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A note on pseudoparallel submanifolds of Lorentzian para-Kenmotsu manifolds

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Abstract. In this article, pseudoparallel submanifolds for Lorentzian para-Kenmotsu manifolds are investigated. The Lorentzian para-Kenmotsu manifold is considered on the W_1-curvature tensor. Submanifolds of these manifolds with properties such as W_1-pseudoparallel, W_1-2 pseudoparallel, W_1-Ricci generalized pseudoparallel, and W_1-2 Ricci generalized pseudoparallel has been characterized.

1. Introduction

Para-Kenmotsu and special para-Kenmotsu manifolds, also known as Almost paracontact metric manifolds, were defined in 1989 by Sinha and Sai Prasad [1]. Sinha and Sai Prasad obtained important characterizations of para-Kenmotsu manifolds. In the following years, para-Kenmotsu manifolds attracted a lot of attention and many authors revealed the important properties of these manifolds. In 2018, Lorentzian para-Kenmotssu manifolds, known as Lorentzian almost paracontact metric manifolds, were introduced [2]. Then, the concept of q-semisymmetry for Lorentzian para-Kenmotsu manifolds is studied [3]. M. Atçeken studied invariant submanifolds of Lorentzian para-Kenmotsu manifolds in 2022 and in this study he gave the necessary and sufficient conditions for the Lorentzian para-Kenmotsu manifold to be total geodesic [4].

Characterizing invariant submanifolds of manifolds is an important problem. Invariant submanifolds of $(LCS)_n-$ manifolds by S.K. Hui et al. [5], invariant submanifolds of LP-Sasakian manifolds by V.Venkatesha et al. [6], invariant submanifolds of Kenmotsu manifolds by S.Sular et al. [7], invariant submanifolds of $(k,\mu)-$contact manifolds by M.S. Siddesha et al [8] have been discussed and revealed many important properties of this submanifolds. Similarly, this problem has been addressed by many other authors ([9],[10],[11],[12],[13]). Similarly, S.K. Hui et al. studied the pseudoparallel contact submanifolds of Kenmotsu manifolds in [14] and the Chaki-pseudoparallel invariant submanifolds of Sasakian manifolds in [15].

Motivated by all these studies, we studied invariant submanifolds of Lorentzian para-Kenmotsu manifolds on curvature tensors in this article. In this article, pseudoparallel submanifolds for Lorentzian para-Kenmotsu manifolds are investigated. The Lorentzian para-Kenmotsu manifold is considered on the W_1-curvature tensor. Submanifolds of these manifolds with properties such as W_1-pseudoparallel, W_1-2

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pseudoparallel, W_1-Ricci generalized pseudoparallel, and W_1-2 Ricci generalized pseudoparallel has been characterized.

2. Preliminary

Let $\hat{M}^n$ be an $n-$dimensional Lorentzian metric manifold. This means that it is endowed with a structure (ϕ, ξ, η, g), where ϕ is a $(1,1)-$type tensor field, ξ is a vector field, η is a $1-$form on $\hat{M}^n$ and g is a Lorentzian metric tensor satisfying;

$$\phi^2 X = X + \eta(X)\xi, g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{1}$$

$$\eta(\xi) = -1, \eta(X) = g(X,\xi) \tag{2}$$

for all vector fields X, Y on $\hat{M}^n$. Then $\hat{M}^n(\phi, \xi, \eta, g)$ is said to be Lorentzian almost paracontact manifold [16].

A Lorentzian almost paracontact manifold $\hat{M}^n(\phi, \xi, \eta, g)$ is called Lorentzian para-Kenmotsu manifold if

$$(\tilde{\nabla}_X \phi) Y = -g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{3}$$

for all $X, Y \in \Gamma(T\hat{M})$, where $\tilde{\nabla}$ and $\Gamma(T\hat{M})$ denote the Levi-Civita connection and differentiable vector fields set on $\hat{M}^n$, respectively.

Lemma 2.1. *Let $\hat{M}^n(\phi, \xi, \eta, g)$ be the $n-$dimensional Lorentzian para-Kenmotsu manifold. The following relations are provided for $\hat{M}^n(\phi, \xi, \eta, g)$.*

$$\tilde{\nabla}_X \xi = -\phi^2 X, \tag{4}$$

$$(\tilde{\nabla}_X \eta) Y = -g(X,Y) - \eta(X)\eta(Y), \tag{5}$$

$$\tilde{R}(\xi, Y) Z = g(Y,Z)\xi - \eta(Z) Y, \tag{6}$$

$$\tilde{R}(\xi, Y)\xi = \eta(Y)\xi + Y, \tag{7}$$

$$\tilde{R}(X, Y)\xi = \eta(Y) X - \eta(X) Y, \tag{8}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{9}$$

$$S(\xi,\xi) = (n-1)\xi, \tag{10}$$

where $\tilde{R}$ and S are the Riemann curvature tensor and Ricci curvature tensor of $\hat{M}^n(\phi, \xi, \eta, g)$, respectively.

