rho}\right)^{\gamma-1}\overline{L}\|y_b - u_b\|_C \\
\leq\ & \frac{K}{\Gamma(\alpha+1)}\left(\frac{b^\rho}{\rho}\right)^{\alpha+1-\gamma}\left(\frac{t^\rho}{\rho}\right)^{\gamma-1}\|y - u\|_\Omega + \overline{L}\|y - u\|_\Omega \\
+\ & \left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t^\rho}{\rho}\right)^{\gamma-1}\overline{L}\|y - u\|_\Omega.
\end{aligned}$$

Hence

$$\begin{aligned}
\left|\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}(Py(t) - Pu(t))\right| \leq\ & \left[\frac{K}{\Gamma(\alpha+1)}\left(\frac{b^\rho}{\rho}\right)^{\alpha+1-\gamma} + \overline{L}\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}\right. \\
& \left. + \overline{L}\left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\right]\|y - u\|_\Omega,
\end{aligned}$$

which implies that

$$\|Py - Pu\|_{C_{1-\gamma,\rho}} \leq \left[\frac{K}{\Gamma(\alpha+1)}\left(\frac{b^\rho}{\rho}\right)^{\alpha+1-\gamma} + 2\overline{L}\left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\right] := \lambda_1\|y - u\|_\Omega.$$

If $t + \theta^* \in [0, b]$ then

$$\|y_t - u_t\|_C \leq \left(\frac{t^\rho}{\rho}\right)^{\gamma-1}\|y - u\|_{C_{1-\gamma,\rho}} = \left(\frac{t^\rho}{\rho}\right)^{\gamma-1}\|y - u\|_\Omega.$$

Therefore, for each $t \in (0, b]$:

$$\begin{aligned}
& |Py(t) - Pu(t)| \\
\leq\ & \frac{K}{\Gamma(\alpha)}\left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t^\rho}{\rho}\right)^{\gamma-1}\int_0^b\left(\frac{b^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\|y_s - u_s\|_C ds \\
+\ & \overline{L}\|y_t - u_t\|_C + \left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t^\rho}{\rho}\right)^{\gamma-1}\overline{L}\|y_b - u_b\|_C \\
\leq\ & K\left(\frac{b^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t^\rho}{\rho}\right)^{\gamma-1}\|y - u\|_\Omega\left({}^\rho I^\alpha_{0^+}\left(\frac{s^\rho}{\rho}\right)^{\gamma-1}\right)(b) \\
+\ & 2\overline{L}\left(\frac{t^\rho}{\rho}\right)^{\gamma-1}\|y - u\|_\Omega.
\end{aligned}$$

By Lemma 2.6, we have:

$$\begin{aligned}
|Py(t) - Pu(t)| \leq\ & \frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho}{\rho}\right)^{\alpha}\left(\frac{t^\rho}{\rho}\right)^{\gamma-1}\|y - u\|_\Omega \\
+\ & 2\overline{L}\left(\frac{t^\rho}{\rho}\right)^{\gamma-1}\|y - u\|_\Omega.
\end{aligned}$$

Hence

$$\left|\left(\frac{t^\rho}{\rho}\right)^{1-\gamma}(Py(t) - Pu(t))\right| \leq \left[\frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho}{\rho}\right)^{\alpha} + 2\overline{L}\right]\|y - u\|_\Omega,$$

which implies that

$$\|Py - Pu\|_{C_{1-\gamma,\rho}} \leq \left[\frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \left(\frac{b^\rho}{\rho} \right)^\alpha + 2\overline{L} \right] \|y-u\|_\Omega := \lambda_2 \|y-u\|_\Omega.$$

Thus

$$\|Py - Pu\|_{C_{1-\gamma,\rho}} = \|Py - Pu\|_\Omega \leq \max(\lambda_1, \lambda_2) \|y-u\|_\Omega.$$

By Equation (17) the operator P is a contraction.

Step 3: Q is compact and continuous.

The continuity of Q follows from the continuity of f. Next we prove that Q is uniformly bounded on B_{η^*}. Let any $u \in B_{\eta^*}$. If $t \in [-r, 0]$ then

$$|Qu(t)| = \frac{1}{2}|\varphi(t)| \leq |\varphi(t)| \leq \sup_{t\in[-r,0]} |\varphi(t)| \leq \eta^*.$$

Which implies that

$$\|Qu\|_\Omega \leq \eta^*.$$

If $t \in (0, b]$ then we have

$$\|Qu\|_\Omega = \|Qu\|_{C_{1-\gamma,\rho}} \leq \max\{T_1, T_2\}.$$

This means that Q is uniformly bounded on B_{η^*}. Next, we show that QB_{η^*} is equicontinuous. Let any $u \in B_{\eta^*}$ and $0 < \tau_1 < \tau_2 \leq b$. Then:

