Miskolc Mathematical Notes
Vol. 24 (2023), No. 2, pp. 1049–1056
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2023.4105

SOME ALGEBRAIC PROPERTIES OF GENERALIZED $q-$ CESÀRO MATRIX $C_g(q)$

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Received 28 January, 2022

Abstract. In this paper, we define the generalized q-Cesàro matrix by using generalized Cesàro matrix and q-Cesàro matrix. We investigate normality, self-adjointedness and Hilbert-Schmidt properties of generalized q-Cesàro matrices.

2010 *Mathematics Subject Classification:* 47A05; 47A16; 47B15; 32A35

Keywords: Cesàro matrix, q-Cesàro matrix, generalized q-Cesàro matrix

1. INTRODUCTION

In [1], Bustoz and Gordillo defined the q-Cesàro matrix. For $0<q<1$, q-Cesàro matrix $C_1(q)$ is defined by

$$c_{nk}=\begin{cases}\frac{q^{n-k}}{1+q+\cdots+q^n} & , \quad 0<n\leq k\\ 0 & , \quad n<k\end{cases}.$$

Since $c_{nk}(q)\rightarrow \dfrac{1}{n+1}$, for $q\rightarrow 1$, the q-analog of $C_1=(c_{nk})$ is matrix $C_1(q)$ where the entries of Cesàro matrix are

$$c_{nk}=\begin{cases}\frac{1}{n+1} & , \quad 0<k\leq n\\ 0 & , \quad n<k\end{cases}$$

and $q-$analog of the integer n is

$$[n]_q=\frac{1-q^n}{1-q}, \quad (q\neq 1).$$

In [3] Durna and Türkay obtained the spectrum of q-Cesàro matrix $C_1(q)$ on the space of convergent sequences c. Also, they calculated spectral decomposition in the sense of Goldberg. In [6] El Shabrawy investigated boundedness, compactness and spectra of q-Cesàro matrix $C_1(q)$.

Let us denote the space $H(\mathbb{D})=\{f:\mathbb{D}\rightarrow\mathbb{C}\text{: } f \text{ is an analytic function}\}$, on $\mathbb{D}$ the standard Hardy space H^p $(1\leq p<\infty)$ where $\mathbb{D}$ is a unit disk, and p-summable complex-valued sequences on the set of non-negative integers ℓ^p. You can see [7] for details.

Previously, some authors have generalized various matrix transformations. For example, Young generalized the Cesàro matrix in his PhD thesis ([10]). For all $f \in H^2$ the generalized Cesàro operator with the symbol g is defined as

$$C_g(f)(z) = \frac{1}{z}\int_0^z f(t)\,g(t)\,dt.$$

Here, g is analytic in $\mathbb{D}$ and has Taylor series representation $g(z) = \sum_{j=0}^{\infty} a_j z^j$, where $\{z^{n-1}\}_{n=1}^{\infty}$ is the standard basis of H^2. The matrix form of C_g in the standard basis is

$$C_g = \begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ \frac{a_1}{2} & \frac{a_0}{2} & 0 & 0 & \cdots \\ \frac{a_2}{3} & \frac{a_1}{3} & \frac{a_0}{3} & 0 & \cdots \\ \frac{a_3}{4} & \frac{a_2}{4} & \frac{a_1}{4} & \frac{a_0}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

And so, one can express entries of matrix C_g as follows:

$$(c_g)_{nj} = \begin{cases} \frac{a_{n-j}}{n} & , \quad n \geq j \\ 0 & , \quad n < j \end{cases}.$$

For all $0 < q < 1$ we can define the generalized $q-$Cesàro matrix $C_g(q)$ by

$$C_g(q) = \begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ \frac{q}{1+q}a_1 & \frac{1}{1+q}a_0 & 0 & 0 & \cdots \\ \frac{q^2}{1+q+q^2}a_2 & \frac{q}{1+q+q^2}a_1 & \frac{1}{1+q+q^2}a_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and we have

$$(c_q(q))_{nk} = \begin{cases} \frac{q^{n-k}}{1+q+\cdots+q^{n-1}}a_{n-k} & , \quad 0 \leq k \leq n \\ 0 & , \quad n < k \end{cases}. \tag{1.1}$$

Since $\lim_{q\to 1^-}(c_g(q))_{nk} = \frac{a_{n-k}}{n}$, matrix $C_g(q)$ is a $q-$analog of generalized Cesàro matrix C_g. If we take $g(z) = \frac{1}{1-z}$ and limit for $q \to 1$, then we obtain $C_g(q) = C(q)$. Thus generalized q-Cesàro matrix contains both of generalized Cesàro matrix and ordinary Cesàro matrix.