For an $n-$dimensional (M, g) semi-Riemannian manifold, the W_1-curvature tensor is defined as

$$W_1(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y]. \tag{11}$$

If we choose $X = \xi, Y = \xi, Z = \xi$ in (11) for the $n-$dimensional Lorentzian para-Kenmotsu manifold, respectively, we obtain

$$W_1(\xi,Y)Z = g(Y,Z)\xi - 2\eta(Z)Y + \frac{1}{n-1}S(Y,Z)\xi, \tag{12}$$

$$W_1(X,Y)\xi = 2[\eta(Y)X - \eta(X)Y], \tag{13}$$

$$W_1(\xi,Y)\xi = 2[\eta(Y)\xi + Y]. \tag{14}$$

Let $\tilde{M}$ be the immersed submanifold of a Lorentzian para–Kenmotsu manifolds $\hat{M}^n\left(\phi,\xi,\eta,g\right)$. Let the tangent and normal subspaces of $\tilde{M}$ in $\hat{M}^n\left(\phi,\xi,\eta,g\right)$ be $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$, respectively. Gauss and Weingarten formulas for $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$ are

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{15}$$

$$\tilde{\nabla}_X V = -A_V X + \nabla_X^{\perp} V, \tag{16}$$

respectively, for all $X,Y \in \Gamma\left(T\tilde{M}\right)$ and $V \in \Gamma\left(T^{\perp}\tilde{M}\right)$, where ∇ and $\nabla^{\perp}$ are the connections on $\tilde{M}$ and $\Gamma\left(T^{\perp}\tilde{M}\right)$, respectively, h and A are the second fundamental form and the shape operator of $\tilde{M}$. There is a relation

$$g(A_V X, Y) = g(h(X,Y), V) \tag{17}$$

between the second basic form and shape operator defined as above. The covariant derivative of the second fundamental form h is defined as

$$(\tilde{\nabla}_X h)(Y,Z) = \nabla_X^{\perp} h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{18}$$

Specifically, if $\tilde{\nabla} h = 0$, $\tilde{M}$ is said to be is parallel second fundamental form [13].

Let R be the Riemann curvature tensor of $\tilde{M}$. In this case, the Gauss equation can be expressed as

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X \\ &+ (\tilde{\nabla}_X h)(Y,Z) - (\tilde{\nabla}_Y h)(X,Z),\end{aligned} \tag{19}$$

for all $X,Y,Z \in \Gamma\left(T\tilde{M}\right)$, where if

$$\left(\tilde{\nabla}_X h\right)(Y,Z) - \left(\tilde{\nabla}_Y h\right)(X,Z) = 0,$$

then it is called curvature-invariant submanifold. Let $\tilde{M}$ be a Riemannian manifold, T is $(0,k)$ −type tensor field and A is $(0,2)$ −type tensor field. In this case, Tachibana tensor field $Q(A,T)$ is defined as

$$\begin{aligned}Q(A,T)(X_1,...,X_k;X,Y) &= -T((X \wedge_A Y)X_1,...,X_k) \\ &-... - T(X_1,...,X_{k-1},(X \wedge_A Y)X_k),\end{aligned} \tag{20}$$

where,

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y, \tag{21}$$

$k \geq 1, X_1, X_2, ..., X_k, X, Y \in \Gamma\left(T\tilde{M}\right)$.

Definition 2.2. *A submanifold of a Riemannian manifold (M,g) is said to be pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel and 2-Ricci generalized pseudopğarallel if*

$\tilde{R}.h$ *and* $Q(g,h)$

$\tilde{R}.\tilde{\nabla}h$ *and* $Q\left(g,\tilde{\nabla}h\right)$

$\tilde{R}.h$ *and* $Q(S,h)$

$\tilde{R}.\tilde{\nabla}h$ *and* $Q\left(S,\tilde{\nabla}h\right)$

are linearly dependent, respectively [5].

3. Invariant Pseudoparalel Submanifolds of Lorentzian para-Kenmotsu Manifolds

Let $\tilde{M}$ be the immersed submanifold of an $n-$dimensional Lorentzian para-Kenmotsu manifold $\hat{M}^n\left(\phi,\xi,\eta,g\right)$. If $\phi\left(T_XM\right)\subset T_XM$ in every X point, the M manifold is called invariant submanifold. We note that all of properties of an invariant submanifold inherit the ambient manifold. From this section of the article, we will assume that the manifold $\tilde{M}$ is the invariant submanifold of the Lorentzian para-Kenmotsu manifold $\hat{M}^n\left(\phi,\xi,\eta,g\right)$. So, it is clear that

$$h\left(X,\xi\right)=0, h\left(\phi X,Y\right)=h\left(X,\phi Y\right)=\phi h\left(X,Y\right) \tag{22}$$

for all $X,Y\in\Gamma\left(TM\right)$.

Lemma 3.1. *Let $\tilde{M}$ be the invariant submanifold of the $n-$dimensional Lorentzian para–Kenmotsu manifold $\hat{M}^n\left(\phi,\xi,\eta,g\right)$. The second fundamental form h of $\tilde{M}$ is parallel if and only if $\tilde{M}$ is the total geodesic submanifold [4].*

Let us now consider the invariant submanifolds of the Lorentz para-Kenmotsu manifold on the W_1-curvature tensor.

