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Inverse problems for discontinuous Dirac operator with eigenparameter dependent boundary and transmission conditions

Yalçın Güldü | **Merve Arslantaş**

Department of Mathematics, Faculty of Science, Cumhuriyet University, Sivas, Turkey

Correspondence
Yalçın Güldü, Department of Mathematics, Faculty of Science, Cumhuriyet University, 58140 Sivas, Turkey.
Email: yguldu@gmail.com

Abstract

In this study, we consider the discontinuous Dirac equations system with eigenparameter dependent boundary and finite number of transmission conditions. First, the space that corresponds to problem is introduced, the norm on this space is defined and the operator model that corresponds to the given problem is constructed on this space. Then the integral equations and asymptotics of eigenfunctions of the problem are obtained. The characteristic function is defined and the asymptotic formula of the characteristic function is given by using obtained asymptotics of eigenfunctions. After the Weyl solution and the Weyl function of the problem are formed. Finally, some uniqueness theorems are proved by using Weyl function and some spectral data.

KEYWORDS

characteristic function, Dirac operator, eigenfunction, eigenvalue, inverse problem, Weyl function, Weyl solution

1 | INTRODUCTION

In this paper, we consider the Dirac equations system

$$\ell[Y(x)] := BY'(x) + \Omega(x)Y(x) = \lambda\rho(x)Y(x) \tag{1}$$

on set (a, b), where $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & r(x) \end{pmatrix}$, $Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $p(x)$, $q(x)$ and $r(x)$ are real valued functions in $L_2(a, b)$, λ is a spectral parameter, for $a = \xi_0 < \xi_1 < \dots < \xi_{n+1} = b$ and $\rho(x) = \rho_i$ as $\xi_i < x < \xi_{i+1}$, $\rho_i \in \mathbb{R}^+, i = 0, 1, \dots, n$.

We denote by L the boundary value problem generated by Equation (1) with the following boundary and transmission conditions

$$l_1 y := a_2(\lambda)y_2(a) - a_1(\lambda)y_1(a) = 0 \tag{2}$$

$$l_2 y := b_2(\lambda)y_2(b) - b_1(\lambda)y_1(b) = 0 \tag{3}$$

$$\begin{cases} l_3 y := y_1(\xi_i + 0) - \theta_i y_1(\xi_i - 0) = 0 \\ l_4 y := y_2(\xi_i + 0) - \theta_i^{-1} y_2(\xi_i - 0) - \gamma_i(\lambda) y_1(\xi_i - 0) = 0 \end{cases} \tag{4}$$

where $a_1(\lambda)$, $a_2(\lambda)$, $b_1(\lambda)$, $b_2(\lambda)$, and $\gamma_i(\lambda)$ are polynomials of λ with real coefficients also $a_1(\lambda)$ and $a_2(\lambda)$ as well as $b_1(\lambda)$ and $b_2(\lambda)$ do not have common zeros, $\theta_i \in \mathbb{R}^+ i = 1, \dots, n$. The polynomials in boundary and transmission conditions have the form $a_1(\lambda) = \sum_{k=0}^{m_1} a_{k1}\lambda^k$, $a_2(\lambda) = \sum_{k=0}^{m_2} a_{k2}\lambda^k$, $b_1(\lambda) = \sum_{k=0}^{m_3} b_{k1}\lambda^k$, $b_2(\lambda) = \sum_{k=0}^{m_4} b_{k2}\lambda^k$ and $\gamma_i(\lambda) = \sum_{k=0}^{r_i} \gamma_{ki}\lambda^k$. Moreover, $m_a := \max\{m_1, m_2\}$, $m_b := \max\{m_3, m_4\}$, $r := \max_{1 \le i \le n}\{\deg(\gamma_i)\}$.

Inverse problems of spectral analysis lead to recovery operators from their spectral characteristics. This kind of problems emerge in mathematics, physics, mechanics, geophysics, electronics, and other branches of natural sciences. Currently, studies on the inverse problem theory are getting more interesting.

The inverse problem of a Sturm–Liouville operator was firstly studied by Ambarzumian in 1929 [1], next by Borg in 1946 [2]. Later, these results were developed in many fields. Borg's this classical paper on inverse spectral theory have the idea of using two sequences of eigenvalues with the same boundary condition at one end and different conditions at the other end. Then, these results were developed to various versions.

Rational conditions $a_1(\lambda)y_1(a) + a_2(\lambda)y_2(a) = 0$ were investigated in [3] and by several following authors, where $a_1(\lambda)$ and $a_2(\lambda)$ are polynomials. When $\deg(a_1) = \deg(a_2) = 1$, this condition depends on spectral parameter as linearly. For instance; in 1973, Walter [4] and in 1977, Fulton [5] studied regular Sturm–Liouville problem involving the eigenvalue parameter λ as linearly in the boundary conditions. Then, boundary value problems which have eigenvalue parameter λ in boundary and transmission conditions were investigated and this kind of problems also arise in various studies of mathematics as well as in applications in [6–15,32,34], where further references and links to applications can be found. Moreover, direct and inverse spectral problems for the appearance of the parameter in boundary and transmission conditions as non-linearly have been investigated fairly completely for classical Sturm–Liouville operator and also Dirac operator [16–24]. Besides all this, in recent years, Sturm–Liouville operators with eigenparameter dependent discontinuity conditions were studied in [25]. And also, complete solutions of various direct and inverse spectral problems for Sturm–Liouville operators with boundary conditions containing rational Herglotz–Nevanlinna functions of the eigenvalue parameter were provided in [26] and [27], and were subsequently extended to distributional [28] and Bessel-type potentials [29,33]. On the other hand, inverse problems for Dirac and Sturm–Liouville operators with eigenparameter dependent nonseparated boundary conditions were solved in [30, 31], respectively.

In this paper, we consider the inverse problem for reconstruction of considered Dirac problem (1)–(4) with boundary conditions and a finite number of transmission conditions dependent polynomials with respect to λ by Weyl function and by spectral data $\{\lambda_n, \mu_n\}_{n\in\mathbb{Z}}$. In this direction, the main result of this paper is that function $\Omega(x)$, the coefficients $b_1(\lambda)$ and $b_2(\lambda)$ of boundary conditions, and the coefficients θ_i, $\gamma_i(\lambda)$ for $i = 1, 2, \dots, n$ in transmission conditions can be uniquely recovered from Weyl function provided that the coefficients $a_1(\lambda)$ and $a_2(\lambda)$ are known a priori.

2 | OPERATOR TREATMENT AND ASYMPTOTICS OF SOLUTIONS

Let us consider the space $H := L_2(a,b) \oplus L_2(a,b) \oplus \mathbb{C}^{m_a} \oplus \mathbb{C}^{m_b} \oplus \sum_{i=1}^{n} \mathbb{C}^r$, $r = \sum_{i=1}^{n} r_i$ and the norm of the element Y in H is defined by

$$\|Y\|^2 := \int_a^b \rho(x) \left\{ |y_1(x)|^2 + |y_2(x)|^2 \right\} dx + \sum_{i=1}^{m_a} Y_i^1 \overline{Y_i^1} + \sum_{j=1}^{m_b} Y_j^2 \overline{Y_j^2} + \sum_{j=1}^{n} \sum_{i=1}^{n} Y_i^{3_j} \overline{Y_i^{3_j}} \tag{5}$$

for $Y = \left(Y(x), Y^1, Y^2, Y^3\right) \in H$, where $Y^1 = \left(Y_1^1, Y_2^1, \dots, Y_{m_a}^1\right)$, $Y^2 = \left(Y_1^2, Y_2^2, \dots, Y_{m_b}^2\right)$, $Y^3 = \left(Y_1^{3_i}, Y_2^{3_i}, \dots, Y_{r_i}^{3_i}\right)$, $i = 1, 2, \dots, n$.

Define an operator T with the domain

$$\begin{aligned} D(T) = \Bigg\{ & Y \in H : Y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} \in AC_{loc}(a,b), lf \in L_2(a,b), \\ & y_1(\xi_i + 0) - \theta_i y_1(\xi_i - 0) = 0, \\ & Y_1^1 = a_{m_a 2} y_2(a) - a_{m_a 1} y_1(a), Y_1^2 = b_{m_b 2} y_2(b) - b_{m_b 1} y_1(b) \\ & Y_1^{3_i} = \gamma_{r_i i} y_1(\xi_i - 0), i = 1, 2, \dots, n \Bigg\} \end{aligned}$$

such that $TY := W = \left(ly, W^1, W^2, W^{3_i}\right)$, with

$$\begin{aligned} W^1 &= \left(W_1^1, W_2^1, \dots, W_{m_a}^1\right), \\ W^2 &= \left(W_1^2, W_2^2, \dots, W_{m_b}^2\right), W^{3_i} = \left(W_1^{3_i}, W_2^{3_i}, \dots, W_{r_i}^{3_i}\right), i = 1, 2, \dots, n, \\ W_i^1 &= Y_{i+1}^1 - a_{m_a - i,2} y_2(a) + a_{m_a - i,1} y_1(a), i = 1, 2, \dots, m_a - 1, \\ W_{m_a}^1 &= -a_{02} y_2(a) + a_{01} y_1(a), \\ W_j^2 &= Y_{j+1}^2 - b_{m_b - j,2} y_2(b) + b_{m_b - j,1} y_1(b), j = 1, 2, \dots, m_b - 1, \\ W_{m_b}^2 &= -b_{02} y_2(b) + b_{01} y_1(b), \\ W_k^{3_i} &= Y_{k+1}^{3_i} - \gamma_{r_i - k,i} y_1(\xi_i - 0), k = 1, 2, \dots, r_i - 1, \qquad i = 1, 2, \dots, n, \\ W_{r_i}^{3_i} &= -\gamma_{0i} y_1(\xi_i - 0) + y_2(\xi_i + 0) - \theta_i^{-1} y_2(\xi_i - 0), \qquad i = 1, 2, \dots, n. \end{aligned}$$

The following theorem can be proved by using definition of T.