The conjugate transpose of the matrix $C_g(q)$ is

$$C_g^*(q) = \begin{pmatrix} \overline{a_0} & \frac{q}{1+q}\overline{a_1} & \frac{q^2}{1+q+q^2}\overline{a_2} & \cdots \\ 0 & \frac{1}{1+q}\overline{a_0} & \frac{q}{1+q+q^2}\overline{a_1} & \cdots \\ 0 & 0 & \frac{1}{1+q+q^2}\overline{a_0} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and so,

$$\left(c_g^*(q)\right)_{nk} = \begin{cases} \frac{q^{n-k}}{1+q+\cdots+q^{n-1}}\overline{a_{n-k}} & , \quad 0 \le n \le k \\ 0 & , \quad k < n \end{cases}. \tag{1.2}$$

Inspired by the relationship between Rhaly and Cesàro matrices, Durna and Yıldırım in [4] defined the generalized Rhaly matrix using the generalized Cesàro matrix. They also investigated the topological properties of the generalized Rhaly matrix. In [5], Durna and Yıldırım show that if an infinite matrix B is commutative with every generalized terraced matrix, then $B = kI$, where I is a unit matrix and $k \in \mathbb{C}$. Also, they investigated normality and self-adjointedness of generalized terraced matrix. Now, let us examine normality and self-adjointedness of generalized q-Cesàro matrix $C_g(q)$.

2. RESULTS

Theorem 1. *$C_g(q)$ is a normal matrix iff for $c \in \mathbb{C}$, $g(z) = c$.*

Proof. We consider $(1,1)$ - entries of $C_g(q)C_g^*(q)$ and $C_g^*(q)C_g(q)$:

$$\begin{aligned} \left[C_g(q)C_g^*(q)\right]_{11} &= \sum_{k=1}^{\infty} (c_g(q))_{1k}\left(c_g^*(q)\right)_{k1} \\ &= (c_g(q))_{11}\left(c_g^*(q)\right)_{11} \\ &= a_0\overline{a_0} = |a_0|^2 \end{aligned}$$

and

$$\begin{aligned} \left[C_g^*(q)C_g(q)\right]_{11} &= \sum_{k=1}^{\infty} \left(c_g^*(q)\right)_{k1}(c_g(q))_{1k} \\ &= a_0\overline{a_0} + \sum_{k=2}^{\infty} \frac{q^{k-1}}{1+q+\cdots+q^{k-1}}\overline{a_{k-1}}\frac{q^{k-1}}{1+q+\cdots+q^{k-1}}a_{k-1} \\ &= |a_0|^2 + \sum_{k=2}^{\infty}\left(\frac{q^{k-1}}{1+q+\cdots+q^{k-1}}\right)^2 |a_{k-1}|^2. \end{aligned}$$

Since $C_g(q)$ is a normal matrix, we get $\sum_{k=2}^{\infty}\left(\frac{q^{k-1}}{1+q+\cdots q^{k-1}}\right)^2 |a_{k-1}|^2 = 0$. From here, we obtain that for all $i > 1$, $a_i = 0$. And so, $g(z) = a_0$, $a_0 \in \mathbb{C}$.