Definition 3.2. *Let $\tilde{M}$ be the invariant submanifold of the $n-$dimensional Lorentzian para-Kenmotsu manifold $\hat{M}^n\left(\phi,\xi,\eta,g\right)$. If $W_1.h$ and $Q\left(g,h\right)$ are linearly dependent, $\tilde{M}$ is called W_1-pseudoparallel submanifold.*

Equivalent to this definition, it can be said that there is a function k_1 on the set $M_1=\left\{x\in\tilde{M}\middle| h\left(x\right)\neq g\left(x\right)\right\}$ such that

$$W_1\cdot h=k_1Q\left(g,h\right).$$

If $k_1=0$ specifically, $\tilde{M}$ is called a W_1-semiparallel submanifold.

Theorem 3.3. *Let $\tilde{M}$ be the invariant submanifold of the $n-$dimensional Lorentzian para–Kenmotsu manifold $\hat{M}^n\left(\phi,\xi,\eta,g\right)$. If $\tilde{M}$ is W_1-pseudoparallel submanifold, then $\tilde{M}$ is either a total geodesic or $k_1=2$.*

Proof. Let's assume that $\tilde{M}$ is a W_1-pseudoparallel submanifold. So, we can write

$$\left(W_1\left(X,Y\right)\cdot h\right)\left(U,V\right)=k_1Q\left(g,h\right)\left(U,V;X,Y\right), \tag{23}$$

for all $X,Y,U,V\in\Gamma\left(T\tilde{M}\right)$. From (23), it is clear that

$$\begin{aligned}&R^{\perp}\left(X,Y\right)h\left(U,V\right)-h\left(W_1\left(X,Y\right)U,V\right)\\&-h\left(U,W_1\left(X,Y\right)V\right)=-k_1\left\{h\left(\left(X\wedge_g Y\right)U,V\right)\right.\\&\left.+h\left(U,\left(X\wedge_g Y\right)V\right)\right\}.\end{aligned}$$

Easily from here, we can write

$$\begin{aligned}&R^{\perp}\left(X,Y\right)h\left(U,V\right)-h\left(W_1\left(X,Y\right)U,V\right)\\&-h\left(U,W_1\left(X,Y\right)V\right)=-k_1\left\{g\left(Y,U\right)h\left(X,V\right)\right.\\&-g\left(X,U\right)h\left(Y,V\right)+g\left(Y,V\right)h\left(U,X\right)\\&\left.-g\left(X,V\right)h\left(U,Y\right)\right\}.\end{aligned} \tag{24}$$

If we choose $V = \xi$ in (24) and make use of (13), (22), we get

$$\begin{aligned} & h(U, 2[\eta(Y)X - \eta(X)Y]) \\ & = k_1[\eta(Y)h(U,X) - \eta(X)h(U,Y)] \end{aligned} \tag{25}$$

IIf we choose $X = \xi$ in (25), we obtain

$$(k_1 - 2)h(U,Y) = 0.$$

This completes the proof. □

Proposition 3.4. *Let $\tilde{M}$ be an invariant pseudoparallel submanifold of the $n-$dimensional Lorentzian para–Kenmotsu manifold $\hat{M}^n(\phi, \xi, \eta, g)$. Then $\tilde{M}$ is W_1-semiparallel if and only if $\tilde{M}$ is totally geodesic.*

Theorem 3.5. *Let $\tilde{M}$ be an invariant pseudoparallel submanifold of the $n-$dimensional Lorentzian para–Kenmotsu manifold $\hat{M}^n(\phi, \xi, \eta, g)$. $Q(g, W_1 \cdot h) = 0$ if and only if $\tilde{M}$ is totally geodesic.*

Proof. Let's assume that $Q(g, W_1 \cdot h) = 0$. So, we can write

$$Q(g, W_1(X,Y) \cdot h)(W,Z;U,V) = 0,$$

for all $X, Y, Z, W, U, V \in \Gamma(T\tilde{M})$. Obviously that,

$$\begin{aligned} & -g(V,W)(W_1(X,Y) \cdot h)(U,Z) \\ & +g(U,W)(W_1(X,Y) \cdot h)(V,Z) \\ & -g(V,Z)(W_1(X,Y) \cdot h)(W,U) \\ & +g(U,Z)(W_1(X,Y) \cdot h)(W,V) = 0, \end{aligned}$$

and and from the last equality we can write

$$\begin{aligned} & -g(V,W)[R^{\perp}(X,Y)h(U,Z) - h(W_1(X,Y)U,Z) \\ & -h(U,W_1(X,Y)Z)] + g(U,W)[R^{\perp}(X,Y)h(V,Z) \\ & -h(W_1(X,Y)V,Z) - h(V,W_1(X,Y)Z)] \\ & -g(V,Z)[R^{\perp}(X,Y)h(W,U) - h(W_1(X,Y)W,U) \\ & -h(W,W_1(X,Y)U)] + g(U,Z)[R^{\perp}(X,Y)h(W,V) \\ & -h(W_1(X,Y)W,V) - h(W,W_1(X,Y)V)] = 0. \end{aligned} \tag{26}$$

If we choose $Y = Z = U = W = \xi$ in (26) and use (22), we get

$$h(V, W_1(X,\xi)\xi) = 0. \tag{27}$$

If we use (14) in (27), we obtain

$$h(V,X) = 0.$$

This completes the proof. The proof of the other side of the theorem is obviously. □