Theorem 1 *The eigenvalue problem of operator T is adequate problem of* (1)–(4), *that is, eigenvalues of operator T and problem* (1)–(4) *coincide.*

Let the functions $S(x,\lambda) = \begin{pmatrix} S_1(x,\lambda) \\ S_2(x,\lambda) \end{pmatrix}$, $C(x,\lambda) = \begin{pmatrix} C_1(x,\lambda) \\ C_2(x,\lambda) \end{pmatrix}$ $\varphi(x,\lambda) = \varphi_i(x,\lambda) = (\varphi_{i1}(x\lambda), \varphi_{i2}(x,\lambda))^T, x \in (\xi_i, \xi_{i+1})$, and $\psi(x,\lambda) = \psi_i(x,\lambda) = (\psi_{i1}(x\lambda), \psi_{i2}(x,\lambda))^T, x \in (\xi_i, \xi_{i+1})$, $i = 0, 1, \dots, n$ such that $\xi_0 = a$ and $\xi_{n+1} = b$ be solutions of Equation (1) satisfy the initial conditions

$$S(a,\lambda)=\begin{pmatrix}0\\1\end{pmatrix},\quad \mathrm{C(a,}\lambda)=\begin{pmatrix}1\\0\end{pmatrix},$$
$$\varphi_0(a,\lambda)=\begin{pmatrix}\varphi_{01}(a,\lambda)\\ \varphi_{02}(a,\lambda)\end{pmatrix}=\begin{pmatrix}a_2(\lambda)\\ a_1(\lambda)\end{pmatrix}, \tag{6}$$
$$\psi_n(b,\lambda)=\begin{pmatrix}\psi_{n1}(b,\lambda)\\ \psi_{n2}(b,\lambda)\end{pmatrix}=\begin{pmatrix}b_2(\lambda)\\ b_1(\lambda)\end{pmatrix}$$

and the transmission conditions (4).

Lemma 1 *The following equations hold for the solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$:*

$$\begin{aligned}
\varphi_{01}(x,\lambda)&=a_2(\lambda)\cos\lambda\rho_0(x-a)-a_1(\lambda)\sin\lambda\rho_0(x-a)\\
&+\int_a^x\left[p(t)\sin\lambda\rho_0(x-t)+q(t)\cos\lambda\rho_0(x-t)\right]\varphi_{01}(t,\lambda)dt\\
&+\int_a^x\left[q(t)\sin\lambda\rho_0(x-t)+r(t)\cos\lambda\rho_0(x-t)\right]\varphi_{02}(t,\lambda)dt\\
\varphi_{02}(x,\lambda)&=a_1(\lambda)\cos\lambda\rho_0(x-a)+a_2(\lambda)\sin\lambda\rho_0(x-a)\\
&+\int_a^x\left[-p(t)\cos\lambda\rho_0(x-t)+q(t)\sin\lambda\rho_0(x-t)\right]\varphi_{01}(t,\lambda)dt\\
&+\int_a^x\left[-q(t)\cos\lambda\rho_0(x-t)+r(t)\sin\lambda\rho_0(x-t)\right]\varphi_{02}(t,\lambda)dt\\
&\textit{for}\quad i=1,2,\ \dots\ ,n;\\
\varphi_{i1}(x,\lambda)&=\theta_i\varphi_{i-1,1}(\xi_i,\lambda)\cos\lambda\rho_i(x-\xi_i)\\
&-\left(\theta_i^{-1}\varphi_{i-1,2}(\xi_i,\lambda)+\gamma_i(\lambda)\varphi_{i-1,1}(\xi_i,\lambda)\right)\sin\lambda\rho_i(x-\xi_i)\\
&+\int_{\xi_i}^x\left[p(t)\sin\lambda\rho_i(x-t)+q(t)\cos\lambda\rho_i(x-t)\right]\varphi_{i1}(t,\lambda)dt\\
&+\int_{\xi_i}^x\left[q(t)\sin\lambda\rho_i(x-t)+r(t)\cos\lambda\rho_i(x-t)\right]\varphi_{i2}(t,\lambda)dt\\
\varphi_{i2}(x,\lambda)&=\theta_i\varphi_{i-1,1}(\xi_i,\lambda)\sin\lambda\rho_i(x-\xi_i)\\
&+\left(\theta_i^{-1}\varphi_{i-1,2}(\xi_i,\lambda)+\gamma_i(\lambda)\varphi_{i-1,1}(\xi_i,\lambda)\right)\cos\lambda\rho_i(x-\xi_i)\\
&+\int_{\xi_i}^x\left[-p(t)\cos\lambda\rho_i(x-t)+q(t)\sin\lambda\rho_i(x-t)\right]\varphi_{i1}(t,\lambda)dt\\
&+\int_{\xi_i}^x\left[-q(t)\cos\lambda\rho_i(x-t)+r(t)\sin\lambda\rho_i(x-t)\right]\varphi_{i2}(t,\lambda)dt.
\end{aligned}$$

and

$$\begin{aligned}
\psi_{n1}(x,\lambda)&=b_2(\lambda)\cos\lambda\rho_n(x-b)-b_1(\lambda)\sin\lambda\rho_n(x-b)\\
&-\int_x^b\left[p(t)\sin\lambda\rho_n(x-t)+q(t)\cos\lambda\rho_n(x-t)\right]\psi_{n1}(t,\lambda)dt\\
&-\int_x^b\left[q(t)\sin\lambda\rho_n(x-t)+r(t)\cos\lambda\rho_n(x-t)\right]\psi_{n2}(t,\lambda)dt\\
\psi_{n2}(x,\lambda)&=b_1(\lambda)\cos\lambda\rho_n(x-b)+b_2(\lambda)\sin\lambda\rho_n(x-b)
\end{aligned}$$

$$+\int_x^b\left[p(t)\cos\lambda\rho_n(x-t)-q(t)\sin\lambda\rho_n(x-t)\right]\psi_{n1}(t,\lambda)dt$$
$$+\int_x^b\left[q(t)\cos\lambda\rho_n(x-t)-r(t)\sin\lambda\rho_n(x-t)\right]\psi_{n2}(t,\lambda)dt$$

for $i=n-1,n-2,\ \dots\ ,0;$

$$\begin{aligned}\psi_{i1}(x,\lambda)=&\ \theta_{i+1}^{-1}\psi_{i+1,1}\left(\xi_{i+1},\lambda\right)\cos\lambda\rho_i\left(x-\xi_{i+1}\right)\\&-\left(\theta_{i+1}\psi_{i+1,2}\left(\xi_{i+1},\lambda\right)-\gamma_{i+1}(\lambda)\psi_{i+1,1}\left(\xi_{i+1},\lambda\right)\right)\sin\lambda\rho_i\left(x-\xi_{i+1}\right)\\&-\int_x^{\xi_{i+1}}\left[p(t)\sin\lambda\rho_i(x-t)+q(t)\cos\lambda\rho_i(x-t)\right]\psi_{i1}(t,\lambda)dt\\&-\int_x^{\xi_{i+1}}\left[q(t)\sin\lambda\rho_i(x-t)+r(t)\cos\lambda\rho_i(x-t)\right]\psi_{i2}(t,\lambda)dt\end{aligned}$$

$$\begin{aligned}\psi_{i2}(x,\lambda)=&\ \theta_{i+1}^{-1}\psi_{i+1,1}\left(\xi_{i+1},\lambda\right)\sin\lambda\rho_i\left(x-\xi_{i+1}\right)\\&+\left(\theta_{i+1}\psi_{i+1,2}\left(\xi_{i+1},\lambda\right)-\gamma_{i+1}(\lambda)\psi_{i+1,1}\left(\xi_{i+1},\lambda\right)\right)\cos\lambda\rho_i\left(x-\xi_{i+1}\right)\\&+\int_x^{\xi_{i+1}}\left[p(t)\cos\lambda\rho_i(x-t)-q(t)\sin\lambda\rho_i(x-t)\right]\psi_{i1}(t,\lambda)dt\\&+\int_x^{\xi_{i+1}}\left[q(t)\cos\lambda\rho_i(x-t)-r(t)\sin\lambda\rho_i(x-t)\right]\psi_{i2}(t,\lambda)dt.\end{aligned}$$