Conversely, if $g(z) = a_0$, $a_0 \in \mathbb{C}$ then we have

$$C_g(q) = \begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{1+q}a_0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{1+q+q^2}a_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

hence $C_g(q)$ is normal. □

Corollary 1. *C_1 is not normal.*

Corollary 2. *$C_1(q)$ is not normal.*

Theorem 2. *$C_g(q)$ is a self-adjoint matrix iff for $c \in \mathbb{R}$, $g(z) = c$.*

Proof. From (1.1) and (1.2), $a_0 = \overline{a_0}$ and

$$\frac{q}{1+q}a_1 = \frac{q^2}{1+q+q^2}a_2 = \frac{q^3}{1+q+q^2+q^3}a_3 \cdots = 0.$$

Since, for all $n \in \mathbb{N}$, $\frac{q^n}{1+q+\cdots+q^n} \neq 0$, then

$$a_0 = \overline{a_0}\,,\ a_1 = a_2 = a_3 = \cdots = 0.$$

Hence $a_0 \in \mathbb{R}$ and $g(z) = a_0 \in \mathbb{R}$.
The other direction is obvious. □

Corollary 3. *C_1 is not self-adjoint.*

Corollary 4. *$C_1(q)$ is not self-adjoint.*

Note that if $n \geq 0$ and if $T_{ij} = 0$ for $j < i+n$, we say that T is an n-triangular operator matrix ([9]).

Lemma 1 ([9, Theorem 2])**.** *Let $T = (T_{ij}) \in \mathcal{L}\left(\overset{\infty}{\underset{n=0}{\oplus}} \mathcal{H}\right)$ be a $1-$triangular operator matrix. Suppose that $T_{i,i+1}$ has dense range for all i. Then T is cyclic.*

Lemma 2 ([2, Proposition 3.6])**.** *Let T be a triangular operator whose diagonal entries with respect to some orthonormal basis for $\mathcal{H}$ are distinct. Then T is cyclic.*

Theorem 3. *$C_g^*(q)$ is cyclic for all $g \in \mathcal{B}$, where*

$$\mathcal{B} = \left\{ g : \mathbb{D} \to \mathbb{C} : \int_0^z g(t)\,dt \ \text{ is a bounded mean oscillation function} \right\}.$$

Proof. Suppose that $g(0) = 0$. From Lemma 1, we obtain the state of the theorem.
Let $g(0) \neq 0$. Therefore the diagonal entries of matrix $C_g^*(q)$ are distinct. And so, from Lemma 2, $C_g^*(q)$ is cyclic. □

Theorem 4. *If $g_\beta(z) = g(\beta z)$ with $|\beta| = 1$, then $C_{g_\beta}(q)$ is unitarily equivalent to $C_g(q)$. We denote this by $C_{g_\beta}(q) \approxeq C_g(q)$.*

Proof. Define the map $U_\beta : H^2 \to H^2$ by $U_\beta(f)(z) = f(\beta z)$. It is easy to see that U_β is unitary with $U_\beta^* = U_{\overline{\beta}}$.

Now, to show the unitary equivalence, we must prove that $U_\beta^* C_{g_\beta}(q) U_\beta = C_g(q)$.

The matrix representation of U_β in the basis $\left\{z^{n-1}\right\}_{n=1}^{\infty}$ is the diagonal matrix $diag\{\beta^n\}$. Moreover, we know that $\left(U_\beta\right)^* = U_{\overline{\beta}} = \left(U_\beta\right)^{-1}$. Using these matrix representations we have $U_\beta^* C_{g_\beta}(q) U_\beta = C_g(q)$ and consequently $C_{g_\beta}(q) \approxeq C_g(q)$ is obtained. □

Corollary 5. *If $\beta \in \partial(\mathbb{D})$, then the spectrum of $C_{\frac{1}{1-\beta z}}(q)$ is*

$$\sigma\left(C_{\frac{1}{1-\beta z}}(q)\right) = \sigma(C(q)) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{1+q}\right| \le \frac{q}{1+q}\right\}.$$

Proof. This is immediate from the unitary equivalence and [6], Theorem 2.2. □

In [8] Mursalenn et al. studied the spectrum and Hilbert Schmidt properties of generalized Rhaly matrices. Now, let us show that $C_g(q)$ is a Hilbert-Schmidt operator. For this, we shall give the following lemma.