Definition 3.6. *Let $\tilde{M}$ be the invariant submanifold of the $n-$dimensional Lorentzian para-Kenmotsu manifold $\hat{M}^n\left(\phi,\xi,\eta,g\right)$. If $W_1\cdot\tilde{\nabla}h$ and $Q\left(g,\tilde{\nabla}h\right)$ are linearly dependent, then $\tilde{M}$ is called W_1-2 pseudoparallel submanifold.*

In this case, it can be said that there is a function k_2 on the set $M_2=\left\{x\in\tilde{M}\middle|\tilde{\nabla}h\left(x\right)\neq g\left(x\right)\right\}$ such that

$$W_1\cdot\tilde{\nabla}h=k_2Q\left(g,\tilde{\nabla}h\right).$$

If $k_2=0$ specifically, M is called a W_1 $2-$semiparallel submanifold.

Theorem 3.7. *Let $\tilde{M}$ be the invariant submanifold of the $n-$dimensional Lorentzian para-Kenmotsu manifold $\hat{M}^n\left(\phi,\xi,\eta,g\right)$. If $\tilde{M}$ is W_1 $2-$pseudoparallel submanifold, then $\tilde{M}$ is either a total geodesic submanifold or $k_2=\frac{3}{2}$.*

Proof. Let's assume that $\tilde{M}$ is a W_1 $2-$pseudoparallel submanifold. So, we can write

$$\left(W_1\left(X,Y\right)\cdot\tilde{\nabla}h\right)\left(U,V,Z\right)=k_2Q\left(g,\tilde{\nabla}h\right)\left(U,V,Z;X,Y\right),\tag{28}$$

for all $X,Y,U,V,Z\in\Gamma\left(TM\right)$. If we choose $X=Z=\xi$ in (28), we can write

$$\begin{aligned}&R^{\perp}\left(\xi,Y\right)\left(\tilde{\nabla}_Uh\right)\left(V,\xi\right)-\left(\tilde{\nabla}_{W_1(\xi,Y)U}h\right)\left(V,\xi\right)\\&-\left(\tilde{\nabla}_Uh\right)\left(W_1\left(\xi,Y\right)V,\xi\right)-\left(\tilde{\nabla}_Uh\right)\left(V,W_1\left(\xi,Y\right)\xi\right)\\&=-k_2\left\{\left(\tilde{\nabla}_{(\xi\wedge_gY)U}h\right)\left(V,\xi\right)+\left(\tilde{\nabla}_Uh\right)\left(\left(\xi\wedge_gY\right)V,\xi\right)\right.\\&\left.+\left(\tilde{\nabla}_Uh\right)\left(V,\left(\xi\wedge_gY\right)\xi\right)\right\}.\end{aligned}\tag{29}$$

Let's calculate all the expressions in (29). So, we can write

$$\begin{aligned}&R^{\perp}\left(\xi,Y\right)\left(\tilde{\nabla}_Uh\right)\left(V,\xi\right)=W_1^{\perp}\left(\xi,Y\right)\left\{\nabla_U^{\perp}h\left(V,\xi\right)\right.\\&\left.-h\left(\nabla_UV,\xi\right)-h\left(V,\nabla_U\xi\right)\right\}\\&=R^{\perp}\left(\xi,Y\right)\phi^2h\left(V,U\right),\end{aligned}\tag{30}$$

$$\begin{aligned}&\left(\tilde{\nabla}_{W_1(\xi,Y)U}h\right)\left(V,\xi\right)=\nabla^{\perp}_{W_1(\xi,Y)U}h\left(V,\xi\right)\\&-h\left(\nabla_{W_1(\xi,Y)U}V,\xi\right)-h\left(V,\nabla_{W_1(\xi,Y)U}\xi\right)\\&=-h\left(V,-\phi^2W_1\left(\xi,Y\right)U\right)\\&=-2\eta\left(U\right)\phi^2h\left(V,Y\right),\end{aligned}\tag{31}$$

$$\begin{aligned}&\left(\tilde{\nabla}_Uh\right)\left(W_1\left(\xi,Y\right)V,\xi\right)=\nabla_U^{\perp}h\left(W_1\left(\xi,Y\right)V,\xi\right)\\&-h\left(\nabla_UW_1\left(\xi,Y\right)V,\xi\right)-h\left(W_1\left(\xi,Y\right)V,\nabla_U\xi\right)\\&=-h\left(g\left(Y,V\right)\xi-2\eta\left(V\right)Y+\tfrac{1}{n-1}S\left(Y,V\right)\xi\right.\\&\left.,-\phi^2U\right)\\&=-2\eta\left(V\right)\phi^2h\left(Y,U\right),\end{aligned}\tag{32}$$