Lemma 2 $\varphi(x,\lambda)=(\varphi_{i1}(x\lambda),\varphi_{i2}(x,\lambda))^T$, $x\in(\xi_i,\xi_{i+1})$, $i=0,\ 1,\ \dots,n$ *are entire functions of* λ *for each fixed x and the following asymptotic relations are true for these solutions as* $|\lambda|\to\infty$:

$$\begin{aligned}&\text{if }\deg(a_2)>\deg(a_1);\\&\varphi_{01}(x,\lambda)=a_{m_2 2}\lambda^{m_2}\cos\lambda\rho_0(x-a)+o\left(\lambda^{m_2}\exp|\mathrm{Im}\lambda|(x-a)\rho_0\right),\\&\varphi_{02}(x,\lambda)=a_{m_2 2}\lambda^{m_2}\sin\lambda\rho_0(x-a)+o\left(\lambda^{m_2}\exp|\mathrm{Im}\lambda|(x-a)\rho_0\right),\\&\varphi_{11}(x,\lambda)=-\gamma_{r_1 1}a_{m_2 2}\lambda^{r_1+m_2}\cos\lambda\rho_0\left(\xi_1-a\right)\sin\lambda\rho_1\left(x-\xi_1\right),\\&\qquad+\ o\left(\lambda^{r_1+m_2}\exp|\mathrm{Im}\lambda|\left(\left(\xi_1-a\right)\rho_0+\left(x-\xi_1\right)\rho_1\right)\right),\\&\varphi_{12}(x,\lambda)=\gamma_{r_1 1}a_{m_2 2}\lambda^{r_1+m_2}\cos\lambda\rho_0\left(\xi_1-a\right)\cos\lambda\rho_1\left(x-\xi_1\right),\\&\qquad+\ o\left(\lambda^{r_1+m_2}\exp|\mathrm{Im}\lambda|\left(\left(\xi_1-a\right)\rho_0+\left(x-\xi_1\right)\rho_1\right)\right),\end{aligned}$$

proceeding in a similar way,

$$\begin{aligned}\varphi_{n1}(x,\lambda)=&\ (-1)^n a_{m_2 2}\lambda^{A+m_2}\prod_{i=1}^{n}\gamma_{r_i i}\cos\lambda\rho_0\left(\xi_1-a\right)\sin\lambda\rho_n\left(x-\xi_n\right)\\&\times\prod_{i=2}^{n}\sin\lambda\rho_{i-1}\left(\xi_i-\xi_{i-1}\right)\\&+\ o\left(\lambda^{A+m_2}\exp|\mathrm{Im}\lambda|\left(\left(b-\xi_n\right)\rho_n+\sum_{i=1}^{n}\left(\xi_i-\xi_{i-1}\right)\rho_{i-1}\right)\right),\end{aligned}$$

$$\begin{aligned}\varphi_{n2}(x,\lambda)=&\ (-1)^{n+1} a_{m_2 2}\lambda^{A+m_2}\prod_{i=1}^{n}\gamma_{r_i i}\cos\lambda\rho_0\left(\xi_1-a\right)\cos\lambda\rho_n\left(x-\xi_n\right)\\&\times\prod_{i=2}^{n}\sin\lambda\rho_{i-1}\left(\xi_i-\xi_{i-1}\right)\\&+\ o\left(\lambda^{A+m_2}\exp|\mathrm{Im}\lambda|\left(\left(b-\xi_n\right)\rho_n+\sum_{i=1}^{n}\left(\xi_i-\xi_{i-1}\right)\rho_{i-1}\right)\right),\end{aligned}$$

$$\textit{if}\,\deg(a_1) > \deg(a_2)\,;$$

$$\begin{aligned}
\varphi_{01}(x,\lambda) &= -a_{m_1 1}\lambda^{m_1}\sin\lambda\rho_0(x-a) + o\left(\lambda^{m_1}\exp|\mathrm{Im}\lambda|(x-a)\rho_0\right),\\
\varphi_{02}(x,\lambda) &= a_{m_1 1}\lambda^{m_1}\cos\lambda\rho_0(x-a) + o\left(\lambda^{m_1}\exp|\mathrm{Im}\lambda|(x-a)\rho_0\right),\\
\varphi_{11}(x,\lambda) &= \gamma_{r_1 1}a_{m_1 1}\lambda^{r_1+m_1}\sin\lambda\rho_0\left(\xi_1-a\right)\sin\lambda\rho_1\left(x-\xi_1\right)\\
&\quad +\ o\left(\lambda^{r_1+m_1}\exp|\mathrm{Im}\lambda|\left(\left(\xi_1-a\right)\rho_0+\left(x-\xi_1\right)\rho_1\right)\right),\\
\varphi_{12}(x,\lambda) &= -\gamma_{r_1 1}a_{m_1 1}\lambda^{r_1+m_1}\sin\lambda\rho_0\left(\xi_1-a\right)\cos\lambda\rho_1\left(x-\xi_1\right)\\
&\quad +\ o\left(\lambda^{r_1+m_1}\exp|\mathrm{Im}\lambda|\left(\left(\xi_1-a\right)\rho_0+\left(x-\xi_1\right)\rho_1\right)\right),
\end{aligned}$$

proceeding in a similar way,

$$\begin{aligned}
\varphi_{n1}(x,\lambda) &= (-1)^{n+1}a_{m_1 1}\lambda^{A+m_1}\prod_{i=1}^{n}\gamma_{r_i i}\sin\lambda\rho_0\left(\xi_1-a\right)\sin\lambda\rho_n\left(x-\xi_n\right)\\
&\quad\times\prod_{i=2}^{n}\sin\lambda\rho_{i-1}\left(\xi_i-\xi_{i-1}\right)\\
&\quad +\ o\left(\lambda^{A+m_1}\exp|\mathrm{Im}\lambda|\left(\left(b-\xi_n\right)\rho_n+\sum_{i=1}^{n}\left(\xi_i-\xi_{i-1}\right)\rho_{i-1}\right)\right),\\
\varphi_{n2}(x,\lambda) &= (-1)^{n}a_{m_1 1}\lambda^{A+m_1}\prod_{i=1}^{n}\gamma_{r_i i}\sin\lambda\rho_0\left(\xi_1-a\right)\cos\lambda\rho_n\left(x-\xi_n\right)\\
&\quad\times\prod_{i=2}^{n}\sin\lambda\rho_{i-1}\left(\xi_i-\xi_{i-1}\right)\\
&\quad +\ o\left(\lambda^{A+m_1}\exp|\mathrm{Im}\lambda|\left(\left(b-\xi_n\right)\rho_n+\sum_{i=1}^{n}\left(\xi_i-\xi_{i-1}\right)\rho_{i-1}\right)\right),
\end{aligned}$$

$$\textit{if}\,\deg(a_1) = \deg(a_2)\,;$$

$$\begin{aligned}
\varphi_{01}(x,\lambda) &= \lambda^{m}\left(a_{m_2 2}\cos\lambda\rho_0(x-a) - a_{m_1 1}\sin\lambda\rho_0(x-a)\right)\\
&\quad +\ o\left(\lambda^{m}\exp|\mathrm{Im}\lambda|(x-a)\rho_0\right),\\
\varphi_{02}(x,\lambda) &= \lambda^{m}\left(a_{m_1 1}\cos\lambda\rho_0(x-a) + a_{m_2 2}\sin\lambda\rho_0(x-a)\right)\\
&\quad +\ o\left(\lambda^{m}\exp|\mathrm{Im}\lambda|(x-a)\rho_0\right),\\
\varphi_{11}(x,\lambda) &= \gamma_{r_1 1}\lambda^{r_1+m}(-a_{m_2 2}\cos\lambda\rho_0\left(\xi_1-a\right)\sin\lambda\rho_1\left(x-\xi_1\right)\\
&\quad +\ a_{m_1 1}\sin\lambda\rho_0\left(\xi_1-a\right)\sin\lambda\rho_1\left(x-\xi_1\right)\\
&\quad +\ o\left(\lambda^{r_1+m}\exp|\mathrm{Im}\lambda|\left(\left(\xi_1-a\right)\rho_0+\left(x-\xi_1\right)\rho_1\right)\right),\\
\varphi_{12}(x,\lambda) &= \gamma_{r_1 1}\lambda^{r_1+m}\Bigg(-a_{m_2 2}\cos\lambda\rho_0\left(\xi_1-a\right)\cos\lambda\rho_1\left(x-\xi_1\right)\\
&\quad +\ a_{m_1 1}\sin\lambda\rho_0\left(\xi_1-a\right)\sin\lambda\rho_1\left(x-\xi_1\right)\Bigg)\\
&\quad +\ o\left(\lambda^{r_1+m}\exp|\mathrm{Im}\lambda|\left(\left(\xi_1-a\right)\rho_0+\left(x-\xi_1\right)\rho_1\right)\right),
\end{aligned}$$

proceeding in a similar way,

$$\varphi_{n1}(x,\lambda) = \lambda^{A+m}\prod_{i=1}^{n}\gamma_{r_i i}\left[(-1)^n a_{m_2 2}\cos\lambda\rho_0(\xi_1-a)\sin\lambda\rho_n(x-\xi_n)\right.$$
$$\times\prod_{i=2}^{n}\sin\lambda\rho_{i-1}(\xi_i-\xi_{i-1}) + (-1)^{n+1}a_{m_1 1}\sin\lambda\rho_0(\xi_1-a)\sin\lambda\rho_n(x-\xi_n)$$
$$\left.\times\prod_{i=2}^{n}\sin\lambda\rho_{i-1}(\xi_i-\xi_{i-1})\right]$$
$$+\ o\left(\lambda^{A+m}\exp|\mathrm{Im}\lambda|\left((b-\xi_n)\rho_n+\sum_{i=1}^{n}(\xi_i-\xi_{i-1})\rho_{i-1}\right)\right),$$