$$\begin{aligned}
\left| \left(\frac{\tau_2^\rho}{\rho}\right)^{1-\gamma} Q(y)(\tau_2) - \left(\frac{\tau_1^\rho}{\rho}\right)^{1-\gamma} Q(y)(\tau_1) \right| &\leq \frac{\left(\frac{\tau_2^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} \left(\frac{\tau_2^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} |g(s)| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} \left| \left[\left(\frac{\tau_2^\rho}{\rho}\right)^{1-\gamma} \left(\frac{\tau_2^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \right.\right. \\
&- \left.\left. \left(\frac{\tau_1^\rho}{\rho}\right)^{1-\gamma} \left(\frac{\tau_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \right] \right| |g(s)| ds \\
&\leq \frac{\max\{M_1; M\}\, \Gamma(\gamma) \left(\frac{\tau_2^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha+\gamma)} \left(\frac{\tau_2^\rho - \tau_1^\rho}{\rho}\right)^{\alpha+\gamma-1} \\
&+ \frac{\max\{M_1; M\}}{\Gamma(\alpha)} \int_0^{\tau_1} \left| \left[\left(\frac{\tau_2^\rho}{\rho}\right)^{1-\gamma} \left(\frac{\tau_2^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \right.\right. \\
&- \left.\left. \left(\frac{\tau_1^\rho}{\rho}\right)^{1-\gamma} \left(\frac{\tau_1^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \right] \right| \left(\frac{s^\rho}{\rho}\right)^{\gamma-1} ds.
\end{aligned}$$

Note that

$$\left| \left(\frac{\tau_2^\rho}{\rho}\right)^{1-\gamma} Q(y)(\tau_2) - \left(\frac{\tau_1^\rho}{\rho}\right)^{1-\gamma} Q(y)(\tau_1) \right| \to 0 \quad \text{as} \quad \tau_2 \to \tau_1.$$

This shows that Q is equicontinuous on $[-r, b]$. Therefore, Q is relatively compact on B_{η^*}. By $C_{1-\gamma}$, type Arzela–Ascoli Theorem Q is compact on B_{η^*}.

As a consequence of Krasnoselskii's fixed point theorem, we conclude that N has at least a fixed point $y^* \in \Omega$. Using Lemma 3.2, we conclude that the problem (1)−(3) has at least one solution in the space Ω.

4. An Example

Consider the following TVP(Terminal Value Problem):

$$ {}^{\frac{1}{2}}D_{1^+}^{\frac{1}{2},0}\left[y_t - \frac{te^t|y_t|}{200\,(1+|y_t|)} - \frac{1}{\sqrt{2\sqrt{t}}}\right] = \frac{\ln(\sqrt{t}+1)}{\sqrt{2\sqrt{t}}} + \frac{2+|y_t|}{84e^{-t+3}\,(1+|y_t|)}, t\in(0,2], \tag{23} $$

$$ y(2) = c \in \mathbb{R}, \tag{24} $$

$$ y(t) = \varphi(t),\ t\in[-r,0],\ r>0. \tag{25} $$

where $\varphi \in C([-r,0],\mathbb{R})$. Set

$$ f(t,u) = \frac{2+|u|}{84e^{-t+3}(1+|u|)} + \frac{\ln(\sqrt{t}+1)}{\sqrt{2\sqrt{t}}},\ \ t\in(0,2],\ \ u\in C([-r,0],\mathbb{R}), $$

and

$$ H(t,w) = \frac{te^t|w|}{200\,(1+|w|)} + \frac{1}{\sqrt{2\sqrt{t}}},\ \ t\in(0,2],\ \ w\in C([-r,0],\mathbb{R}). $$

We have

$$ C^{\beta(1-\alpha)}_{1-\gamma,\rho}([0,2]) = C^0_{\frac{1}{2},\frac{1}{2}}([0,2]) = C_{\frac{1}{2},\frac{1}{2}}([0,2]), $$

with $\gamma=\alpha=\rho=\frac{1}{2}$ and $\beta=0$. Clearly, the functions $f\in C_{\frac{1}{2},\frac{1}{2}}([0,2])$ and $H\in C^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}([0,2])$. Hence condition (B1) is satisfied.
For each $u,\bar{u}\in C([-r,0],\mathbb{R})$ and $t\in(0,2]$:

$$ \begin{aligned} |f(t,u)-f(t,\bar{u})| &\le \frac{1}{84e^{-t+3}}\|u-\bar{u}\|_C \\ &\le \frac{1}{84e}\|u-\bar{u}\|_C \end{aligned} $$

and

$$ |H(t,u)-H(t,\bar{u})| \le \frac{e^2}{100}\|u-\bar{u}\|_C. $$

Hence condition $(B2)$ is satisfied with $K=\frac{1}{84e}$ and $\overline{L}=\frac{e^2}{100}$.
The condition

$$ \max\left\{\overline{L} + \frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{b^\rho}{\rho}\right)^{\alpha}; \overline{L}\left(\frac{b^\rho}{\rho}\right)^{1-\gamma} + \frac{K}{\Gamma(\alpha+1)}\left(\frac{b^\rho}{\rho}\right)^{1-\gamma+\alpha}\right\} \approx 0.1358 < \frac{1}{2}, $$

is satisfied with $b=2$. It follows from Theorem 3.3 that problem (24)-(25) has a unique solution.

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