$$
\begin{aligned}
&(\tilde{\nabla}_U h)(V, W_1(\xi, Y)\xi) = (\tilde{\nabla}_U h)(V, 2[\eta(Y)\xi + Y]) \\
&= 2(\tilde{\nabla}_U h)(V, \eta(Y)\xi) + 2(\tilde{\nabla}_U h)(V, Y) \\
&= -2h(V, U\eta(Y)\xi + \eta(Y)\nabla_U \xi) + 2(\tilde{\nabla}_U h)(V, Y) \\
&= 2\eta(Y)\phi^2 h(V, U) + 2(\tilde{\nabla}_U h)(V, Y),
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
&\left(\tilde{\nabla}_{(\xi\wedge_g Y)U} h\right)(V, \xi) = \nabla^{\perp}_{(\xi\wedge_g Y)U} h(V, \xi) \\
&-h\left(\nabla_{(\xi\wedge_g Y)U} V, \xi\right) - h\left(V, \nabla_{(\xi\wedge_g Y)U}\xi\right) \\
&= -h\left(V, -\phi^2 [g(Y, U)\xi\right. \\
&\left. -g(\xi, U)Y]\right) \\
&= -\eta(U)\phi^2 h(V, Y),
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
&(\tilde{\nabla}_U h)\left(\left(\xi \wedge_g Y\right)V, \xi\right) = \nabla^{\perp}_U h\left(\left(\xi \wedge_g Y\right)V, \xi\right) \\
&-h\left(\nabla_U \left(\xi \wedge_g Y\right)V, \xi\right) - h\left(\left(\xi \wedge_g Y\right)V, \nabla_U \xi\right) \\
&= -h(g(Y, V)\xi - g(\xi, V)Y, \\
&\left. -\phi^2 U\right) \\
&= -\eta(V)\phi^2 h(Y, U),
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
&(\tilde{\nabla}_U h)\left(V, \left(\xi \wedge_g Y\right)\xi\right) = (\tilde{\nabla}_U h)(V, \eta(Y)\xi + Y) \\
&= (\tilde{\nabla}_U h)(V, \eta(Y)\xi) + (\tilde{\nabla}_U h)(V, Y) \\
&= \eta(Y)\phi^2 h(V, U) + (\tilde{\nabla}_U h)(V, Y).
\end{aligned} \tag{36}
$$

If we substitute (30), (31), (32), (33), (34), (35), (36) in (29), we obtain

$$
\begin{aligned}
&R^{\perp}(\xi, Y)\phi^2 h(V, U) + 2\eta(U)\phi^2 h(V, Y) \\
&+2\eta(V)\phi^2 h(Y, U) - 2\eta(Y)\phi^2 h(V, U) \\
&-2(\tilde{\nabla}_U h)(V, Y) = -k_2\left\{-\eta(V)\phi^2 h(U, Y)\right. \\
&-\eta(U)\phi^2 h(Y, V) + \eta(Y)\phi^2 h(V, U) \\
&\left. +(\tilde{\nabla}_U h)(V, Y)\right\}.
\end{aligned} \tag{37}
$$

If we choose $V = \xi$ in (27) and by using (22), also if we use,

$$(\tilde{\nabla}_U h)(\xi, Y) = \phi^2 h(U, Y),$$

we get

$$[3-2k_2]\,h(Y,U)=0.$$

This completes of the proof. □

Proposition 3.8. *Let $\tilde{M}$ be an invariant pseudoparallel submanifold of the $n-$dimensional Lorentzian para–Kenmotsu manifold $\hat{M}^n\left(\phi,\xi,\eta,g\right)$. Then $\tilde{M}$ is W_1 $2-$semiparallel if and only if $\tilde{M}$ is totally geodesic.*

Definition 3.9. *Let $\tilde{M}$ be the invariant submanifold of the $n-$dimensional Lorentzian para-Kenmotsu manifold $\hat{M}^n\left(\phi,\xi,\eta,g\right)$. If $W_1\cdot h$ and $Q(S,h)$ are linearly dependent, $\tilde{M}$ is called W_1-Ricci generalized pseudoparallel submanifold.*

In this case, there is a function k_3 on the set $M_3=\left\{x\in\tilde{M}\middle|\,h(x)\neq S(x)\right\}$ such that

$$W_1\cdot h=k_3Q(S,h).$$

If $k_3=0$ specifically, M is called a W_1 Ricci generalized semiparallel submanifold.

Theorem 3.10. *Let $\tilde{M}$ be the invariant submanifold of the $n-$dimensional Lorentzian para-Kenmotsu manifold $\hat{M}^n\left(\phi,\xi,\eta,g\right)$. If $\tilde{M}$ is W_1-Ricci generalized pseudoparallel submanifold, then $\tilde{M}$ is either a total geodesic or $k_3=\dfrac{2}{n-1}$.*

Proof. Let's assume that $\tilde{M}$ is a W_1-Ricci generalized pseudoparallel submanifold. So, we can write

$$(W_1(X,Y)\cdot h)(U,V)=k_3Q(S,h)(U,V;X,Y),\tag{38}$$

that is

$$R^{\perp}(X,Y)h(U,V)-h(W_1(X,Y)U,V)$$
$$-h(U,W_1(X,Y)V)=-k_3\{h((X\wedge_S Y)U,V)$$
$$+h(U,(X\wedge_S Y)V)\}.$$

for all $X,Y,U,V\in\Gamma(TM)$. If we choose $X=V=\xi$ in from the last equality and make use of (22), we get

$$-h(U,W_1(\xi,Y)\xi)=k_3S(\xi,\xi)h(U,Y).\tag{39}$$

If we use (10) and (14) in (39), we have

$$[2-(n-1)k_3]\,h(U,Y)=0.$$

This completes the proof. □

Proposition 3.11. *Let $\tilde{M}$ be an invariant pseudoparallel submanifold of the $n-$dimensional Lorentzian para–Kenmotsu manifold $\hat{M}^n\left(\phi,\xi,\eta,g\right)$. Then $\tilde{M}$ is W_1 Ricci generalized semiparallel if and only if $\tilde{M}$ is totally geodesic.*