Lemma 3 ([8]). *If $\alpha \in \mathbb{Z}$, $B_{j-1} = 0$, $B_k = \sum_{k=j}^{n} \beta^k$ for $k = j, \dots, n$ and (a_k) is a positive decreasing sequence, then*

$$\left|\sum_{k=j}^{n} \frac{\beta^k}{(a_k)^\alpha}\right| \le \frac{2}{|1-\beta|}\left(\frac{1}{a_n^\alpha} + \left|\frac{1}{a_j^\alpha} - \frac{1}{a_n^\alpha}\right|\right).$$

The set of all Hilbert Schmidt operators on H is denoted by $B_2(H)$.

Theorem 5. *If $\beta_1 \neq \beta_2$, then $C_\beta(q) C(q)$ is a Hilbert-Schmidt operator.*

Proof. Notice that

$$C_{\beta_1}(q) C_{\beta_2}(q) \in B_2\left(H^2\right) \Leftrightarrow U_{\overline{\beta_2}} C_{\beta_1}(q) U_{\beta_2} U_{\overline{\beta_2}} C_{\beta_2}(q) U_{\beta_2} \in B_2\left(H^2\right).$$

Since $U_{\overline{\beta_2}} C_{\beta_2}(q) U_{\beta_2} = C(q)$, then we can assume $\beta_2 = 1$. Rewrite β_1 as β. Since $C_\beta(q) = C_{\frac{1}{1-\beta z}}(q)$, we have

$$\left[C_\beta(q)\right]_{nj} = \begin{cases} \frac{(\beta q)^{n-j}}{1+q+\cdots+q^{n-1}}, & n \ge j \\ 0, & n < j \end{cases}, \; n, j = 1, 2, \dots$$

Since $[C(q)]_{nk} = \begin{cases} \frac{q^{n-k}}{1+q+\cdots+q^{n-1}}, & n \ge k \\ 0, & n < k \end{cases}$, $n, k = 1, 2, \dots$, we get that

$$\left[C_\beta(q) C(q)\right]_{nj} = \begin{cases} \frac{1}{1+q+\cdots+q^{n-1}} \sum_{k=j}^{n} (\beta q)^{n-k} \frac{q^{k-j}}{1+\cdots+q^{k-1}}, & n \ge j \\ 0, & n < j \end{cases}, \; n, j = 1, 2, \dots$$

$$= \begin{cases} \frac{\beta^n q^n q^{-j}}{1+q+\cdots+q^{n-1}} \sum_{k=j}^{n} \frac{\overline{\beta}^k}{1+\cdots+q^{k-1}}, & n \geq j \\ 0, & n < j \end{cases}, \; n, j = 1, 2, \ldots$$

From here,

$$\begin{aligned} \left\| C_\beta(q) C(q) \right\|_{H.S}^2 &= \sum_{n,j} \left| \left[C_\beta(q) C(q) \right]_{nj} \right|^2 \\ &= \sum_{n=1}^{\infty} \sum_{j=1}^{n} \left| \left[C_\beta(q) C(q) \right]_{nj} \right|^2 \\ &= \sum_{n=1}^{\infty} \frac{\beta^{2n} q^{2n}}{(1+q+\cdots+q^{n-1})^2} \sum_{j=1}^{n} q^{-2j} \left| \frac{\overline{\beta}^k}{1+q+\cdots+q^{k-1}} \right|^2 . \end{aligned}$$

From Lemma 3,

$$\begin{aligned} & \left| \sum_{k=j}^{n} \frac{\overline{\beta}^k}{1+\cdots+q^{k-1}} \right| \\ & \leq \frac{2}{\left|1-\overline{\beta}\right|} \left(\frac{1}{1+q+\cdots+q^{n-1}} + \left| \frac{1}{1+\cdots+q^{j-1}} - \frac{1}{1+q+\cdots+q^{n-1}} \right| \right) \\ & \leq \frac{2}{\left|1-\overline{\beta}\right| (1+q+\cdots+q^{j-1})}. \end{aligned}$$