$$\varphi_{n2}(x,\lambda) = \lambda^{A+m}\prod_{i=1}^{n}\gamma_{r_i i}\left[(-1)^{n+1} a_{m_2 2}\cos\lambda\rho_0(\xi_1-a)\cos\lambda\rho_n(x-\xi_n)\right.$$
$$\times\prod_{i=2}^{n}\sin\lambda\rho_{i-1}(\xi_i-\xi_{i-1}) + (-1)^{n}a_{m_1 1}\sin\lambda\rho_0(\xi_1-a)\cos\lambda\rho_n(x-\xi_n)$$
$$\left.\times\prod_{i=2}^{n}\sin\lambda\rho_{i-1}(\xi_i-\xi_{i-1})\right]$$
$$+\ o\left(\lambda^{A+m}\exp|\mathrm{Im}\lambda|\left((b-\xi_n)\rho_n+\sum_{i=1}^{n}(\xi_i-\xi_{i-1})\rho_{i-1}\right)\right),$$

where $A = r_1 + r_2 + \cdots + r_n$.

Lemma 3 $\psi(x,\lambda) = (\psi_{i1}(x\lambda), \psi_{i2}(x,\lambda))^T$, $x \in (\xi_i, \xi_{i+1})$, $i = 0, 1, \ldots, n$ *are entire functions of* λ *for each fixed x and the following asymptotic relations are true for these solutions as* $|\lambda| \to \infty$:

if $\deg(b_2) > \deg(b_1)$;

$$\psi_{01}(x,\lambda) = (-1)^{n-1} b_{m_4 2}\lambda^{A+m_4}\prod_{i=1}^{n}\gamma_{r_i i}$$
$$\times\cos\lambda\rho_n(\xi_n-b)\sin\lambda\rho_0(x-\xi_1)\prod_{i=2}^{n}\sin\lambda\rho_{i-1}(\xi_i-\xi_{i-1})$$
$$+\ o\left(\lambda^{A+m_4}\exp|\mathrm{Im}\lambda|\left((\xi_1-a)\rho_0+\sum_{i=2}^{n}(\xi_i-\xi_{i-1})\rho_{i-1}\right)\right),$$

$$\psi_{02}(x,\lambda) = (-1)^{n} b_{m_4 2}\lambda^{A+m_4}\prod_{i=1}^{n}\gamma_{r_i i}$$
$$\times\cos\lambda\rho_n(\xi_n-b)\cos\lambda\rho_0(x-\xi_1)\prod_{i=2}^{n}\sin\lambda\rho_{i-1}(\xi_i-\xi_{i-1})$$
$$+\ o\left(\lambda^{A+m_4}\exp|\mathrm{Im}\lambda|\left((\xi_1-a)\rho_0+\sum_{i=2}^{n}(\xi_i-\xi_{i-1})\rho_{i-1}\right)\right),$$

$$
\begin{aligned}
\psi_{11}(x,\lambda) &= (-1)^n b_{m_4 2}\lambda^{r_2+\dots+r_n+m_4}\prod_{i=2}^{n}\gamma_{r_i i} \\
&\times\ \cos\lambda\rho_n(\xi_n-b)\sin\lambda\rho_1(x-\xi_2)\prod_{i=3}^{n}\sin\lambda\rho_{i-1}(\xi_i-\xi_{i-1}) \\
&+\ o\left(\lambda^{r_2+\dots+r_n+m_4}\exp|\mathrm{Im}\lambda|\left((\xi_2-\xi_1)\rho_1+\sum_{i=3}^{n}(\xi_i-\xi_{i-1})\rho_{i-1}\right)\right), \\
\psi_{12}(x,\lambda) &= (-1)^{n-1} b_{m_4 2}\lambda^{r_2+\dots+r_n+m_4}\prod_{i=2}^{n}\gamma_{r_i i} \\
&\times\cos\lambda\rho_n(\xi_n-b)\cos\lambda\rho_1(x-\xi_2)\prod_{i=3}^{n}\sin\lambda\rho_{i-1}(\xi_i-\xi_{i-1}) \\
&+\ o\left(\lambda^{r_2+\dots+r_n+m_4}\exp|\mathrm{Im}\lambda|\left((\xi_2-\xi_1)\rho_1+\sum_{i=3}^{n}(\xi_i-\xi_{i-1})\rho_{i-1}\right)\right),
\end{aligned}
$$

proceeding in a similar way,

$$
\begin{aligned}
\psi_{n1}(x,\lambda) &= b_{m_4 2}\lambda^{m_4}\cos\lambda\rho_n(x-b)+o\left(\lambda^{m_4}\exp|\mathrm{Im}\lambda|(x-b)\rho_n\right), \\
\psi_{n2}(x,\lambda) &= b_{m_4 2}\lambda^{m_4}\sin\lambda\rho_n(x-b)+o\left(\lambda^{m_4}\exp|\mathrm{Im}\lambda|(x-b)\rho_n\right),
\end{aligned}
$$

if $\deg(b_1) > \deg(b_2)$;

$$
\begin{aligned}
\psi_{01}(x,\lambda) &= (-1)^n b_{m_3 1}\lambda^{A+m_3}\prod_{i=1}^{n}\gamma_{r_i i} \\
&\times\sin\lambda\rho_n(\xi_n-b)\sin\lambda\rho_0(x-\xi_1)\prod_{i=2}^{n}\sin\lambda\rho_{i-1}(\xi_i-\xi_{i-1}) \\
&+\ o\left(\lambda^{A+m_3}\exp|\mathrm{Im}\lambda|\left((\xi_1-a)\rho_0+\sum_{i=2}^{n}(\xi_i-\xi_{i-1})\rho_{i-1}\right)\right), \\
\psi_{02}(x,\lambda) &= (-1)^{n-1} b_{m_3 1}\lambda^{A+m_3}\prod_{i=1}^{n}\gamma_{r_i i} \\
&\times\sin\lambda\rho_n(\xi_n-b)\cos\lambda\rho_0(x-\xi_1)\prod_{i=2}^{n}\sin\lambda\rho_{i-1}(\xi_i-\xi_{i-1}) \\
&+\ o\left(\lambda^{A+m_3}\exp|\mathrm{Im}\lambda|\left((\xi_1-a)\rho_0+\sum_{i=2}^{n}(\xi_i-\xi_{i-1})\rho_{i-1}\right)\right), \\
\psi_{11}(x,\lambda) &= (-1)^{n-1} b_{m_3 1}\lambda^{r_2+\dots+r_n+m_3}\prod_{i=2}^{n}\gamma_{r_i i} \\
&\times\sin\lambda\rho_n(\xi_n-b)\sin\lambda\rho_1(x-\xi_2)\prod_{i=3}^{n}\sin\lambda\rho_{i-1}(\xi_i-\xi_{i-1}) \\
&+\ o\left(\lambda^{r_2+\dots+r_n+m_3}\exp|\mathrm{Im}\lambda|\left((\xi_2-\xi_1)\rho_1+\sum_{i=3}^{n}(\xi_i-\xi_{i-1})\rho_{i-1}\right)\right), \\
\psi_{12}(x,\lambda) &= (-1)^{n} b_{m_3 1}\lambda^{r_2+\dots+r_n+m_3}\prod_{i=2}^{n}\gamma_{r_i i} \\
&\times\sin\lambda\rho_n(\xi_n-b)\cos\lambda\rho_1(x-\xi_2)\prod_{i=3}^{n}\sin\lambda\rho_{i-1}(\xi_i-\xi_{i-1}) \\
&+\ o\left(\lambda^{r_2+\dots+r_n+m_3}\exp|\mathrm{Im}\lambda|\left((\xi_2-\xi_1)\rho_1+\sum_{i=3}^{n}(\xi_i-\xi_{i-1})\rho_{i-1}\right)\right),
\end{aligned}
$$

proceeding in a similar way,

$$\psi_{n1}(x,\lambda) = -b_{m_3 1}\lambda^{m_3}\sin\lambda\rho_n(x-b) + o\left(\lambda^{m_3}\exp|\mathrm{Im}\lambda|(x-b)\rho_n\right),$$
$$\psi_{n2}(x,\lambda) = b_{m_3 1}\lambda^{m_3}\cos\lambda\rho_n(x-b) + o\left(\lambda^{m_3}\exp|\mathrm{Im}\lambda|(x-b)\rho_n\right),$$

if $\deg(b_1) = \deg(b_2)$;

$$\begin{aligned}\psi_{01}(x,\lambda) = {} & \lambda^{A+m_b}\prod_{i=1}^{n}\gamma_{r_i i}\sin\lambda\rho_0\,(x-\xi_1) \\ & \times\prod_{i=2}^{n}\sin\lambda\rho_{i-1}\,(\xi_i-\xi_{i-1})\left[(-1)^n b_{m_3 1}\sin\lambda\rho_n\,(\xi_n-b)\right. \\ & \left.+(-1)^{n-1}b_{m_4 2}\cos\lambda\rho_n\,(\xi_n-b)\right] \\ & +\ o\left(\lambda^{A+m_b}\exp|\mathrm{Im}\lambda|\left((\xi_1-a)\,\rho_0+\sum_{i=2}^{n}(\xi_i-\xi_{i-1})\,\rho_{i-1}\right)\right),\end{aligned}$$