Theorem 3.12. *Let $\tilde{M}$ be an invariant pseudoparallel submanifold of the $n-$dimensional Lorentzian para–Kenmotsu manifold $\hat{M}^n\left(\phi,\xi,\eta,g\right)$. $Q(S,W_1\cdot h)=0$ if and only if $\tilde{M}$ is totally geodesic.*

Proof. Let's assume that $Q(S, W_1 \cdot h) = 0$. So, we can write

$$Q(S, W_1(X,Y) \cdot h)(W, Z; U, V) = 0,$$

for all $X, Y, Z, W, U, V \in \Gamma\left(T\tilde{M}\right)$. Obviously that,

$$\begin{aligned}
&-S(V,W)(W_1(X,Y).h)(U,Z)\\
&+S(U,W)(W_1(X,Y).h)(V,Z)\\
&-S(V,Z)(W_1(X,Y).h)(W,U)\\
&+S(U,Z)(W_1(X,Y).h)(W,V) = 0,
\end{aligned}$$

and and from the last equality we can write

$$\begin{aligned}
&-S(V,W)[R^{\perp}(X,Y)h(U,Z) - h(W_1(X,Y)U,Z)\\
&-h(U,W_1(X,Y)Z)] + S(U,W)[R^{\perp}(X,Y)h(V,Z)\\
&-h(W_1(X,Y)V,Z) - h(V,W_1(X,Y)Z)]\\
&-S(V,Z)[R^{\perp}(X,Y)h(W,U) - h(W_1(X,Y)W,U)\\
&-h(W,W_1(X,Y)U)] + S(U,Z)[R^{\perp}(X,Y)h(W,V)\\
&-h(W_1(X,Y)W,V) - h(W,W_1(X,Y)V)] = 0.
\end{aligned} \tag{40}$$

If we choose $Y = Z = U = W = \xi$ in (40) and use (22), we get

$$S(\xi,\xi)h(V, W_1(X,\xi)\xi) = 0. \tag{41}$$

If we use (10) and (14) in (41), we obtain

$$h(V,X) = 0.$$

This completes the proof. □

Definition 3.13. *Let $\tilde{M}$ be an invariant pseudoparallel submanifold of the $n-$dimensional Lorentzian para–Kenmotsu manifold $\hat{M}^n\left(\phi,\xi,\eta,g\right)$. If $W_1 \cdot \tilde{\nabla}h$ and $Q(S,\tilde{\nabla}h)$ are linearly dependent, $\tilde{M}$ is called W_1 $2-$Ricci generalized pseudoparallel submanifold.*

Then, there is a function k_4 on the set $M_4 = \left\{x \in \tilde{M} \middle| \tilde{\nabla}h(x) \neq S(x)\right\}$ such that

$$W_1 \cdot \tilde{\nabla}h = k_4 Q(S,\tilde{\nabla}h).$$

If $k_4 = 0$ specifically, M is called a W_1 $2-$Ricci generalized semiparallel submanifold.

Theorem 3.14. *Let $\tilde{M}$ be an invariant pseudoparallel submanifold of the $n-$dimensional Lorentzian para–Kenmotsu manifold $\hat{M}^n\left(\phi,\xi,\eta,g\right)$. If $\tilde{M}$ is W_1 $2-$ Ricci generalized pseudoparallel submanifold, then $\tilde{M}$ is either a total geodesic or $k_4 = \dfrac{4}{n-2}$.*

Proof. Let's assume that $\tilde{M}$ is a W_1 2−Ricci generalized pseudoparallel submanifold. So, we can write

$$(W_1(X,Y)\cdot\tilde{\nabla}h)(U,V,Z)=k_4Q(S,\tilde{\nabla}h)(U,V,Z;X,Y), \tag{42}$$

for all $X,Y,U,V,Z\in\Gamma(TM)$. If we choose $X=V=\xi$ in (42), we can write

$$\begin{aligned}
&R^{\perp}(\xi,Y)(\tilde{\nabla}_Uh)(\xi,Z)-\left(\tilde{\nabla}_{W_1(\xi,Y)U}h\right)(\xi,Z)\\
&-(\tilde{\nabla}_Uh)(W_1(\xi,Y)\xi,Z)-(\tilde{\nabla}_Uh)(\xi,W_1(\xi,Y)Z)\\
&=-k_4\left\{\left(\tilde{\nabla}_{(\xi\wedge_S Y)U}h\right)(\xi,Z)+(\tilde{\nabla}_Uh)((\xi\wedge_S Y)\xi,Z)\right.\\
&\left.+(\tilde{\nabla}_Uh)(\xi,(\xi\wedge_S Y)Z)\right\}.
\end{aligned}\tag{43}$$