We use this in the last equation,

$$\begin{aligned} \left\| C_\beta(q) C(q) \right\|_{H.S}^2 &= \sum_{n=1}^{\infty} \frac{\beta^{2n} q^{2n}}{(1+q+\cdots+q^{n-1})^2} \sum_{j=1}^{n} q^{-2j} \left| \frac{\overline{\beta}^k}{1+q+\cdots+q^{k-1}} \right|^2 \\ &\leq \frac{4}{\left|1-\overline{\beta}\right|^2} \sum_{n=1}^{\infty} \frac{\beta^{2n} q^{2n}}{(1+q+\cdots+q^{n-1})^2} \sum_{j=1}^{n} \frac{q^{-2j}}{\left|1+q+\cdots+q^{k-1}\right|^2} \\ &\leq \frac{4}{\left|1-\overline{\beta}\right|^2} \sum_{n=1}^{\infty} \frac{\beta^{2n} q^{2n}}{(1+q+\cdots+q^{n-1})^2} < \infty. \end{aligned}$$

Thus $C_\beta(q) C(q)$ is a Hilbert-Schmidt operator. □

Theorem 6. *If* $\beta_1 \neq \beta_2$, *then* $\left(C_{\beta_1}(q)\right)^* C_{\beta_2}(q)$ *and* $C_{\beta_1}(q) \left(C_{\beta_2}(q)\right)^*$ *are Hilbert-Schmidt operators.*

Proof. Without loss of generality, let us take $\beta_1 = \beta$. Firstly, we will show that $\left(C_{\beta_1}(q)\right)^* C_{\beta_2}(q) \in B_2\left(H^2\right)$.

$$\left[\left(C_\beta(q)\right)^*\right]_{nj} = \begin{cases} 0, & n > j \\ \frac{\overline{\beta}^{n-j} q^{n-j}}{1+q+\cdots+q^{n-1}}, & n \le j \end{cases}, \; n, j = 1, 2, \ldots$$

Therefore,

$$\left[\left(C_\beta(q)\right)^* C(q)\right]_{nj} = \frac{\overline{\beta}^n q^n q^{-j}}{1+q+\cdots+q^{n-1}} \sum_{k=\max\{n,j\}}^{\infty} \frac{\beta^k q^k}{1+q+\cdots+q^{k-1}}, \; n, j = 1, 2, \ldots$$

Since

$$\begin{aligned} &\sum_{k=n}^{m} \frac{\beta^k q^k}{1+q+\cdots+q^{k-1}} \\ &\le \frac{2}{|1-q\beta|}\left(\frac{q^{n-1}}{1+q+\cdots+q^{n-1}} + \left|\frac{q^{j-1}}{1+q+\cdots+q^{j-1}} - \frac{q^{n-1}}{1+q+\cdots+q^{n-1}}\right|\right) \\ &\le \frac{2q^{j-1}}{|1-q\beta|\left(1+q+\cdots+q^{j-1}\right)}, \end{aligned}$$

we have

$$\begin{aligned} \left[\left(C_\beta(q)\right)^* C(q)\right]_{nj} &= \sum_{n,j=1}^{\infty} \left|\sum_{k=j}^{\infty} \frac{\beta^k q^k}{1+q+\cdots+q^{k-1}}\right|^2 \\ &\le \sum_{n,j=1}^{\infty} \frac{4}{|1-q\beta|^2} \frac{q^{2j-2}}{\left(1+q+\cdots+q^{k-1}\right)^2} \\ &= \frac{4}{|1-q\beta|^2} \sum_{n,j=1}^{\infty} \frac{q^{2j-2}}{\left(1+q+\cdots+q^{k-1}\right)^2} < \infty. \end{aligned}$$

Thus $\left(C_{\beta_1}(q)\right)^* C_{\beta_2}(q)$ is a Hilbert-Shcmidt operator.