$$\begin{aligned}\psi_{02}(x,\lambda) = {} & \lambda^{A+m_b}\prod_{i=1}^{n}\gamma_{r_i i}\cos\lambda\rho_0\,(x-\xi_1) \\ & \times\prod_{i=2}^{n}\sin\lambda\rho_{i-1}\,(\xi_i-\xi_{i-1})\left[(-1)^{n-1} b_{m_3 1}\sin\lambda\rho_n\,(\xi_n-b)\right. \\ & \left.+(-1)^{n}b_{m_4 2}\cos\lambda\rho_n\,(\xi_n-b)\right] \\ & +\ o\left(\lambda^{A+m_b}\exp|\mathrm{Im}\lambda|\left((\xi_1-a)\,\rho_0+\sum_{i=2}^{n}(\xi_i-\xi_{i-1})\,\rho_{i-1}\right)\right),\end{aligned}$$

$$\begin{aligned}\psi_{11}(x,\lambda) = {} & \lambda^{r_2+\dots+r_n+m_b}\prod_{i=2}^{n}\gamma_{r_i i}\sin\lambda\rho_1\,(x-\xi_2) \\ & \times\prod_{i=3}^{n}\sin\lambda\rho_{i-1}\,(\xi_i-\xi_{i-1})\left[(-1)^{n-1} b_{m_3 1}\sin\lambda\rho_n\,(\xi_n-b)\right. \\ & \left.+(-1)^{n}b_{m_4 2}\cos\lambda\rho_n\,(\xi_n-b)\right] \\ & +\ o\left(\lambda^{r_2+\dots+r_n+m_b}\exp|\mathrm{Im}\lambda|\left((\xi_2-\xi_1)\,\rho_1+\sum_{i=3}^{n}(\xi_i-\xi_{i-1})\,\rho_{i-1}\right)\right),\end{aligned}$$

$$\begin{aligned}\psi_{12}(x,\lambda) = {} & \lambda^{r_2+\dots+r_n+m_b}\prod_{i=2}^{n}\gamma_{r_i i}\cos\lambda\rho_1\,(x-\xi_2) \\ & \times\prod_{i=3}^{n}\sin\lambda\rho_{i-1}\,(\xi_i-\xi_{i-1})\left[(-1)^{n} b_{m_3 1}\sin\lambda\rho_n\,(\xi_n-b)\right. \\ & \left.+(-1)^{n-1}b_{m_4 2}\cos\lambda\rho_n\,(\xi_n-b)\right] \\ & +\ o\left(\lambda^{r_2+\dots+r_n+m_b}\exp|\mathrm{Im}\lambda|\left((\xi_2-\xi_1)\,\rho_1+\sum_{i=3}^{n}(\xi_i-\xi_{i-1})\,\rho_{i-1}\right)\right),\end{aligned}$$

proceeding in a similar way,

$$\begin{aligned}\psi_{n1}(x,\lambda) = {} & \lambda^{m_b}\left(b_{m_4 2}\cos\lambda\rho_n(x-b) - b_{m_3 1}\sin\lambda\rho_n(x-b)\right) \\ & +\ o\left(\lambda^{m_b}\exp|\mathrm{Im}\lambda|(x-b)\rho_n\right), \\ \psi_{n2}(x,\lambda) = {} & \lambda^{m_b}\left(b_{m_4 2}\sin\lambda\rho_n(x-b) + b_{m_3 1}\cos\lambda\rho_n(x-b)\right) \\ & +\ o\left(\lambda^{m_b}\exp|\mathrm{Im}\lambda|(x-b)\rho_n\right),\end{aligned}$$

where $A = r_1 + r_2 + \dots + r_n$.

3 | CHARACTERISTIC FUNCTION

The Wronskian of the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ is defined by $W\{\psi,\varphi\}=\psi_1(x,\lambda)\varphi_2(x,\lambda)-\varphi_1(x,\lambda)\psi_2(x,\lambda)$ and since functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ satisfy the Equation (1), the following equality is valid;

$$\begin{aligned}\frac{\partial}{\partial x}W\{\psi,\varphi\} &= \psi_1'(x,\lambda)\varphi_2(x,\lambda)+\psi_1(x,\lambda)\varphi_2'(x,\lambda)\\ &\quad-\psi_2'(x,\lambda)\varphi_1(x,\lambda)-\psi_2(x,\lambda)\varphi_1'(x,\lambda)\\ &=\left[q(x)\psi_1(x,\lambda)+r(x)\psi_2(x,\lambda)-\lambda\rho(x)\psi_2(x,\lambda)\right]\varphi_2(x,\lambda)\\ &\quad+\left[-p(x)\varphi_1(x,\lambda)-q(x)\varphi_2(x,\lambda)+\lambda\rho(x)\varphi_1(x,\lambda)\right]\psi_1(x,\lambda)\\ &\quad-\left[-p(x)\psi_1(x,\lambda)-q(x)\psi_2(x,\lambda)+\lambda\rho(x)\psi_1(x,\lambda)\right]\varphi_1(x,\lambda)\\ &\quad-\left[q(x)\varphi_1(x,\lambda)+r(x)\varphi_2(x,\lambda)-\lambda\rho(x)\varphi_2(x,\lambda)\right]\psi_2(x,\lambda)=0.\end{aligned}$$

Since solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ satisfy the transmission conditions (4),

$$\begin{aligned}W\{\psi,\varphi\}(\xi_i+0)&=\psi_1(\xi_i+0,\lambda)\varphi_2(\xi_i+0,\lambda)-\psi_2(\xi_i+0,\lambda)\varphi_1(\xi_i+0,\lambda)\\ &=\theta_i\psi_1(\xi_i-0,\lambda)\left[\theta_i^{-1}\varphi_2(\xi_i-0,\lambda)+\gamma_i(\lambda)\varphi_1(\xi_i-0,\lambda)\right]\\ &\quad-\left[\theta_i^{-1}\psi_2(\xi_i-0,\lambda)+\gamma_i(\lambda)\psi_1(\xi_i-0,\lambda)\right]\theta_i\varphi_1(\xi_i-0,\lambda)\\ &=\psi_1(\xi_i-0,\lambda)\varphi_2(\xi_i-0,\lambda)-\psi_2(\xi_i-0,\lambda)\varphi_1(\xi_i-0,\lambda)\\ &=W\{\psi,\varphi\}(\xi_i-0)\end{aligned}$$

is valid.

$W\{\psi,\varphi\}$ does not depend on x and $\varphi(x,\lambda)$, $\psi(x,\lambda)$ are linearly independent if $W\{\varphi,\psi\}\neq 0$. The characteristic function of the problem (1)–(4) is defined as follows:

$$\begin{aligned}\Delta(\lambda)&=\psi_1(b,\lambda)\varphi_2(b,\lambda)-\psi_2(b,\lambda)\varphi_1(b,\lambda)\\ &=\psi_1(a,\lambda)\varphi_2(a,\lambda)-\psi_2(a,\lambda)\varphi_1(a,\lambda)\\ &=b_2(\lambda)\varphi_2(b,\lambda)-b_1(\lambda)\varphi_1(b,\lambda)\\ &=a_1(\lambda)\psi_1(a,\lambda)-a_2(\lambda)\psi_2(a,\lambda).\end{aligned}$$

$\Delta(\lambda)$ is analytic function of λ and its zeros are precisely eigenvalues of the problem L.

Lemma 4 *From the asymptotics formulae of $\varphi(x,\lambda)$and $\psi(x,\lambda)$, the following asymptotic formulae hold as $|\lambda|\to\infty$:*

I) for $\deg(a_2)>\deg(a_1)$

if $\deg(b_2)>\deg(b_1)$;

$$\begin{aligned}\Delta(\lambda)&=(-1)^{n+1}a_{m_2 2}b_{m_4 2}\lambda^{A+m_2+m_4}\\ &\quad\times\prod_{i=1}^{n}\gamma_{r_i i}\cos\lambda\rho_0(\xi_1-a)\cos\lambda\rho_n(b-\xi_n)\prod_{i=2}^{n}\sin\lambda\rho_{i-1}(\xi_i-\xi_{i-1})\\ &\quad+o\left(\lambda^{A+m_2+m_4}\exp|\mathrm{Im}\lambda|((b-\xi_n)\rho_n+\sum_{i=1}^{n}(\xi_i-\xi_{i-1})\rho_{i-1}\right),\end{aligned}$$

if $\deg(b_1)>\deg(b_2)$;

$$\Delta(\lambda)=(-1)^{n+1}a_{m_2 2}b_{m_3 1}\lambda^{A+m_2+m_3}$$

$$\times \prod_{i=1}^{n}\gamma_{r_i i}\cos\lambda\rho_0(\xi_1-a)\sin\lambda\rho_n(b-\xi_n)\prod_{i=2}^{n}\sin\lambda\rho_{i-1}(\xi_i-\xi_{i-1})$$
$$+\ o\left(\lambda^{A+m_2+m_3}\exp|\mathrm{Im}\lambda|\left((b-\xi_n)\rho_n+\sum_{i=1}^{n}(\xi_i-\xi_{i-1})\rho_{i-1}\right)\right),$$

if $\deg(b_1)=\deg(b_2)$;

$$\Delta(\lambda)=(-1)^{n+1}\lambda^{A+m_2+m_b}a_{m_2 2}\prod_{i=1}^{n}\gamma_{r_i i}$$
$$\times\cos\lambda\rho_0(\xi_1-a)\prod_{i=2}^{n}\sin\lambda\rho_{i-1}(\xi_i-\xi_{i-1})\left[b_{m_4 2}\cos\lambda\rho_n(b-\xi_n)\right.$$
$$\left.+\ b_{m_3 1}\sin\lambda\rho_n(b-\xi_n)\right]$$
$$+\ o\left(\lambda^{A+m_2+m_b}\exp|\mathrm{Im}\lambda|\left((b-\xi_n)\rho_n+\sum_{i=1}^{n}(\xi_i-\xi_{i-1})\rho_{i-1}\right)\right),$$