Let's calculate all the expressions in (43). Firstly, we can write

$$\begin{aligned}
&R^{\perp}(\xi,Y)(\tilde{\nabla}_Uh)(\xi,Z)=W_1^{\perp}(\xi,Y)\left\{\nabla_U^{\perp}h(\xi,Z)\right.\\
&\left.-h(\nabla_UZ,\xi)-h(Z,\nabla_U\xi)\right\}\\
&=R^{\perp}(\xi,Y)\phi^2h(Z,U),
\end{aligned}\tag{44}$$

$$\begin{aligned}
&\left(\tilde{\nabla}_{W_1(\xi,Y)U}h\right)(\xi,Z)=\nabla^{\perp}_{W_1(\xi,Y)U}h(\xi,Z)\\
&-h\left(\nabla_{W_1R(\xi,Y)U}\xi,Z\right)-h\left(\xi,\nabla_{W_1(\xi,Y)U}Z\right)\\
&=-h\left(-\phi^2W_1(\xi,Y)U,Z\right)\\
&=-2\eta(U)\phi^2h(Y,Z),
\end{aligned}\tag{45}$$

$$\begin{aligned}
&(\tilde{\nabla}_Uh)(W_1(\xi,Y)\xi,Z)=(\tilde{\nabla}_Uh)(2[\eta(Y)\xi+Y],Z)\\
&=2(\tilde{\nabla}_Uh)(\eta(Y)\xi,Z)+2(\tilde{\nabla}_Uh)(Y,Z)\\
&=2\eta(Y)\phi^2h(U,Z)+2(\tilde{\nabla}_Uh)(Y,Z),
\end{aligned}\tag{46}$$

$$\begin{aligned}
&(\tilde{\nabla}_Uh)(\xi,W_1(\xi,Y)Z)=\nabla_U^{\perp}h(\xi,W_1(\xi,Y)Z)\\
&-h(\nabla_U\xi,W_1(\xi,Y)Z)-h(\xi,\nabla_UW_1(\xi,Y)Z)\\
&=-h\left(-\phi^2U,[g(Y,Z)\xi-2\eta(Z)Y\right.\\
&\left.+\tfrac{1}{n-1}S(Y,Z)\xi]\right)\\
&=-2\eta(Z)\phi^2h(U,Y)
\end{aligned}\tag{47}$$

$$\begin{aligned}&\left(\tilde{\nabla}_{(\xi\wedge_S Y)U}h\right)(\xi,Z)=\nabla^{\perp}_{(\xi\wedge_S Y)U}h(\xi,Z)\\&-h\left(\nabla_{(\xi\wedge_S Y)U}\xi,Z\right)-h\left(\xi,\nabla_{(\xi\wedge_S Y)U}Z\right)\\&=-S(Y,U)h(\nabla_\xi\xi,Z)+S(\xi,U)h(\nabla_Y\xi,Z)\\&=-(n-1)\eta(U)\phi^2h(Y,Z),\end{aligned}\tag{48}$$

$$\begin{aligned}&(\tilde{\nabla}_U h)((\xi\wedge_S Y)\xi,Z)=(\tilde{\nabla}_U h)(S(Y,\xi)\xi-S(\xi,\xi)Y,Z)\\&=(\tilde{\nabla}_U h)((n-1)\eta(Y)\xi+(n-1)Y,Z)\\&=(n-1)\left\{\nabla^{\perp}_U h(\eta(Y)\xi,Z)-h(\nabla_U\eta(Y)\xi,Z)\right.\\&\left.-h(\eta(Y)\xi,\nabla_U Z)+(\tilde{\nabla}_U h)(Y,Z)\right\}\\&=(n-1)(\tilde{\nabla}_U h)(Y,Z)\\&+(n-1)\eta(Y)\phi^2h(U,Z),\end{aligned}\tag{49}$$

$$\begin{aligned}&(\tilde{\nabla}_U h)(\xi,(\xi\wedge_S Y)Z)=(\tilde{\nabla}_U h)(\xi,S(Y,Z)\xi-S(\xi,Z)Y)\\&=(\tilde{\nabla}_U h)(\xi,S(Y,Z)\xi)-(n-1)(\tilde{\nabla}_U h)(\xi,\eta(Z)Y)\\&=\eta(Z)\phi^2h(U,Y).\end{aligned}\tag{50}$$

If we substitute (44), (45), (46), (47), (48), (49), (50) in (43), we obtain

$$\begin{aligned}&R^{\perp}(\xi,Y)\phi^2h(Z,U)+2\eta(U)\phi^2h(Y,Z)\\&-2\eta(Y)\phi^2h(U,Z)+2\eta(Z)\phi^2h(U,Y)\\&-2(\tilde{\nabla}_U h)(Y,Z)=-k_4\left\{-(n-1)\eta(U)\phi^2h(Y,Z)\right.\\&+(n-1)\eta(Y)\phi^2h(U,Z)+\eta(Z)\phi^2h(U,Y)\\&\left.+(n-1)(\tilde{\nabla}_U h)(Y,Z)\right\}.\end{aligned}\tag{51}$$