Now, let us show that $C_\beta(q)\left(C(q)\right)^* \in B_2\left(H^2\right)$.

$$\begin{aligned} \left[C_\beta(q)\left(C(q)\right)^*\right]_{nj} &= \sum_{k=1}^{\min\{n,j\}} \frac{\beta^{n-k} q^{n-k}}{1+q+\cdots+q^{n-1}} \frac{q^{k-j}}{1+q+\cdots+q^{k-1}} \\ &= \frac{\beta^n q^{n-j}}{1+q+\cdots+q^{n-1}} \sum_{k=1}^{\min\{n,j\}} \frac{\overline{\beta}^k}{1+q+\cdots+q^{k-1}} \end{aligned}$$

and so,

$$\left|\frac{\beta^n q^{n-j}}{1+q+\cdots+q^{n-1}} \sum_{k=1}^{\min\{n,j\}} \frac{\overline{\beta}^k}{1+q+\cdots+q^{k-1}}\right| \le \frac{q^{n-j}}{1+q+\cdots+q^{n-1}} \sum_{k=1}^{\min\{n,j\}} \left|\overline{\beta}^k\right|$$

$$= \frac{q^{n-j}}{1+q+\cdots+q^{n-1}}\left|\overline{\beta}\right|\frac{\left|1-\overline{\beta}^{\min\{n,j\}}\right|}{\left|1-\overline{\beta}\right|} \leq \frac{2q^{n-j}}{(1+q+\cdots+q^{n-1})\left|1-\overline{\beta}\right|}.$$

Therefore we obtain that

$$\left\|C_{\beta}(q)(C(q))^{*}\right\|_{H.S} = \sum_{n,j=1}^{\infty}\left|\frac{\beta^{n}q^{n-j}}{1+\ldots+q^{n-1}}\sum_{k=1}^{\min\{n,j\}}\frac{\overline{\beta}^{k}}{1+\ldots+q^{k-1}}\right|^{2}$$
$$\leq \frac{4}{\left|1-\overline{\beta}\right|^{2}}\sum_{n,j=1}^{\infty}\frac{q^{2n}q^{-2j}}{(1+\ldots+q^{n-1})^{2}} < \infty.$$

Thus $C_{\beta}(q)(C(q))^{*} \in B_{2}\left(H^{2}\right)$ is a Hilbert-Shcmidt operator. □

REFERENCES

[1] J. Bustoz and L. F. Gordillo, "q-Hausdorff summability." *J. Comput. Anal. Appl.*, vol. 7, no. 1, pp. 35–48, 2005.

[2] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, and D. R. Wood, "On the metric dimension of Cartesian products of graphs." *SIAM J. Discrete Math.*, vol. 21, no. 2, pp. 423–441, 2007, doi: 10.1137/050641867.

[3] N. Durna and M. E. Turkay, "The spectrum of q-Cesàro matrices on c and its various spectral decomposition for $0 < q < 1$." *Operators and Matrices*, vol. 15, no. 3, pp. 795–813, 2021, doi: 10.7153/oam-2021-15-55.

[4] N. Durna and M. Yildirim, "Topological results about the set of generalized Rhaly matrices." *Gen. Math. Notes*, vol. 17, no. 1, pp. 1–7, 2013.

[5] N. Durna and M. Yildirim, "Generalized terraced matrices." *Miskolc Math. Notes*, vol. 17, no. 1, pp. 201–208, 2016, doi: 10.18514/MMN.2016.1272.

[6] S. R. El-Shabrawy, "On q-Cesàro operators: Boundedness, compactness and spectra." *Numer. Func. Anal. Optim.*, vol. 42, no. 9, pp. 1019–1037, 2021, doi: 10.1080/01630563.2021.1933027.

[7] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities.* Cambridge, University Press, 1952.

[8] M. Mursaleen, M. Yildirim, and N. Durna, "On the spectrum and Hilbert Schmidt properties of generalized Rhaly matrices." *Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat.*, vol. 68, no. 1, pp. 712–723, 2019, doi: 10.31801/cfsuasmas.464177.

[9] W. R. Wogen, "On some operators with cyclic vectors." *Indiana Univ. Math. J.*, vol. 27, no. 1, pp. 163–171, 1978, doi: 10.1512/iumj.1978.27.27014.

[10] S. W. Young, *Algebraic and spectral properties of generalized Cesàro operators.* PhD Thesis, The University of North Carolina at Chapel Hill: ProQuest LLC, Ann Arbor, MI, 2002.

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