II) for $\deg(a_1)>\deg(a_2)$,

if $\deg(b_2)>\deg(b_1)$;

$$\Delta(\lambda)=(-1)^n a_{m_1 1}b_{m_4 2}\lambda^{A+m_1+m_4}$$
$$\times\prod_{i=1}^{n}\gamma_{r_i i}\sin\lambda\rho_0(\xi_1-a)\cos\lambda\rho_n(b-\xi_n)\prod_{i=2}^{n}\sin\lambda\rho_{i-1}(\xi_i-\xi_{i-1})$$
$$+\ o\left(\lambda^{A+m_1+m_4}\exp|\mathrm{Im}\lambda|\left((b-\xi_n)\rho_n+\sum_{i=1}^{n}(\xi_i-\xi_{i-1})\rho_{i-1}\right)\right),$$

if $\deg(b_1)>\deg(b_2)$;

$$\Delta(\lambda)=(-1)^n a_{m_1 1}b_{m_3 1}\lambda^{A+m_1+m_3}$$
$$\times\prod_{i=1}^{n}\gamma_{r_i i}\sin\lambda\rho_0(\xi_1-a)\sin\lambda\rho_n(b-\xi_n)\prod_{i=2}^{n}\sin\lambda\rho_{i-1}(\xi_i-\xi_{i-1})$$
$$+\ o\left(\lambda^{A+m_1+m_3}\exp|\mathrm{Im}\lambda|\left((b-\xi_n)\rho_n+\sum_{i=1}^{n}(\xi_i-\xi_{i-1})\rho_{i-1}\right)\right),$$

if $\deg(b_1)=\deg(b_2)$;

$$\Delta(\lambda)=(-1)^n\lambda^{A+m_1+m_b}a_{m_1 1}\times\prod_{i=1}^{n}\gamma_{r_i i}$$
$$\times\sin\lambda\rho_0(\xi_1-a)\prod_{i=2}^{n}\sin\lambda\rho_{i-1}(\xi_i-\xi_{i-1})\left[b_{m_4 2}\cos\lambda\rho_n(b-\xi_n)\right.$$
$$\left.+\ b_{m_3 1}\sin\lambda\rho_n(b-\xi_n)\right]$$
$$+\ o\left(\lambda^{A+m_1+m_b}\exp|\mathrm{Im}\lambda|\left((b-\xi_n)\rho_n+\sum_{i=1}^{n}(\xi_i-\xi_{i-1})\rho_{i-1}\right)\right),$$

III) for $\deg(a_1)=\deg(a_2)$,

if $\deg(b_2)>\deg(b_1)$;

$$\Delta(\lambda) = \lambda^{A+m+m_4} \prod_{i=1}^{n} \gamma_{r_i i} \prod_{i=2}^{n} \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1})$$
$$\left[(-1)^{n+1} a_{m_2 2} b_{m_4 2} \cos \lambda \rho_0 (\xi_1 - a) \cos \lambda \rho_n (b - \xi_n) + (-1)^n a_{m_1 1} b_{m_4 2} \sin \lambda \rho_0 (\xi_1 - a) \cos \lambda \rho_n (b - \xi_n)\right]$$
$$+ o\left(\lambda^{A+m+m_4} \exp |\mathrm{Im}\lambda| \left((b - \xi_n) \rho_n + \sum_{i=1}^{n} (\xi_i - \xi_{i-1}) \rho_{i-1}\right)\right),$$

if $\deg(b_1) > \deg(b_2)$;

$$\Delta(\lambda) = \lambda^{A+m+m_3} \prod_{i=1}^{n} \gamma_{r_i i} \prod_{i=2}^{n} \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1})$$
$$\left[(-1)^{n+1} a_{m_2 2} b_{m_3 1} \cos \lambda \rho_0 (\xi_1 - a) \sin \lambda \rho_n (b - \xi_n) + (-1)^{n+1} a_{m_1 1} b_{m_3 1} \sin \lambda \rho_0 (\xi_1 - a) \sin \lambda \rho_n (b - \xi_n)\right]$$
$$+ o\left(\lambda^{A+m+m_3} \exp |\mathrm{Im}\lambda| \left((b - \xi_n) \rho_n + \sum_{i=1}^{n} (\xi_i - \xi_{i-1}) \rho_{i-1}\right)\right),$$

if $\deg(b_1) = \deg(b_2)$;

$$\Delta(\lambda) = \lambda^{A+m+m_b} \prod_{i=1}^{n} \gamma_{r_i i} \prod_{i=2}^{n} \sin \lambda \rho_{i-1} (\xi_i - \xi_{i-1})$$
$$\left[(-1)^{n+1} a_{m_2 2} \cos \lambda \rho_0 (\xi_1 - a) \left(b_{m_4 2} \cos \lambda \rho_n (b - \xi_n) + b_{m_3 1} \sin \lambda \rho_n (b - \xi_n)\right) + (-1)^n a_{m_1 1} \sin \lambda \rho_0 (\xi_1 - a) \left(b_{m_4 2} \cos \lambda \rho_n (b - \xi_n) + b_{m_3 1} \sin \lambda \rho_n (b - \xi_n)\right)\right]$$
$$+ o\left(\lambda^{A+m+m_b} \exp |\mathrm{Im}\lambda| \left((b - \xi_n) \rho_n + \sum_{i=1}^{n} (\xi_i - \xi_{i-1}) \rho_{i-1}\right)\right),$$

where $A = r_1 + r_2 + \cdots + r_n$.

It can be proved as in [3] that the problem L has countable many eigenvalues such that only finitely many of them are non-real or non-simple and the eigenvalues of the problem L are distributed symmetrically respect to real axis.

Let $\Phi(x, \lambda) = \begin{pmatrix} \Phi_1(x, \lambda) \\ \Phi_2(x, \lambda) \end{pmatrix}$ be solution of Equation (1) under the conditions $l_1(\Phi) = 1$, $l_2(\Phi) = 0$ and jump conditions (4). Since the Wronskian $W\{\varphi, \Phi\}$ does not depend on x, taking $x = a$ we get

$$\begin{aligned} W\{\varphi, \Phi\}|_{x=a} &= \varphi_1(a, \lambda)\Phi_2(a, \lambda) - \varphi_2(a, \lambda)\Phi_1(a, \lambda) \\ &= a_2(\lambda)\Phi_2(a, \lambda) - a_1(\lambda)\Phi_1(a, \lambda) \\ &= a_2(\lambda) (1 + a_1(\lambda)\Phi_1(a, \lambda)) \frac{1}{a_2(\lambda)} - a_1(\lambda)\Phi_1(a, \lambda) \\ &= 1 \end{aligned}$$

Thus, φ and Φ are linearly independent. Since $l_2(\Phi) = l_2(\psi) = 0$, we may suppose $\Phi(x, \lambda) = k\psi(x, \lambda)$, where k is a constant independent of x. By the relation $l_1(\Phi) = 1$, we obtain $k\,[a_1(\lambda)\psi_1(a, \lambda) - a_2(\lambda)\psi_2(a, \lambda)] = -1$. In view of

$$\Delta(\lambda) = -l_1(\psi) = a_1(\lambda)\psi_1(a, \lambda) - a_2(\lambda)\psi_2(a, \lambda) = -\frac{1}{k}$$

we get for $\lambda \neq \lambda_n \Phi(x, \lambda) = -\frac{\psi(x,\lambda)}{\Delta(\lambda)}$, that is, $\Phi_i(x, \lambda) = -\frac{\psi_i(x,\lambda)}{\Delta(\lambda)}$ for $i = 1, 2$.

Denote

$$Z(x,\lambda) := \frac{1}{a_2(\lambda)}(S(x,\lambda) + M(\lambda)\varphi(x,\lambda))$$

where $S(x,\lambda)$ and $\varphi(x,\lambda)$ defined by (6). Let us show that $Z(x,\lambda) \equiv \Phi(x,\lambda)$.