If we choose $Z=\xi$ in (51) and use (15) and

$$(\tilde{\nabla}_U h)(Y,\xi)=\phi^2h(U,Y),$$

we get

$$[4-(n-2)k_4]\,h(U,Y)=0.$$

This completes the proof. □

Proposition 3.15. *Let $\tilde{M}$ be an invariant pseudoparallel submanifold of the $n-$dimensional Lorentzian para–Kenmotsu manifold $\hat{M}^n(\phi,\xi,\eta,g)$. Then $\tilde{M}$ is W_1 $2-$Ricci generalized semiparallel if and only if $\tilde{M}$ is totally geodesic.*

Let us now give an example that satisfies the theorems we have given above. In [4], M. Atceken took a 5−dimensional Lorentzian para-Kenmotsu manifold and obtained its 3−dimensional submanifold. And then he showed that the 3−dimensional manifold he obtained satisfies the conditions for being pseudoparallel, 2−pseudoparallel, Ricci generalized pseudoparallel, and 2−Ricci generalized pseudoparallel in [4]. Let us show that the same manifold satisfies the conditions of the theorems we proved above.

Example 3.16. *Let us consider the* 5−*dimensional manifold*

$$\hat{M}^5 = \{(x_1, x_2, x_3, x_4, z) | z > 0\},$$

where (x_1, x_2, x_3, x_4, z) *denote the standard coordinates of* $\mathbb{R}^5$. *Then let* e_1, e_2, e_3, e_4, e_5 *be vector fields on* $\hat{M}^5$ *given by*

$$e_1 = z\frac{\partial}{\partial x_1}, e_2 = z\frac{\partial}{\partial x_2}, e_3 = \frac{\partial}{\partial x_3}, e_4 = \frac{\partial}{\partial x_4}, e_5 = \frac{\partial}{\partial z}$$

which are linearly independent at each point of $\hat{M}^5$ *and we define a Lorentzian metric tensor* g *on* $\hat{M}^5$ *as*

$$g(e_i, e_i) = 1, 1 \leq i \leq 4$$

$$g\left(e_i, e_j\right) = 0, 1 \leq i \neq j \leq 5$$

$$g(e_5, e_5) = -1.$$

Let η *be the* 1−*form defined by* $\eta(X) = g(X, e_5)$ *for all* $X \in \Gamma\left(T\hat{M}\right)$. *Now, we define the tensor field* $(1,1)$−*type* φ *such that*

$$\varphi e_1 = -e_2, \varphi e_3 = -e_4, \varphi e_5 = 0.$$

Then for $X = x_i e_i, Y = y_j e_j \in \Gamma\left(T\hat{M}\right), 1 \leq i, j \leq 5$, *we can easily see that*

$$\varphi^2 X = X + \eta(X)\xi, \xi = e_5, \eta(X) = g(X, \xi)$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y).$$

By direct calculations, only non-vanishing components are

$$[e_i, e_5] = -e_i, 1 \leq i \leq 4.$$

From Kozsul's formula, we can compute

$$\tilde{\nabla}_{e_i} e_5 = -e_i, 1 \leq i \leq 4.$$

Thus for $X = x_i e_i, Y = y_j e_j \in \Gamma\left(T\hat{M}\right)\hat{;}$ *we have*

$$\tilde{\nabla}_X \xi = -X - \eta(X)\xi,$$

and

$$(\tilde{\nabla}_X \varphi) Y = -g(\varphi X, Y)\xi - \eta(Y)\varphi X,$$

that is, $\hat{M}^5(\varphi, \xi, \eta, g)$ *is a Lorentzian para-Kenmotsu manifold.*
Now, we consider the 3−*dimensional submanifold* $\tilde{M}^3$ *of* $\hat{M}^5(\varphi, \xi, \eta, g)$ *given by* ψ−*immersion*

$$\psi : \tilde{M}^3 \to \hat{M}^5(\varphi, \xi, \eta, g)$$

$$\psi\left(x_1, x_2, z\right) = \left(zx_1, zx_2, zx_1, zx_2, \frac{1}{2}z^2\right).$$

Then the tangent space of submanifold $\tilde{M}$ is spanned by the vector fields

$$U = z\frac{\partial}{\partial x_1} + z\frac{\partial}{\partial x_3 = e_1 + e_3}, V = z\frac{\partial}{\partial x_2} + z\frac{\partial}{\partial x_4} = e_2 + e_4, \xi = z\frac{\partial}{\partial z}.$$

Thus we can see that

$$\varphi U = \varphi\left(e_1 + e_3\right) = -e_2 - e_4 = -V.$$

This verifies $\tilde{M}$ is a $3-$dimensional invariant submanifold of a Lorentzian para-Kenmotsu manifold $\hat{M}^5\left(\varphi, \xi, \eta, g\right)$. On the other hand, we can easily that

$$\tilde{\nabla}_U V = \hat{\nabla}_V U = 0, \tilde{\nabla}_U \xi = -U, \tilde{\nabla}_V \xi = -V.$$

Also this tell us that $\tilde{M}$ is W_1-pseudoparallel, $W_1 - 2$ pseudoparallel, W_1- Ricci generalized pseudoparallel and $W_1 - 2$ Ricci generalized pseudoparallel.

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