Indeed

$$Z_1(a,\lambda) = \frac{1}{a_2(\lambda)}\left(S_1(a,\lambda) + M(\lambda)\varphi_1(a,\lambda)\right) = M(\lambda)$$

$$\begin{aligned} Z_2(a,\lambda) &= \frac{1}{a_2(\lambda)}\left(S_2(a,\lambda) + M(\lambda)\varphi_2(a,\lambda)\right) \\ &= \frac{1}{a_2(\lambda)}\left(1 + M(\lambda)a_1(\lambda)\right) \end{aligned}$$

and consequently

$$\begin{aligned} l_1 Z &= a_2(\lambda)Z_2(a,\lambda) - a_1(\lambda)Z_1(a,\lambda) \\ &= a_2(\lambda)\left(S_2(a,\lambda) + M(\lambda)\varphi_2(a,\lambda)\right)\frac{1}{a_2(\lambda)} - M(\lambda)a_1(\lambda) \\ &= 1 \end{aligned}$$

Thus, the functions $Z(x,\lambda)$ and $\Phi(x,\lambda)$ satisfy Equation (1) and $Z(a,\lambda) = \varphi(a,\lambda)$, hence

$$\Phi(x,\lambda) = \frac{1}{a_2(\lambda)}\left(S(x,\lambda) + \Phi_1(a,\lambda)\varphi(x,\lambda)\right).$$

The function $\Phi(x,\lambda)$ is called Weyl solution and the function $M(\lambda) = \Phi_1(a,\lambda)$ is called Weyl function. The functions $Z(x, \lambda)$ and $\Phi(x,\lambda)$, satisfy Equation (1) and $Z_1(a,\lambda) = \Phi_1(a,\lambda)$, $Z_2(a,\lambda) = \Phi_2(a,\lambda)$ for $\forall x \in [a,\xi_1)$. Moreover, for ξ_i, $i = 1, 2, \ldots, n$ and for $x \in [a,b]$

$$\begin{aligned} Z_1(\xi_i + 0,\lambda) &= \theta_i Z_1(\xi_i - 0,\lambda) = \theta_i \Phi_1(\xi_i - 0,\lambda) = \Phi_1(\xi_i + 0,\lambda) \\ Z_2(\xi_i + 0,\lambda) &= \theta_i^{-1} Z_2(\xi_i - 0,\lambda) - \gamma_i(\lambda) Z_1(\xi_i - 0,\lambda) \\ &= \theta_i^{-1}\Phi_2(\xi_i - 0,\lambda) - \gamma_i(\lambda)\Phi_1(\xi_i - 0,\lambda) \\ &= \Phi_2(\xi_i + 0,\lambda). \end{aligned}$$

We get, $Z(x,\lambda) \equiv \Phi(x,\lambda)$. Hence

$$\Phi(x,\lambda) = \frac{1}{a_2(\lambda)}(S(x,\lambda) + M(\lambda)\varphi(x,\lambda)).$$

4 | INVERSE PROBLEMS

In this section, we investigate the inverse problem of the reconstruction of a boundary value problem L from the Weyl function and two different eigenvalues sequences.

Let us consider the boundary-value problem

$$\begin{aligned} &\tilde{\ell}[Y(x)] \overset{def}{=} BY'(x) + \tilde{\Omega}(x)Y(x) = \lambda\rho(x)Y(x), \quad x \in (a,b), \\ &\tilde{l}_1 y := \tilde{a}_2(\lambda)y_2(a) - \tilde{a}_1(\lambda)y_1(a) = 0 \\ &\tilde{l}_2 y := \tilde{b}_2(\lambda)y_2(b) - \tilde{b}_1(\lambda)y_1(b) = 0 \\ &\tilde{l}_3 y := y_1(\xi_i + 0) - \tilde{\theta}_i y_1(\xi_i - 0) = 0 \\ &\tilde{l}_4 y := y_2(\xi_i + 0) - \tilde{\theta}_i^{-1} y_2(\xi_i - 0) - \tilde{\gamma}_i(\lambda)y_1(\xi_i - 0) = 0 \end{aligned}$$

$\widetilde{L} = L\left(\widetilde{\Omega}, \widetilde{f}, \widetilde{a}_j(\lambda), \xi_i, \widetilde{\theta}_i, \widetilde{\gamma}_i(\lambda)\right)$, where $\widetilde{\Omega}(x) = \begin{pmatrix} \widetilde{p}(x) & q(x) \\ q(x) & \widetilde{r}(x) \end{pmatrix}$, $f(\lambda) = \frac{b_1(\lambda)}{b_2(\lambda)}$ and $j = 1, 2$, $i = 1, 2, \ldots, n$. It is assumed in what follows that if a certain symbol s denotes an object related to the problem L then $\widetilde{s}$ denotes the corresponding object related to the problem $\widetilde{L}$.

Theorem 2 *The boundary value problem L is uniquely determined by the Weyl function, that is, if $M(\lambda) = \widetilde{M}(\lambda)$, $a_i(\lambda) = \widetilde{a}_i(\lambda)$, $i = 1, 2$ then $\Omega(x) = \widetilde{\Omega}(x)$, a.e., $f(\lambda) = \widetilde{f}(\lambda)$, $\theta_i = \widetilde{\theta}_i$, and $\gamma_i(\lambda) = \widetilde{\gamma}_i(\lambda)$ for $i = 1, 2, \ldots, n$.*

Proof. Introduce a matrix $P(x, \lambda) = \left[P_{kj}(x, \lambda)\right]_{i,j=1,2}$ by the formulae

$$P(x, \lambda) = \phi(x, \lambda)\widetilde{\phi}^{-1}(x, \lambda) \tag{7}$$

where $\phi(x, \lambda) = \begin{pmatrix} \varphi_2(x, \lambda) & \frac{\psi_2(x,\lambda)}{\Delta(\lambda)} \\ \varphi_1(x, \lambda) & \frac{\psi_1(x,\lambda)}{\Delta(\lambda)} \end{pmatrix}$ and $\widetilde{\phi}(x, \lambda) = \begin{pmatrix} \widetilde{\varphi}_2(x, \lambda) & \frac{\widetilde{\psi}_2(x,\lambda)}{\widetilde{\Delta}(\lambda)} \\ \widetilde{\varphi}_1(x, \lambda) & \frac{\widetilde{\psi}_1(x,\lambda)}{\widetilde{\Delta}(\lambda)} \end{pmatrix}$.

Write (7) openly, to get

$$\begin{cases} P_{11}(x, \lambda) = \varphi_2(x, \lambda)\dfrac{\widetilde{\psi}_1(x, \lambda)}{\widetilde{\Delta}(\lambda)} - \widetilde{\varphi}_1(x, \lambda)\dfrac{\psi_2(x, \lambda)}{\Delta(\lambda)} \\ P_{12}(x, \lambda) = \widetilde{\varphi}_2(x, \lambda)\dfrac{\psi_2(x, \lambda)}{\Delta(\lambda)} - \varphi_2(x, \lambda)\dfrac{\widetilde{\psi}_2(x, \lambda)}{\widetilde{\Delta}(\lambda)} \\ P_{21}(x, \lambda) = \varphi_1(x, \lambda)\dfrac{\widetilde{\psi}_1(x, \lambda)}{\widetilde{\Delta}(\lambda)} - \widetilde{\varphi}_1(x, \lambda)\dfrac{\psi_1(x, \lambda)}{\Delta(\lambda)} \\ P_{22}(x, \lambda) = \widetilde{\varphi}_2(x, \lambda)\dfrac{\psi_1(x, \lambda)}{\Delta(\lambda)} - \varphi_1(x, \lambda)\dfrac{\widetilde{\psi}_2(x, \lambda)}{\widetilde{\Delta}(\lambda)} \end{cases} \tag{8}$$

Since $\frac{\psi_i(x,\lambda)}{\Delta(\lambda)} = -\Phi_i(x, \lambda) = -\left[\frac{1}{a_2(\lambda)}\left(S_i(x, \lambda) + M(\lambda)\varphi_i(x, \lambda)\right)\right]$ and $\varphi_i(x, \lambda) = a_2(\lambda)C_i(x, \lambda) + a_1(\lambda)S_i(x, \lambda), i = 1, 2, M(\lambda) = \widetilde{M}(\lambda)$, the relations

$$\begin{cases} P_{11}(x, \lambda) = \widetilde{C}_1(x, \lambda)S_2(x, \lambda) - C_2(x, \lambda)\widetilde{S}_1(x, \lambda) \\ P_{12}(x, \lambda) = C_2(x, \lambda)\widetilde{S}_2(x, \lambda) - \widetilde{C}_2(x, \lambda)S_2(x, \lambda) \\ P_{21}(x, \lambda) = \widetilde{C}_1(x, \lambda)S_1(x, \lambda) - C_1(x, \lambda)\widetilde{S}_1(x, \lambda) \\ P_{22}(x, \lambda) = C_1(x, \lambda)\widetilde{S}_2(x, \lambda) - \widetilde{C}_2(x, \lambda)S_1(x, \lambda) \end{cases} \tag{9}$$

are obtained from (8). Hence the functions $P_{ij}(x, \lambda)$ are entire on λ as $M(\lambda) = \widetilde{M}(\lambda)$. In addition, $P_{ij}(x, \lambda)$ are bounded with respect to λ. Therefore, it is obvious from Liouville's theorem that, these functions depend only on x .

On the other hand, from (8),

$$P_{11}(x, \lambda) - 1 = \varphi_2(x, \lambda)\left(\frac{\widetilde{\psi}_1(x, \lambda)}{\widetilde{\Delta}(\lambda)} - \frac{\psi_1(x, \lambda)}{\Delta(\lambda)}\right) - \frac{\psi_2(x, \lambda)\left(\widetilde{\varphi}_1(x\lambda) - \varphi_1(x, \lambda)\right)}{\Delta(\lambda)}$$

$$P_{12}(x, \lambda) = \widetilde{\varphi}_2(x, \lambda)\left(\frac{\psi_2(x, \lambda)}{\Delta(\lambda)} - \frac{\widetilde{\psi}_2(x, \lambda)}{\widetilde{\Delta}(\lambda)}\right) - \frac{\widetilde{\psi}_2(x, \lambda)\left(\varphi_2(x\lambda) - \widetilde{\varphi}_2(x, \lambda)\right)}{\widetilde{\Delta}(\lambda)}$$

$$P_{21}(x,\lambda)=\varphi_1(x,\lambda)\left(\frac{\widetilde{\psi}_1(x,\lambda)}{\widetilde{\Delta}(\lambda)}-\frac{\psi_1(x,\lambda)}{\Delta(\lambda)}\right)-\frac{\psi_1(x,\lambda)\left(\widetilde{\varphi}_1(x\lambda)-\varphi_1(x,\lambda)\right)}{\Delta(\lambda)}$$

$$P_{22}(x,\lambda)-1=\widetilde{\varphi}_2(x,\lambda)\left(\frac{\psi_1(x,\lambda)}{\Delta(\lambda)}-\frac{\widetilde{\psi}_1(x,\lambda)}{\widetilde{\Delta}(\lambda)}\right)-\frac{\widetilde{\psi}_2(x,\lambda)\left(\varphi_1(x\lambda)-\widetilde{\varphi}_1(x,\lambda)\right)}{\widetilde{\Delta}(\lambda)}.$$

Therefore, due to the fact that,

$$\lim_{\lambda\in\mathbb{R},\lambda\to\infty}\frac{\psi_2(x,\lambda)\left(\widetilde{\varphi}_1(x\lambda)-\varphi_1(x,\lambda)\right)}{\Delta(\lambda)}=0,\ \lim_{\lambda\in\mathbb{R},\lambda\to\infty}\varphi_2(x,\lambda)\left(\frac{\widetilde{\psi}_1(x,\lambda)}{\widetilde{\Delta}(\lambda)}-\frac{\psi_1(x,\lambda)}{\Delta(\lambda)}\right)=0$$

for all $x\in[a,b]$ and for all cases of degrees $a_i(\lambda)$ and $b_i(\lambda)$, $i=1,2$

$$\lim_{\substack{\lambda\to\infty\\ \lambda\in R}}[P_{11}(x,\lambda)-1]=0$$

uniformly with respect to x. Thus $P_{11}(x,\ \lambda)\equiv 1$ and similarly, $P_{22}(x,\ \lambda)\equiv 1 P_{12}(x,\ \lambda)=P_{21}(x,\lambda)\equiv 0$. Substitute these relations in (8), to get

$$\varphi_1(x,\lambda)\equiv\widetilde{\varphi}_1(x,\lambda),\varphi_2(x,\lambda)\equiv\widetilde{\varphi}_2(x,\lambda),$$
$$\frac{\psi_1(x,\lambda)}{\Delta(\lambda)}\equiv\frac{\widetilde{\psi}_1(x,\lambda)}{\widetilde{\Delta}(\lambda)},\frac{\psi_2(x,\lambda)}{\Delta(\lambda)}\equiv\frac{\widetilde{\psi}_2(x,\lambda)}{\widetilde{\Delta}(\lambda)}.$$

Hence $\Omega(x)=\widetilde{\Omega}(x)$, a.e., since $\frac{\psi_i(x,\lambda)}{\Delta(\lambda)}=\frac{\widetilde{\psi}_i(x,\lambda)}{\widetilde{\Delta}(\lambda)}$, for $i=1,2$ and

$$b_2(\lambda)\psi_2(b,\lambda)-b_1(\lambda)\psi_1(b,\lambda)=0$$
$$\widetilde{b}_2(\lambda)\widetilde{\psi}_2(b,\lambda)-\widetilde{b}_1(\lambda)\widetilde{\psi}_1(b,\lambda)=0,$$

$\frac{b_1(\lambda)}{b_2(\lambda)}=\frac{\widetilde{b}_1(\lambda)}{\widetilde{b}_2(\lambda)}$, that is, $f(\lambda)=\widetilde{f}(\lambda)$, since $\varphi_i(x,\lambda)=\widetilde{\varphi}_i(x,\lambda)$, $i=1,2$, from transmission conditions (4) we get $\theta_i=\widetilde{\theta}_i$, $\gamma_i(\lambda)=\widetilde{\gamma}_i(\lambda)$ for $i=1,2,\ \ldots,n$. Thus, $L=\widetilde{L}$.. ■

Consider the following boundary value problem L_1 $(\Omega(x),f(\lambda),\theta_i,\gamma_i(\lambda))$:

$$\ell[Y(x)]:=BY'(x)+\Omega(x)Y(x)=\lambda\rho(x)Y(x),\quad x\in(a,b),$$
$$l_1y:=y_1(a)=0$$
$$l_2y:=b_2(\lambda)y_2(b)-b_1(\lambda)y_1(b)=0$$
$$l_3y:=y_1(\xi_i+0)-\theta_i y_1(\xi_i-0)=0$$
$$l_4y:=y_2(\xi_i+0)-\theta_i^{-1}y_2(\xi_i-0)-\gamma_i(\lambda)y_1(\xi_i-0)=0$$

Let $\{\mu_n\}_{n\geq 0}$ be the eigenvalues of the problem L_1. It is clear that,

$$\Delta_1(\lambda)=W\{\psi,S\}=\psi_1(a,\lambda)S_2(a,\lambda)-\psi_2(a,\lambda)S_1(a,\lambda)$$
$$=\psi_1(a,\lambda)$$

is the characteristic function of L_1.

Theorem 3 *If $\lambda_n=\widetilde{\lambda}_n$ and $\mu_n=\widetilde{\mu}_n$ for all $n\in\mathbb{Z}$, $a_i(\lambda)=\widetilde{a}_i(\lambda)$, $i=1,2$, $b_{m_3 1}=\widetilde{b}_{m_3 1}$, $b_{m_4 2}=\widetilde{b}_{m_4 2}$ and $\gamma_{r_i i}=\widetilde{\gamma}_{r_i i}$, $i=1,2,\ \ldots,n$ then $\Omega(x)=\widetilde{\Omega}(x)$, a.e., $f(\lambda)=\widetilde{f}(\lambda)$, $\gamma_i(\lambda)=\widetilde{\gamma}_i(\lambda)$, $\theta_i=\widetilde{\theta}_i$, $i=1,2,\ \ldots\ ,n$.*

Proof. Since $\lambda_n=\widetilde{\lambda}_n$, and $\mu_n=\widetilde{\mu}_n$, then $\frac{\Delta(\lambda)}{\widetilde{\Delta}(\lambda)}$ and $\frac{\Delta_1(\lambda)}{\widetilde{\Delta}_1(\lambda)}$ are entire function in λ. On the other hand, for all cases of degrees $a_i(\lambda)$ and $b_i(\lambda)$, $i=1,2$, since $\lim_{\lambda\to-\infty}\frac{\Delta(\lambda)}{\widetilde{\Delta}(\lambda)}=$

$\lim_{\lambda\to-\infty} \frac{\Delta_1(\lambda)}{\widetilde{\Delta}_1(\lambda)} = 1$, then $\Delta(\lambda) \equiv \widetilde{\Delta}(\lambda)$ and $\Delta_1(\lambda) \equiv \widetilde{\Delta}_1(\lambda)$. Therefore, $\psi_1(a,\lambda) \equiv \widetilde{\psi}_1(a,\lambda)$ and then from $\Phi_1(x,\lambda) = -\frac{\psi_1(x,\lambda)}{\Delta(\lambda)}$ and $M(\lambda) = \Phi_1(a,\lambda)$,

$M(\lambda) = \widetilde{M}(\lambda)$. Thus, the proof is completed by Theorem 2. ∎

5 | CONCLUSION

In this study, Dirac operator with discontinuous coefficient are considered and in which is one of the broadest generalizations of the classical Dirac operator in that both the boundary conditions and the discontinuity conditions with a finite number of discontinuity points depend polynomially on λ parameter.

We prove that if the coefficient $a_i(\lambda)$, $i = 1, 2$ in the first boundary condition is known, the other coefficients of the boundary value problem L can be uniquely determined by the Weyl function $M(\lambda)$ and the sequence $\{\lambda_n, \mu_n\}$ which is two given spectra. Therefore, the results obtained in this study will make significant contributions to the literature.

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DATA AVAILABILITY STATEMENT

All data generated or analyzed during this study are included in this article.

ORCID

Yalçın Güldü https://orcid.org/0000-0001-9678-8629
Merve Arslantaş https://orcid.org/0000-0002-0493-4551

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